

Recall:

## • Div test

$$- \lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

$$- \lim_{n \rightarrow \infty} a_n = 0 \Rightarrow \text{INCONCLUSIVE}$$

• Ratio test:  $\sum_{n=1}^{\infty} a_n$ 

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

$$- \rho > 1 \Rightarrow \sum a_n \text{ divergent}$$

$$- \rho < 1 \Rightarrow \sum a_n \text{ convergent}$$

$$- \rho = 1 \Rightarrow \text{INCONCLUSIVE}$$

• Integral test:  $f$  continuous, positive, decreasing function.

$$a_n = f(n)$$

$$- \int_1^{\infty} f(x) dx \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent}$$

$$- \int_1^{\infty} f(x) dx \text{ divergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ divergent}$$

• p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1$$

and diverges otherwise.

Ex: Does  $\sum_{n=1}^{\infty} \frac{n}{n^2-1}$  converge or diverge?

① Div test:  $\lim_{n \rightarrow \infty} \frac{n}{n^2-1} = 0$   
INCONCLUSIVE

② Ratio test:  $a_n = \frac{n}{n^2-1}$

$$\begin{aligned}
 f &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)/(n+1)^2 - 1}{n/(n^2-1)} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{(n+1)(n^2-1)}{n(n+1)^2-1} = \lim_{n \rightarrow \infty} \frac{n^3+n^2-n-1}{n(n^2+2n)} \\
 &= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}}{1 + \frac{2}{n}} = 1
 \end{aligned}$$

INCONCLUSIVE

③ Integral test?

$f(x) = \frac{x}{x^2-1}$  is continuous and positive for  $x \geq 1$ .

Decreasing?  $f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2}$

$$= \frac{-1-x^2}{(x^2+1)^2} < 0 \text{ for } x \geq 1.$$

So it's decreasing. We can use the integral test.

$$\int_{x=1}^{x=\infty} \frac{x}{x^2-1} dx = \left[ \begin{array}{l} u = x^2 - 1 \\ du = 2x dx \\ x=1 \Rightarrow u=0 \\ x \rightarrow \infty \Rightarrow u \rightarrow \infty \end{array} \right] = \int_{u=0}^{u=\infty} \frac{1}{2} \cdot \frac{1}{u} du$$

$$= \frac{1}{2} [\ln|u|]_{u=0}^{\infty} = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2-1} \text{ diverges.}$$


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## Comparison test

The idea: the series  $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$  looks very similar to a series that we know, namely  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ .

We notice that the terms in

$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$  are smaller (larger

denominator) than those in  $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\text{So } \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

geometric series  
with  $r = \frac{1}{2}$

therefore  $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \leq 2$  so it  
must converge.

Thm (Comparison test)

If  $\sum a_n$  and  $\sum b_n$  are series w/  
positive terms and  $a_n \leq b_n$   
for all  $n \geq 1$ , then

① If  $\sum b_n$  is convergent  
then  $\sum a_n$  is convergent

② If  $\sum a_n$  is divergent  
then  $\sum b_n$  is divergent.

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Ex Consider  $\sum_{n=1}^{\infty} \frac{n}{n^2-1}$  from before.

It "almost" looks like  $\sum \frac{n}{n^2} = \sum \frac{1}{n}$   
 So we would like to compare the series.

$$\sum_{n=1}^{\infty} \frac{n}{n^2-1} \geq \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

larger denominator  
 → smaller quotient

$$\sum_{n=1}^{\infty} \frac{n}{n^2-1} \text{ divergent.}$$

Ex:  $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

changing -1 to +1 should not change its convergence properties, but the comparison

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^2} = \left[ \sum_{n=1}^{\infty} \frac{1}{n} \right]$$

DIVERGENT

but because  $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$  is  
smaller we can not use the  
comparison test. However:

$$\frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{n}{2n^2} = \frac{1}{2n}$$

Since  $n \geq 1$

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{so}$$

$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$  diverges by the  
comparison test.

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