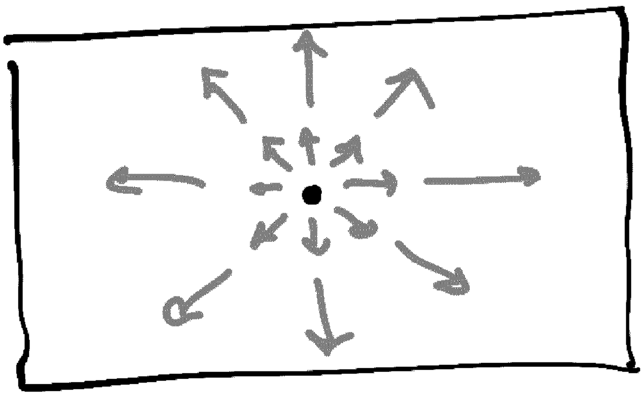
MAT 203 L20 Tue Nov 18

Recall: • Vector field: an assignment of a vector at every point in space, or the plane.

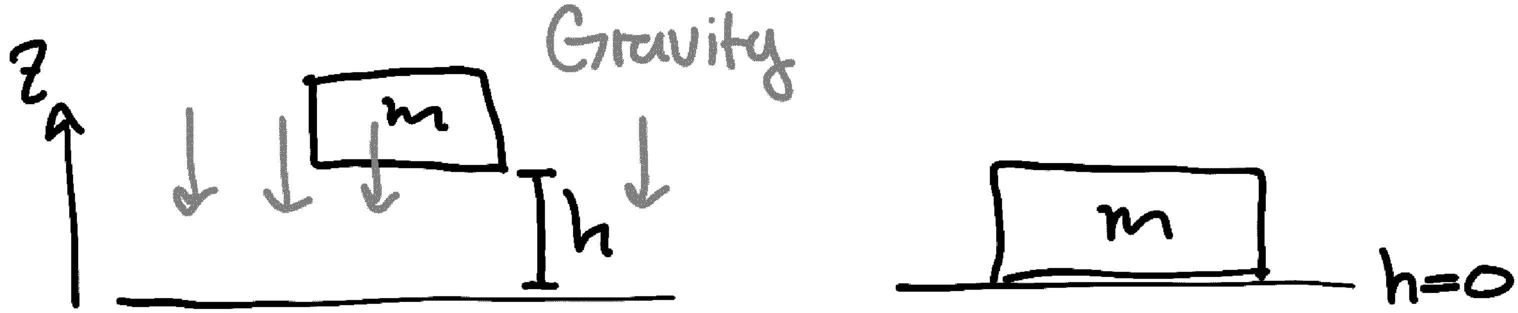


• If f is a differentiable function, its gradient is a vector field

• If $\vec{F}$ is a vector field such that $\vec{F} = \nabla f$ for some function f, $\vec{F}$ is called conservative, and f is called a potential.

§6.2 Line integrals

Energy/Work in physics.



box w/ mass m

The work done by gravity on the box is

$$W = \text{Force} \cdot \Delta\text{height}$$
$$= \langle 0,0,-mg \rangle \cdot \langle 0,0,0-h \rangle$$
$$= mgh$$

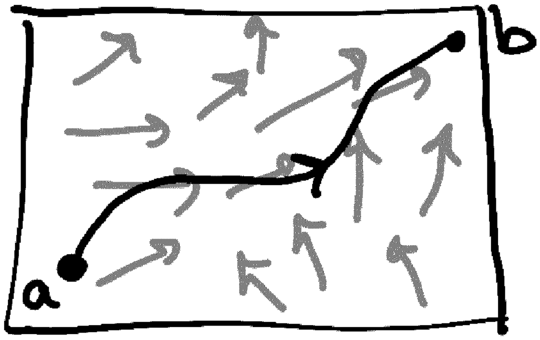
Another way to phrase this is as an integral:

$$\int_0^h mg\, dz = \int_0^h \text{Force}\, dz$$

This kind of integral is the special case of a more general integral:

If $\vec{F}$ is a vector field in the plane (let's say),

$\vec{F}$



then if C is a parametrized curve in the plane

one can compute the work done requires to walk from a to b.

How do we do it?

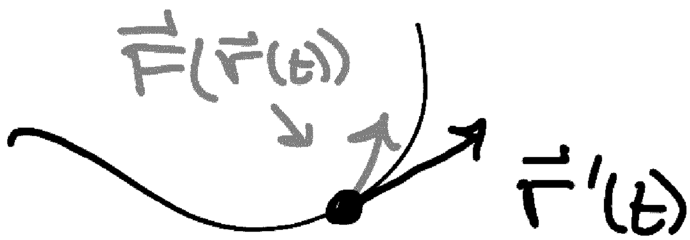
Let's assume the parametrization of C is given by:

$$\vec{r}(t), \quad t_0 \le t \le t_1$$

So that $\begin{cases} \vec{r}(t_0) = a \\ \vec{r}(t_1) = b \end{cases}$

Then Energy $= \displaystyle\int_{t_0}^{t_1} \text{Force}(\vec{r}(t))\,dt$

but what is Force at $\vec{r}(t)$?



$\text{Force}(\vec{r}(t)) = \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)$. So

$$\boxed{\text{Energy} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt.}$$

Now, we can write this as

$$\boxed{\text{Energy} = \int_C \vec{F} \bullet d\vec{r}}$$

This is an example of a line integral.

Def: Let C be a parametrized curve in space, and let $\vec{F}$ be a vector field. Suppose the parametrization of C is $\vec{r}(t)$ for $t_0 \le t \le t_1$. Then

$$\boxed{\int_C \vec{F} \bullet d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt}$$

Ex: Consider the curve C parametrized by $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \pi$, and consider the vector field $\vec{F}(x,y) = \langle -y, x \rangle$.

Let's compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}(t)\, dt \quad (*)$$

First $\vec{F}(\vec{r}(t)) = \vec{F}(\cos t, \sin t)$

$= \langle -\sin t, \cos t \rangle$, and

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$, so

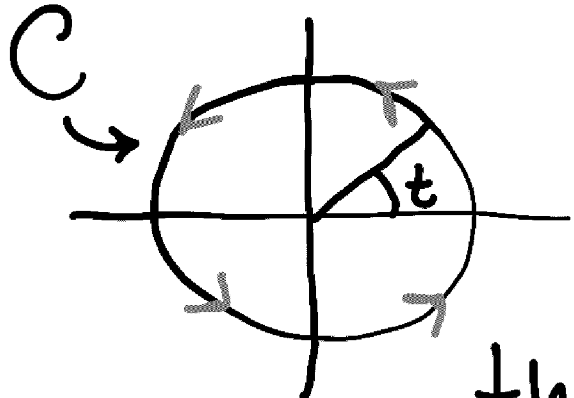
$$(*) = \int_0^{\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi} (-\sin t)^2 + (\cos t)^2 \, dt = \int_0^{\pi} dt = \pi.$$

Recall that a curve w/ a

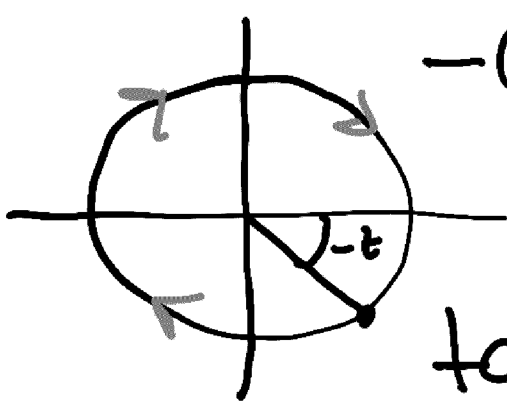
parametrization has an <u>orientation</u>

<u>Ex</u>: $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$



orientation is <u>counterclockwise</u> (ccw) since t measures the angle, and as t increases, the rotation goes ccw.

If we change the parametrization to $\vec{s}(t) = \langle \cos(-t), \sin(-t) \rangle$ we get the same curve but with the <u>opposite orientation</u>



$-C$ We call the curve C with the opposite orientation $-C$.

Ex: Let $\vec{F}(x,y) = \langle -y, x \rangle$ and let $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $(0 \le t \le \pi)$ be a parametrization of the curve C. In the previous example we computed $\int_C \vec{F} \cdot d\vec{r}$.

Let's now compute $\int_{-C} \vec{F} \cdot d\vec{r}$.

The curve $-C$ is parametrized by

$$\vec{s}(t) = \vec{r}(-t) = \langle \cos(-t), \sin(-t) \rangle$$
$$= \langle \cos t, -\sin t \rangle.$$

$0 \le t \le \pi$ becomes $0 \le -t \le \pi$

$\Leftrightarrow \; 0 \ge t \ge -\pi$.

$$\int_{-C} \vec{F}(\vec{s}(t)) \cdot \vec{s}\,'(t)\, dt$$

$$= \int_{-\pi}^{0} \vec{F}(\cos t, -\sin t) \cdot \langle -\sin t, -\cos t \rangle \, dt$$

$$= \int_{-\pi}^{0} \langle \sin t, \cos t\rangle \cdot \langle -\sin t, -\cos t\rangle \, dt$$

$$= \int_{-\pi}^{0} -\sin^2 t - \cos^2 t \, dt = \int_{-\pi}^{0} -1 \, dt = -\pi .$$

<u>Properties of line integrals</u>:

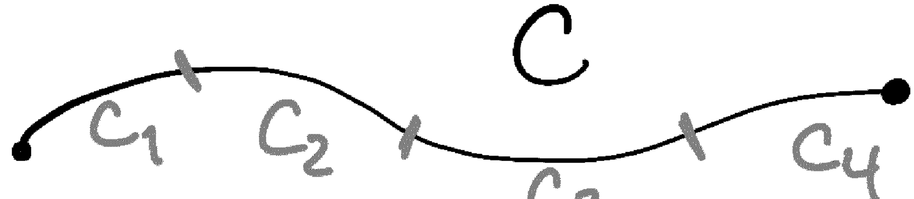
(1) $\displaystyle\int_C (\vec{F}+\vec{G}) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C \vec{G} \cdot d\vec{r}$

(2) $\displaystyle\int_C k\vec{F} \cdot d\vec{r} = k \int_C \vec{F} \cdot d\vec{r}$

scalar

(3) $\displaystyle\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$

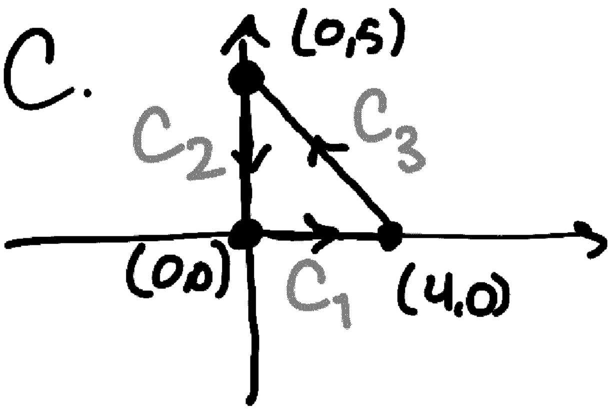
(4) If C is the concatenation of the curves $C_1, \dots, C_n$



$$\int_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n} \int_{C_i} \vec{F} \cdot d\vec{r} .$$

Ex: Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \langle y^2, 2xy+1\rangle$, and C is the triangle in the plane with vertices $(0,0)$, $(4,0)$ and $(0,5)$ oriented counterclockwise.

Sol: We first have to parametrize C.



$\boxed{C_1}$ $\vec{r}_1(t) = \langle t, 0\rangle \quad 0 \le t \le 4$

$\boxed{C_2}$ $\vec{r}_2(t) = \langle 0, -t\rangle \quad -5 \le t \le 0$

$\boxed{C_3}$ Direction vector:

$$\langle 0,5\rangle - \langle 4,0\rangle = \langle -4, 5\rangle$$

Start pt: $(4,0)$

$$\vec{r}_3(t) = \langle 4,0\rangle + t\langle -4,5\rangle, \quad 0 \le t \le 1$$

$$= \langle 4-4t, 5t \rangle, \quad 0 \le t \le 1.$$

<u>Integrals</u>:

① $\vec{F}(\vec{r}_1(t)) = \vec{F}(\langle t, 0 \rangle) = \langle 0, 1 \rangle$

$$\int_{C_1} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt = \int_0^4 \langle 0,1 \rangle \cdot \langle 1,0 \rangle\,dt$$

$$= 0$$

② $\vec{F}(\vec{r}_2(t)) = \vec{F}(\langle 0, -t \rangle) = \langle t^2, 1 \rangle$

$$\int_{C_2} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt = \int_{-5}^{0} \langle t^2, 1 \rangle \cdot \langle 0, -1 \rangle\,dt$$

$$= \int_{-5}^{0} -1\,dt = \int_{0}^{-5} dt = -5$$

③ $\vec{F}(\vec{r}_3(t)) = \vec{F}(\langle 4-4t, 5t \rangle)$

$$= \langle 25t^2, 2(5t)(4-4t)+1 \rangle = \langle 25t^2, 40t - 40t^2 + 1 \rangle$$

$$\int_{C_3} \vec{F}(\vec{r}_3(t)) \cdot \vec{r}_3{}'(t)\,dt = \int_0^1 \langle 25t^2, 40t - 40t^2 + 1 \rangle \cdot \langle -4, 5 \rangle\,dt$$

$$= \int_0^1 -100t^2 + 5(40t - 40t^2 + 1)\,dt$$

$$= \int_0^1 -100t^2 + 200t - 200t^2 + 5\,dt$$

$$= \left[-100t^3 + 100t^2 + 5t\right]_0^1 = 5$$

SO: $$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$+ \int_{C_3} \vec{F} \cdot d\vec{r} = 0 - 5 + 5 = 0.$$
