MAT 127 Lec 7 M Feb 12

Alternating series test

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$

Actually converges.

Alt series test

If $\sum_{n=1}^{\infty} (-1)^n a_n$ is so that

(i) $a_{n+1} \leq a_n$ for all n
(a_n is decreasing)

(i) $\lim_{n\to\infty} a_n = 0$

then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Ex $a_n = \frac{1}{n}$ decreasing &
$\lim_{n\to\infty} a_n = 0$
so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$.

$a_n = \frac{3n}{4n-1}$. Is it decreasing?

$f(x) = \frac{3x}{4x-1}$. Want to see if $f'(x) < 0$ for $x \geq 1$.

$$f'(x) = \frac{3(4x-1) - 4 \cdot 3x}{(4x-1)^2} = -\frac{3}{(4x-1)^2} < 0$$

so a_n is decreasing. However

$$\lim_{n \to \infty} \frac{3n}{4n-1} = \lim_{n \to \infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4}$$

so the alternating series test doesn't apply!

However the div test is conclusive in this case.

$$\lim_{n \to \infty} (-1)^n \cdot \frac{3n}{4n-1} = \left(\lim_{n \to \infty} (-1)^n\right) \cdot \left(\lim_{n \to \infty} \frac{3n}{4n-1}\right)$$

$$= \frac{3}{4} \cdot \lim_{n \to \infty} (-1)^n \text{ doesn't exist}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1} \text{ diverges.}$$

Power Series (§8.5 Stewart)

Until now: only numerical series:

$\sum_{n=0}^{\infty} \frac{1}{n!}$ is numerical

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \text{some number.}$$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a power series

(x is a variable.

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} = \cdots$$

For each x its a numerical series.

<u>Def</u>: A <u>power series</u> is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where $\{c_n\}$ is a sequence of real numbers, x is a variable, and a is a constant.

"power series centered at a"

<u>Ex</u>: We have already seen & studied $\sum_{n=0}^{\infty} x^n$ for varying values of x.

<u>Geometric Series</u>

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{1-x} \text{ for } |x| < 1$$

and it diverges if $|x| \geq 1$.

<u>Ex</u>: For what values of x does $\sum_{n=0}^{\infty} n!\,x^n$ converge?

$a_n = n!\,x^n$.

<u>Ratio test</u>:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}}{\cancel{n!}}\cdot x\right| = \lim_{n\to\infty}(n+1)\,|x|$$

If $|x| \neq 0$, then $\rho = \infty$ & ratio test tells us that it div.

If $|x| = 0$ then $\rho = 0$ & ratio test tells us that it conv.

$$\boxed{\sum_{n=0}^{\infty} n!\,x^n \quad \begin{array}{l}\text{conv if } |x| = 0\\ \text{div if } |x| \neq 0\end{array}}$$

Ex: For which values does $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Ratio test: $a_n = \frac{x^n}{n}$.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right|$$

$$= \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x| \cdot \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{=1} = |x|$$

- $\rho = |x| < 1$ then $\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv
- $\rho = |x| > 1$ then $\sum_{n=1}^{\infty} \frac{x^n}{n}$ div.
- $\rho = |x| = 1$ (meaning $x = \pm 1$) INCONCLUSIVE.

CHECK THESE VALUES SEPARATELY!

$\boxed{x=1}$ $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ div

$\boxed{x=-1}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv by the alternating series test.

(see previous example)

To summarize:

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv. for $-1 \le x < 1$

and div. otherwise.

Thm: For a power series $\sum_{n=1}^{\infty} c_n (x-a)^n$ there is 1 of 3 possibilities:

1. it converges only at $x=a$
2. it converges for all x
3. There's a positive number R such that it conv for

$|x-a| < R$ and diverges
for $|x-a| > R$
(And anything can happen
for $|x-a| = R \Leftrightarrow x = a \pm R$)

R = radius of convergence

in case 1 $R = 0$
in case 2 $R = \infty$
The interval of convergence is the interval that consists of all values of x for which the series conv.

Ex $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$. Find radius & interval of conv.

Ratio test: (this is similar to the previous calculation)

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}/(n+1)}{(x-3)^n/n} \right|$$

$$= \lim_{n\to\infty} |x-3| \cdot \frac{n}{n+1} = |x-3|.$$

<u>Conv</u> if $\rho = |x-3| < 1$

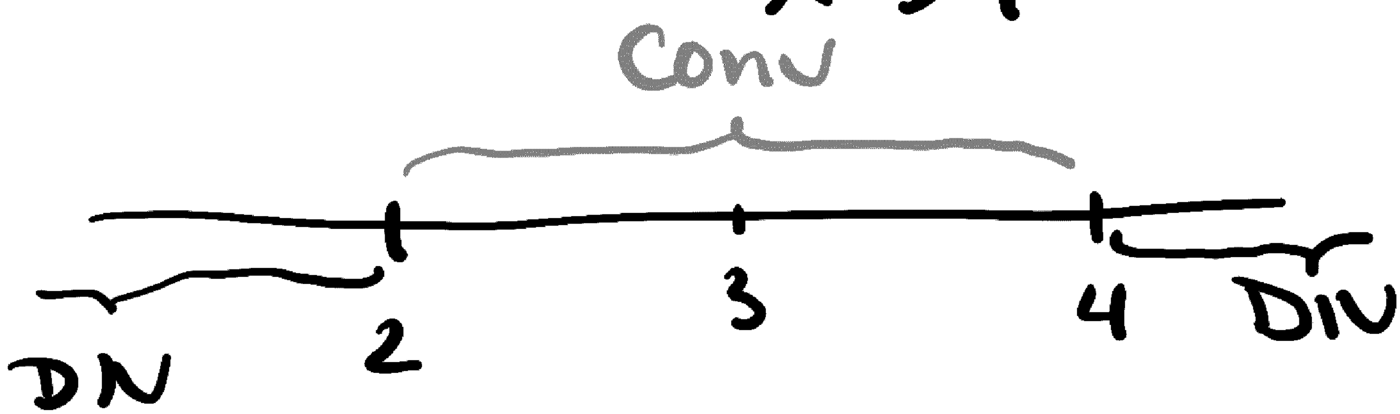
$$\iff -1 < x-3 < 1$$

$$\iff 2 < x < 4$$

<u>div</u> if $\rho = |x-3| > 1$

$$\iff x-3 < -1 \text{ or } x-3 > 1$$

$$\iff x < 2 \text{ or } x > 4$$



Need to check $x = 2$ and $x = 4$ separately.

$\underline{x=2}$: $\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ <u>conv</u>

$\underline{x=4}$: $\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ <u>div.</u>

- Radius of convergence is 1.
- Interval of convergence is $[2,4)$

<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ find radius & interval of conv.

<u>Ratio test</u>:

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} |x| \cdot \frac{1}{n} = 0$$

for all values of x.

so $R=\infty$ and interval of convergence is $(-\infty, \infty)$.

Ex: Find interval & radius of conv of $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^n}$.

Ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)(x+2)^{n+1}/3^{n+1}}{n(x+2)^n/3^n} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \cdot |x+2| \cdot \frac{1}{3} = \frac{1}{3} \cdot |x+2|.$$

Conv if $\rho = \frac{1}{3}|x+2| < 1$

$$|x+2| < 3 = R$$

$$\Leftrightarrow -3 < x+2 < 3$$

$$\Leftrightarrow -5 < x < 1$$

div if $\rho = \frac{1}{3}|x+2| > 1$

$$\Leftrightarrow |x+2| > 3$$

Endpoints:

$x = -5$

$$\sum_{n=1}^{\infty} \frac{n \cdot (-5+2)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n$$

<u>div</u> by div test

$\lim_{n\to\infty} (-1)^n n$ doesn't exist!

<u>$x=1$</u> $\sum_{n=1}^{\infty} \frac{n\cdot(1+2)^n}{3^n} = \sum_{n=1}^{\infty} n$

also <u>div</u> by div test.

R (radius of conv) $= 3$

interval of conv $= (-5,1)$.