

Recall:

- $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ exists if $|f(x,y) - L|$ can be made arbitrarily small by making the distance between (x,y) and (a,b) very small.

§4.2 Continuity

$f(x,y)$ is continuous at (a,b) if

- (1) $f(a,b)$ exists (2) $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ exists

(3) $f(a,b) = \lim_{(x,y) \rightarrow (a,b)} f(x,y)$

Thm: Let $f(x,y)$ and $g(x,y)$ be continuous at $(x,y) = (a,b)$. Then

- (1) $f(x,y) \pm g(x,y)$ is continuous

at $(x,y) = (a,b)$

(2) $f(x,y)g(x,y)$ is continuous at $(x,y) = (a,b)$

(3) If $h(z)$ is continuous at $z = f(a,b)$, then $h(f(x,y))$ is continuous at $(x,y) = (a,b)$

Ex: (1) $f(x,y) = 4x^2y^3$, $h(z) = \cos(z)$ are both continuous everywhere, so $h(f(x,y)) = \cos(4x^2y^3)$ is also continuous.

(2) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

$f(0,0)$ exists, but we saw last time that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$ doesn't exist, so $f(x,y)$ is not continuous at $(x,y) = (0,0)$.

§4.3 Partial derivatives

Def: Let $f(x,y)$ be a function of two variables.

The partial derivative of f , with respect to x , written $\frac{\partial f}{\partial x}$, or f_x is defined as:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}$$

The partial derivative of f with respect to y is defined as

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h \rightarrow 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$\frac{\partial f}{\partial x}$ is the derivative of $f(x,y)$ with respect to x , and treating y as if it was a constant!

(and vice versa for $\frac{\partial f}{\partial y}$)

Ex: $\frac{\partial}{\partial x} (3xy + x + 2y) = 3y + 1$

$$\frac{\partial}{\partial y} (3xy + x + 2y) = 3x + 2$$

All rules for derivatives we learned in Calc I applies.

Ex: (1) $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$

$$\frac{\partial f}{\partial x} = f_x = 2x - 3y - 4$$

$$\frac{\partial f}{\partial y} = f_y = -3x + 4y + 5$$

(2) $g(x, y) = \sin(x^2y - 2x + 4)$

$$\frac{\partial g}{\partial x} = \cos(x^2y - 2x + 4) \cdot (2xy - 2)$$

$$\frac{\partial g}{\partial y} = \cos(x^2y - 2x + 4) \cdot (x^2).$$

Everything also works the same w/ 3 or more variables.

EX: $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

$$f_x = \frac{1}{x^2 + y^2 + z^2} \cdot 2x$$

$$f_y = \frac{1}{x^2 + y^2 + z^2} \cdot 2y$$

$$f_z = \frac{1}{x^2 + y^2 + z^2} \cdot 2z$$

Higher-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

$= f_{xx} \qquad \qquad \qquad = f_{yy}$

We also have mixed partial derivatives:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{yx}$$

y first, then x

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{xy}.$$

$$\underline{\text{Ex:}} \quad f(x,y) = x e^{-3y} + \sin(2x-5y)$$

$$f_x = e^{-3y} + \cos(2x-5y) \cdot 2$$

$$f_{xx} = -2 \sin(2x-5y) \cdot 2 = -4 \sin(2x-5y)$$

$$\begin{aligned} f_{xy} &= -3e^{-3y} - 2 \sin(2x-5y) \cdot (-5) \\ &= -3e^{-3y} + 10 \sin(2x-5y) \end{aligned}$$

$$f_y = -3x e^{-3y} + \cos(2x-5y) \cdot (-5)$$

$$\begin{aligned} f_{yy} &= 9x e^{-3y} + 5 \sin(2x-5y) \cdot (-5) \\ &= 9x e^{-3y} - 10 \sin(2x-5y) \end{aligned}$$

$$\begin{aligned} f_{yx} &= -3e^{-3y} + 5 \sin(2x-5y) \cdot 2 \\ &= -3e^{-3y} + 10 \sin(2x-5y) \end{aligned}$$

Note $f_{xy} = f_{yx}$

In fact, this is always true!

Thm: If $f(x,y)$ is a function of 2 variables such that f_{xy} and f_{yx} exist. Then $f_{xy} = f_{yx}$.

Ex: Let $f(x,y) = \frac{1}{2}(e^y - e^{-y})\sin x$.

Show $f_{xx} + f_{yy} = 0$

$$f_x = \frac{1}{2}(e^y - e^{-y})\cos x$$

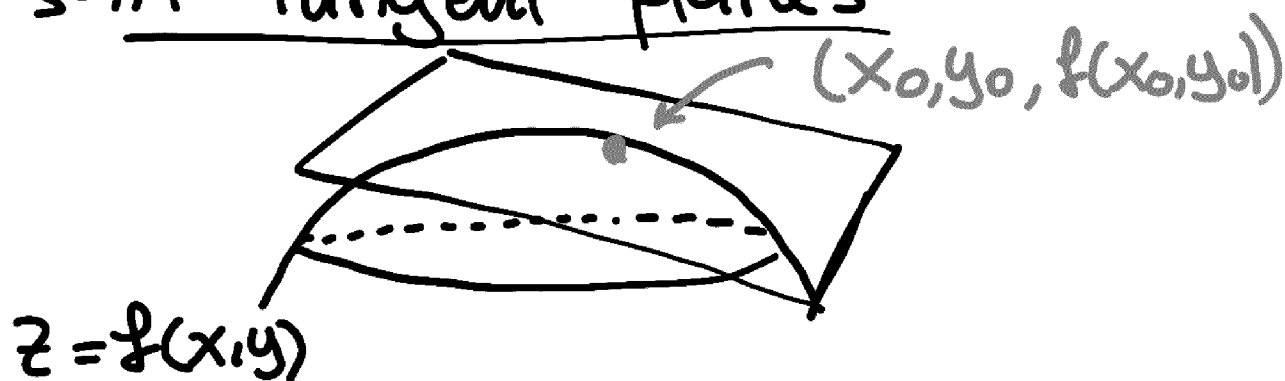
$$f_{xx} = -\frac{1}{2}(e^y - e^{-y})\sin x$$

$$f_y = \frac{1}{2}(e^y - e^{-y})(-1)\sin x = -\frac{1}{2}(e^y + e^{-y})\sin x$$

$$f_{yy} = \frac{1}{2}(e^y + e^{-y})(-1)\sin x$$

$$= \frac{1}{2}(e^y - e^{-y})\sin x = -f_{xx}.$$

§4.4 Tangent planes



Def: If S is the surface $z = f(x, y)$ in space, the tangent plane at (x_0, y_0) to S is given by the eqn

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

Remark: A normal vector to S is $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$.

Ex: $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$
let's find the tangent plane at $(x, y) = (2, -1)$.

$$f_x(x, y) = 4x - 3y + 2$$

$$f_y(x, y) = -3x + 16y - 4$$

$$f_x(2, -1) = 8 + 3 + 2 = 13$$

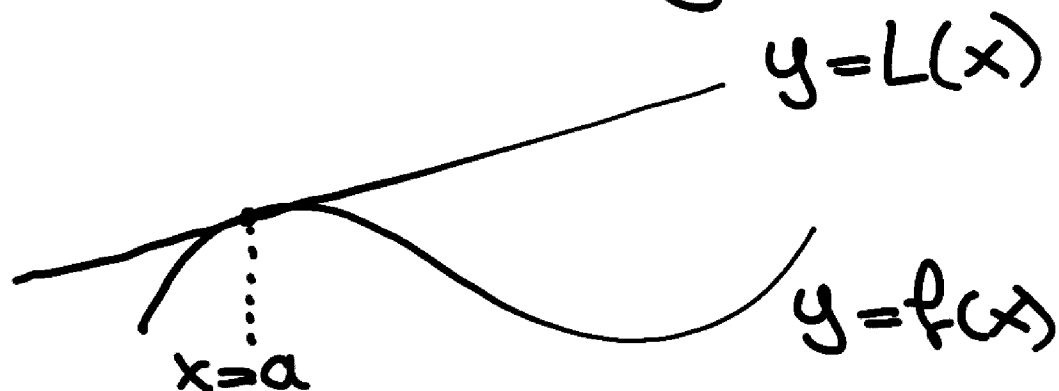
$$f_y(2, -1) = -6 - 16 - 4 = -26$$

$$f(2, -1) = 2 \cdot 4 - 3 \cdot 2 \cdot (-1) + 8(-1)^2 + 2 \cdot 2 \\ - 4(-1) + 4 = 26$$

Tangent plane at $(2, -1)$ is:

$$13(x-2) - 26(y+1) - (z-26) = 0$$

From Calc I again:



$L(x) = f(a) + f'(a)(x-a)$
 is the linear approximation of $f(x)$ at $x=a$.

Similarly,

$$L(x, y) = f(a, b) + f_x(a, b)(x-a) \\ + f_y(a, b)(y-b)$$

is the linear approximation of $f(x,y)$ at $(x,y) = (a,b)$.

The idea is that if f is differentiable, it has a linear approximation at $x=a$.

Def: $f(x,y)$ is differentiable at $(x,y) = (a,b)$ if for all points (x,y) near (a,b) , we can write

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + E(x,y)$$

error term $\rightarrow$ satisfies

$$\lim_{(x,y) \rightarrow (a,b)} \frac{E(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

Important: Just because f_x and f_y exist, it does not mean f is differentiable.

Ex: $f(x,y) = 2x^2 - 4y$ is differentiable at the origin.

$$f_x(x,y) = 4x, \quad f_x(0,0) = 0$$

$$f_y(x,y) = -4, \quad f_y(0,0) = -4$$

$$f(0,0) = 0.$$

$$E(x,y) = \underbrace{(f(x,y) - f(0,0))}_{=0} - \underbrace{f_x(0,0)}_{=0}(x-0) - \underbrace{f_y(0,0)}_{=-4}(y-0)$$

$$= 2x^2 - 4y + 4y = 2x^2.$$

Now we have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} = 0 \text{ because}$$

$$0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} \leq \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2}{\sqrt{x^2+0}}$$

$$= \lim_{x \rightarrow 0} \frac{2x^2}{|x|} = 0$$

$E(x,y) \geq 0$
and the
square root ≥ 0