MAT 127 Lec 24 W Apr 17

Last topic of the Semester:

Linear homogeneous
2^{nd} order ODES with
constant coefficients.

These are fancy words used to describe 2^{nd} order ODES
of the form:

$$\boxed{\begin{array}{c} Ay'' + By' + Cy = 0 \\ A, B, C \text{ constants} \end{array}}$$

When Solving a first order ODE
(e.g. via separation of variables)
we get a general solution

depending on one constant. This is to say there is a single degree of freedom (we can choose whatever value of the constant as we wish).

- For initial-value problems such as

$$e^y y' = 2x, \quad y(0) = 0$$

there is a single initial condition because there is 1 constant to determine.

2^{nd} order ODES:

- We will always expect two constants, and solutions are of the form

$$y = C_1 y_1(x) + C_2 y_2(x).$$

- Initial-value problems need to come with two conditions:

For instance

$$\boxed{y''+y=0\,,\ y(0)=1\,,\ y'(0)=-1}$$

we saw two lectures ago using power series that

$$y = C_1 \cos x + C_2 \sin x$$ is

a general solution to $y''+y=0$.

So via the initial conditions we can determine both constants:

$$y(0) = \underline{C_1 = 1}$$

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

$$y'(0) = \underline{C_2 = -1}$$

Specific solution:

$y = \cos x - \sin x.$

<u>Ex</u> $y'' = 0$ is a 2^{nd} order ODE of the kind we will consider.

Integrate twice to solve it!

$y' = C_1$

$$\boxed{y = C_1 x + C_2}$$

Note that since we integrated twice we got two constants.

<u>Ex</u> $y'' - y = 0$.

Let's guess some solutions!

- $y_1(x) = e^x \Rightarrow y_1'(x) = e^x$

$\Rightarrow y_1''(x) = e^x$

So $y_1'' = y_1$.

- $y_2(x) = e^{-x} \Rightarrow y_2'(x) = -e^{-x}$

$\Rightarrow y_2''(x) = e^{-x}$

So $y_2'' = y_2$

In fact we see that

$c_1 y_1(x)$ and $c_2 y_2(x)$

constants

are both solutions too!
(They don't affect the derivatives...)
The general solution is their

linear combination

$$y(x) = c_1 e^{x} + c_2 e^{-x}.$$

This is generally true:

If $y_1(x)$ and $y_2(x)$ are both solutions to

$$Ay'' + By' + Cy = 0$$

then so is

$$c_1 y_1(x) + c_2 y_2(x)$$

<u>Proof</u>:

$$y(x) := c_1 y_1(x) + c_2 y_2(x)$$

$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$$

$$y''(x) = c_1 y_1''(x) + c_2 y_2''(x)$$

so

$$Ay'' + By' + Cy$$

$$= A\left(c_1 y_1''(x) + c_2 y_2''(x)\right)$$
$$+ B\left(c_1 y_1'(x) + c_2 y_2'(x)\right)$$
$$+ C\left(c_1 y_1(x) + c_2 y_2(x)\right)$$

$$= c_1\left(A y_1''(x) + B y_1'(x) + C y_1(x)\right)$$
$$+ c_2\left(A y_2''(x) + B y_2'(x) + C y_2(x)\right)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0. \qquad \square$$

<u>Ex</u>: Let's try to solve

$$y'' - 9y = 0$$

We only changed the "1" in front of y to a "9", so we would guess that there are solutions of the form

$$e^{rx} \text{ for some } r.$$

Let's try to find r!

$y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

$$y'' - 9y = r^2 e^{rx} - 9e^{rx}$$
$$= e^{rx}(r^2 - 9) = 0$$

Since we want this to be $= 0$ as a function (so for all x) we must have

$$r^2 - 9 = 0 \iff r = \pm 3.$$

Therefore

$y_1 = e^{3x}$, $y_2 = e^{-3x}$

so $y(x) = C_1 e^{3x} + C_2 e^{-3x}$ is the general solution to $y'' - 9y = 0$.

Ex $y'' - 5y' + 6y = 0$.

Let's again look for solutions of the form $y = e^{rx}$.

$y' = re^{rx}, \quad y'' = r^2 e^{rx}$.

$y'' - 5y' + 6y = r^2 e^{rx} - 5re^{rx} + 6e^{rx}$

$= e^{rx}(r^2 - 5r + 6) = 0$

For the same reason as before we want

$$r^2 - 5r + 6 = 0$$

$$r = \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 6}$$

$$= \frac{5}{2} \pm \sqrt{\frac{25-24}{4}}$$

$$= \frac{5}{2} \pm \frac{1}{2}$$

$r_1 = 2, \quad r_2 = 3$

so e^{2x} and e^{3x} are solutions.

General solution is

$$y(x) = C_1 e^{2x} + C_2 e^{3x}.$$

Ex: Solve the IVP

$$\boxed{\begin{array}{l} y'' - 3y' + 2y = 0 \\ y(0) = -1, \quad y'(0) = 1 \end{array}}$$

First look for solutions of the eqn of the form e^{rx}.

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

Then

$$y'' - 3y' + 2y = r^2 e^{rx} - 3re^{rx} + 2e^{rx}$$

$= e^{rx}(r^2 - 3r + 2) = 0$

$\Rightarrow r = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}$

$= \frac{3}{2} \pm \frac{1}{2}$

$r_1 = 1, \ r_2 = 2$

e^x and e^{2x} are solutions

$\rightsquigarrow$ General Solution is

$y = c_1 e^x + c_2 e^{2x}$.

Initial values:

$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 1$

$y'(x) = c_1 e^x + 2c_2 e^{2x}$

$y'(0) = c_1 + 2c_2 = 2$

$$\begin{cases} C_1 + C_2 = -1 & (1) \\ C_1 + 2C_2 = 1 & (2) \end{cases}$$

(1) $C_1 = -1 - C_2$ Plug into (2)

$\rightsquigarrow -1 - C_2 + 2C_2 = -1 + C_2 = 1$

$\Leftrightarrow \boxed{C_2 = 2}$

$\Rightarrow \boxed{C_1 = -1 - 2 = -3}$

Specific solution to the IVP is therefore

$$\boxed{y(x) = -3e^{x} + 2e^{2x}}$$