MAT 127 Lec 4, W Jan 31

Recall:

- Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- Geometric series

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r|<1}$$

Ex: $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$

Ex: $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=0}^{\infty} \frac{2 \cdot 2^n}{5^n} = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

$= 2 \cdot \frac{1}{1-2/5} = 2 \cdot \frac{1}{3/5} = 2 \cdot \frac{5}{3} = \frac{10}{3}.$

Ex: $\sum_{n=1}^{\infty} \frac{1}{e^{2n}} = \sum_{n=1}^{\infty} \frac{1}{(e^2)^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n$

Almost geometric with $r = \frac{1}{e^2}$ but

Sum starts at $n=1$.

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^{n+1} = \frac{1}{e^2}\sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^n$$

$$= \frac{1}{e^2}\cdot\frac{1}{1-1/e^2} = \frac{1}{e^2}\cdot\frac{1}{\frac{e^2}{e^2}-\frac{1}{e^2}} = \frac{1}{\cancel{e^2}}\cdot\frac{\cancel{e^2}}{e^2-1}$$

$$= \frac{1}{e^2-1}$$

<u>Telescoping</u> <u>Series</u>

<u>Ex</u>: Compute $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

Note (by partial fraction decomposition)

$\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, so

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= 1\underbrace{-\frac{1}{2}+\frac{1}{2}}_{\text{cancel}}\underbrace{-\frac{1}{3}+\frac{1}{3}}_{\text{cancel}}\underbrace{-\frac{1}{4}+\frac{1}{4}}_{\text{cancel}}-\cdots$$

N-th partial sum

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(1-\underbrace{\frac{1}{2}\right)+\cdots+\left(\frac{1}{N}}_{\text{All these cancel}}-\frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1}.$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$

so $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$.

$$\boxed{\text{Thm: If } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n\to\infty} a_n = 0.}$$

Warning! If we know $\lim_{n\to\infty} a_n = 0$ then we can not draw any conclusion!

Ex: Harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges but $\lim_{n\to\infty}\frac{1}{n} = 0$.

Ex: $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$ as seen previously and $\lim_{n\to\infty}\frac{1}{n(n+1)} = 0$.

A logically equivalent version of the theorem:

Test for divergence

If $\lim_{n\to\infty} a_n$ does not exist, or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

When faced with a problem of finding out whether a series converges or diverges, we should always try this test first!

Ex: Does $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converge or diverge?

Div test:

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0$$

So the series diverges.

Let us prove it!

<u>Thm</u>: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

<u>Proof</u>: $\sum_{n=1}^{\infty} a_n$ convergent means that if $S_N = \sum_{n=1}^{N} a_n$ is the N-th partial sum, the limit $\boxed{\lim_{N\to\infty} S_N \text{ exists}}$.

Note

$$S_N = a_1 + a_2 + \cdots + a_{N-1} + a_N$$

$$S_{N-1} = a_1 + a_2 + \cdots + a_{N-1}$$

$$\rightarrow S_N - S_{N-1} = a_N \quad \text{So}$$

$$\lim_{N\to\infty} a_N = \lim_{N\to\infty}(S_N - S_{N-1})$$

$$= \underbrace{\lim_{N\to\infty} S_N}_{=L} - \underbrace{\lim_{N\to\infty} S_{N-1}}_{=L} = L - L = 0$$

$\square$

Ex: Does $\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}$ converge?

NO! Because $\lim_{n\to\infty} \frac{3^n}{2^n+1}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{2}{3}\right)^n + 1} = 1 \neq 0.$

Ex: Does $\sum_{n=1}^{\infty} \frac{3^n - 2^n}{e^n}$ converge or diverge?

$\lim_{n\to\infty} \frac{3^n - 2^n}{e^n} = \lim_{n\to\infty} \frac{3^n}{e^n} - \frac{2^n}{e^n} = \lim_{n\to\infty} \left(\frac{3}{e}\right)^n$

$- \lim_{n\to\infty} \left(\frac{2}{e}\right)^n = \infty - 0 = \infty$

so divergence test gives

$\sum_{n=1}^{\infty} \frac{3^n - 2^n}{e^n}$ diverges.