MAT 127 Lec 3, M Jan 29

<u>Recall</u>: $\{a_n\}$ sequence

- <u>increasing</u>: $a_n \leq a_{n+1}$ $n \geq 1$
- <u>decreasing</u>: $a_n \geq a_{n+1}$ $n \geq 1$
- <u>bounded</u>: $m \leq a_n \leq M$ $n \geq 1$
- <u>squeeze thm</u>:

  $\{a_n\}, \{b_n\}, \{c_n\}$ sequences such that

  $$a_n \leq b_n \leq c_n \quad n \geq 1$$

  If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ exists, then $\lim_{n\to\infty} b_n = L$.

Here is a useful fact that we can prove as a consequence:

<u>Thm</u> If $\{a_n\}$ is a sequence such that $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

<u>Proof</u>: We always have

$-|a_n| \le a_n \le |a_n|$ for all $n \ge 1$ for any sequence $\{a_n\}$, so if $\lim_{n\to\infty} |a_n| = 0$, also $\lim_{n\to\infty} -|a_n| = 0$. The squeeze thm then implies $\lim_{n\to\infty} a_n = 0$.

$\square$

<u>Ex</u>: Find $\lim_{n\to\infty} \frac{\sin(n)}{n}$.

$b_n = \frac{\sin(n)}{n}$. We know that

$-1 \le \sin(n) \le 1$, so $\underbrace{-\frac{1}{n}}_{a_n} \le \underbrace{\frac{\sin(n)}{n}}_{b_n} \le \underbrace{\frac{1}{n}}_{c_n}$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$.

So the squeeze thm gives

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0.$$

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§8.2 Stewart

§Infinite series

Comes naturally eg from the desire to describe numbers with infinite decimal expansions such as π or e.

$$\pi = 3.14159265\ldots$$
$$= 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \cdots$$

Series $\neq$ **Sequence**

infinite sum　　　infinite list

Sequence: $\{a_n\} = \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

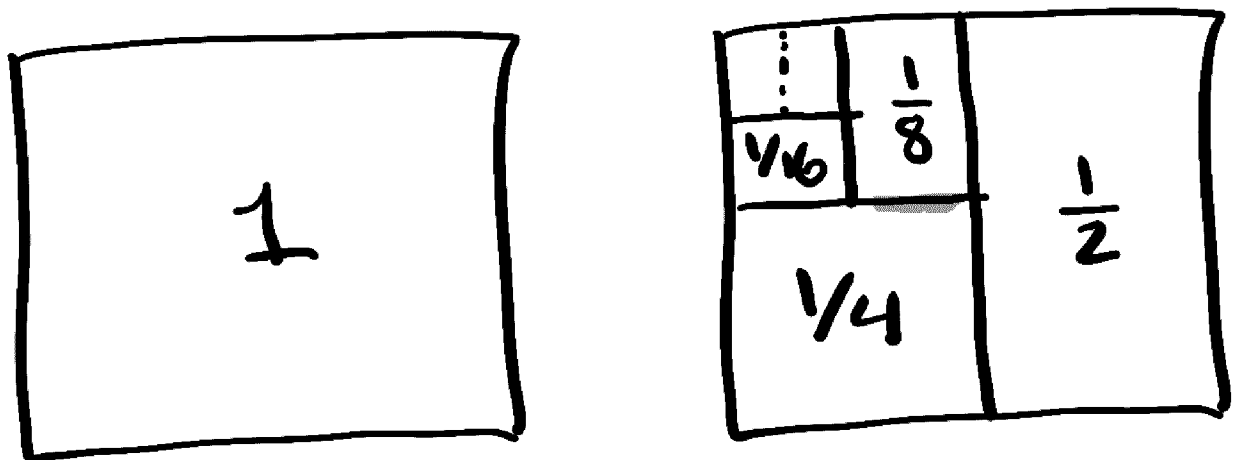
Series:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^k}$$

$$= \lim_{n\to\infty} \Big(\underbrace{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}_{\text{geometric sum}}\Big) = 2$$

Geometrically:



Algebraically:

we have the sequence

$$b_n = \sum_{k=1}^{n} a_n = \sum_{k=1}^{n} \frac{1}{2^k}$$

$$\{b_n\} = 1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{4},\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \dots$$

$$\{b_n\} = 1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{31}{16} \dots$$

$b_n = \dfrac{2^{n+1}-1}{2^n}$ & we find

$$\lim_{n\to\infty} \frac{2^{n+1}-1}{2^n} = \lim_{n\to\infty} \frac{1-\frac{1}{2^{n+1}}}{\frac{1}{2}} = 2.$$

$\to 0$

Def: An (infinite) series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots$$

There is a sequence of partial sums $\{S_n\}$ where $S_n = \sum_{k=1}^{n} a_k$. Then we define

$$\boxed{\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n \quad \text{if it converges}}$$

If $\lim_{n\to\infty} S_n$ doesn't exist, then $\sum_{k=1}^{\infty} a_k$ diverges.

Ex: Consider $\sum_{k=1}^{n} k$.

Then its partial sums are

$$S_1 = \sum_{k=1}^{1} = 1$$

$$S_2 = \sum_{k=1}^{2} k = 1+2 = 3$$

$$S_3 = \sum_{k=1}^{3} k = 1+2+3 = 6$$

$$\vdots$$

ARITHMETIC SUM

$$S_n = \sum_{k=1}^{n} k = 1 + \cdots + n = \frac{n(n-1)}{2}$$

And of course

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n-1)}{2} = \infty$$

so $\sum_{k=1}^{\infty} k$ diverges (to ∞).

Ex: Consider $\sum_{k=1}^{\infty} (-1)^{k+1}$

Partial sums:

$S_1 = 1$, $S_2 = 1-1 = 0$, $S_3 = 1-1+1 = 1$

so $\{S_n\} = 1, 0, 1, 0, 1, \ldots$

$\lim_{n\to\infty} S_n$ does not exist, so

$\sum_{k=1}^{\infty} (-1)^{k+1}$ is divergent.

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Grows very slowly.

- Takes over 12,000 terms for $S_n \geq 10$.
- To get $S_n \geq 54$ it requires as many terms as stars in the universe $\sim 1.6 \cdot 10^{23}$

Thm The harmonic series diverges to ∞.

Proof: Look at partial sums $S_2, S_4, S_8, S_{16}, S_{32}, \dots S_{2^n}$

$$S_2 = 1+\frac{1}{2}$$

$$S_4 = 1+\frac{1}{2}+\boxed{\frac{1}{3}+\frac{1}{4}} > 1+\frac{1}{2}+\boxed{\frac{1}{4}+\frac{1}{4}}$$

$$= 1+\frac{1}{2}+\frac{1}{2} = 1+\frac{2}{2}$$

$$S_8 = 1+\frac{1}{2}+\boxed{\frac{1}{3}+\frac{1}{4}}+\boxed{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}$$

$$> 1+\frac{1}{2}+\frac{1}{2}+\underbrace{\boxed{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}}_{1/2}$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$= 1+\frac{3}{2} \quad (8=2^3) \quad \text{so}$$

$$\boxed{S_{2^n} = 1+\frac{n}{2}}$$

As n increases, partial sums go to ∞, so $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ $\square$

Geometric Series:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$
$$= a \boxed{\sum_{n=0}^{\infty} r^n}$$

(a, r some constants)

$r=1$: $\sum_{k=0}^{\infty} 1^k = 1+1+1+\cdots \longrightarrow \infty$

DIVERGES

So assume $r \neq 1$.

n-th partial sum:

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$
$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$rS_n + 1 = \underbrace{\boxed{1 + r + r^2 + r^3 + \cdots + r^n}}_{= S_n} + r^{n+1}$$

$$rS_n + 1 = S_n + r^{n+1}$$

$$1 - r^{n+1} = S_n - rS_n = S_n(1-r$$

$$\boxed{S_n = \frac{1-r^{n+1}}{1-r}}$$

we know $\lim_{n\to\infty} \frac{1-r^{n+1}}{1-r}$

$$= \frac{1 - \lim_{n\to\infty} r^{n+1}}{1-r} \boxed{=} \frac{1}{1-r}$$

If $-1 < r < 1$
(diverges otherwise)

$$\boxed{\sum_{k=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r| < 1}$$