MAT 127 Lec 23 M Apr 15

<u>Next topic</u>: second order ODEs "with constant coefficients".

In general we can in fact solve

$ay'' + by' + cy = 0$ where

a, b, c are <u>constants</u>

Before delving into this topic we will review complex numbers.

<u>Complex numbers</u>

Recall that the solutions to the eqn $x^2 - 4 = 0$ are $x = \pm 2$.

Only working with real numbers makes it impossible to solve

$x^2 + 4 = 0$.

Imaginary unit: $i = \sqrt{-1}$.

Then

$$x^2 = -4 \Rightarrow x = \pm\sqrt{-2} = \pm 2i$$

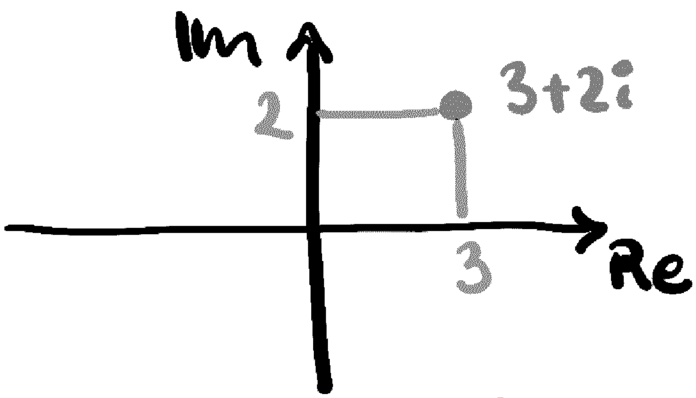
So it has solutions in the complex numbers.

The imaginary unit satisfies

$\boxed{i^2 = -1}$ so if $x = 2i$ we get

$x^2 = (2i)^2 = 2^2 \cdot i^2 = -4$ and this is indeed a solution to $x^2 + 4 = 0$.

A general complex number is one of the form $\boxed{a+bi}$



a = real part

b = imaginary part.

– Both a and b are real.

<u>Operations</u>: "Treat i as a variable"

- $(a+bi)+(c+di)=(a+c)+(b+d)i$
- $(a+bi)-(c+di)=(a-c)+(b-d)i$
- $(a+bi)(c+di) = ac+adi+bci+bdi^2$

$=(ac-bd)+(ad+bc)i$

- $$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

$$= \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

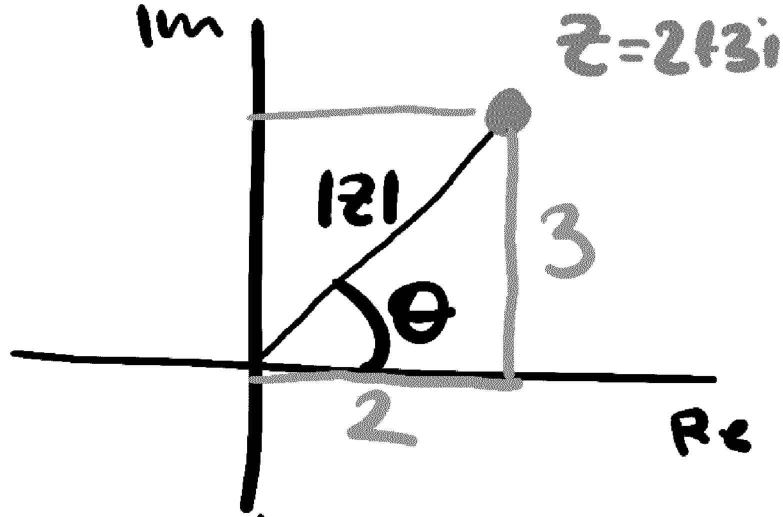
$z=a+bi$ then its <u>conjugate</u> is $\bar{z}=a-bi$

note that $z\bar{z}$ is always a real number!

$$z\bar{z} = (a+bi)(a-bi) = a^2 - (bi)^2$$
$$= a^2 - b^2 i^2 = a^2 + b^2$$

$z = a+bi$ then its modulus ("size") is

$$|z| := \sqrt{a^2+b^2}$$



Can also measure the angle θ: it's called the <u>argument</u>

$$\arg z = \theta = \arctan\left(\frac{b}{a}\right)$$

Can describe complex numbers using <u>polar coordinates</u> consisting of its modulus and its argument

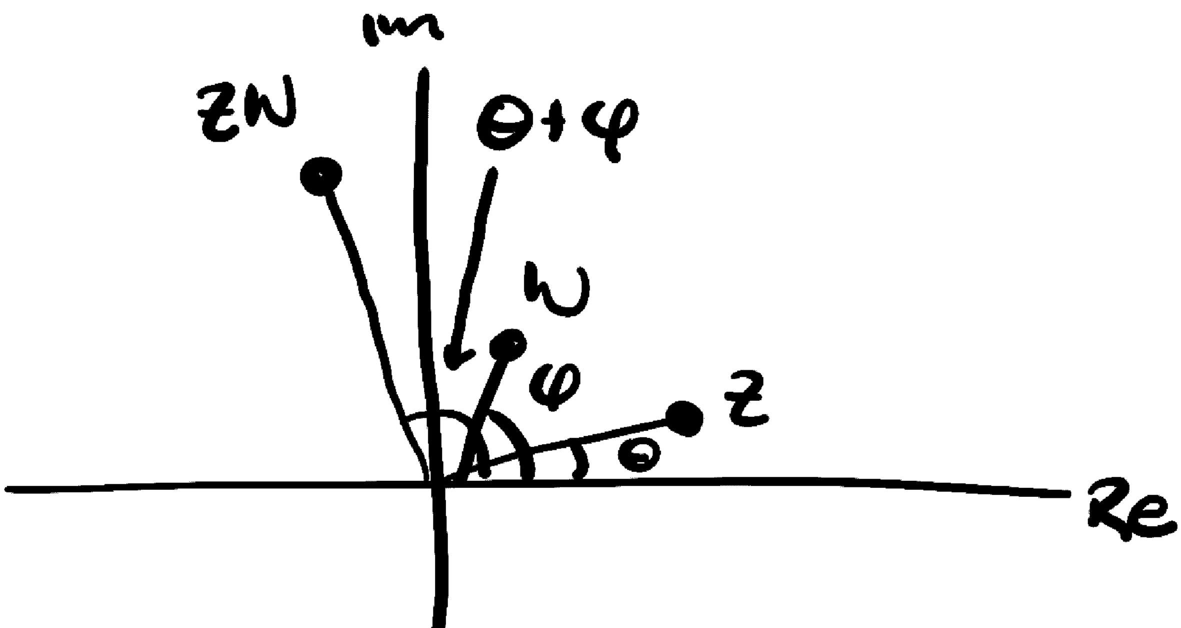
$z = a+bi$, $r = |z|$, $\theta = \arg z$

then $\boxed{z = re^{i\theta}}$

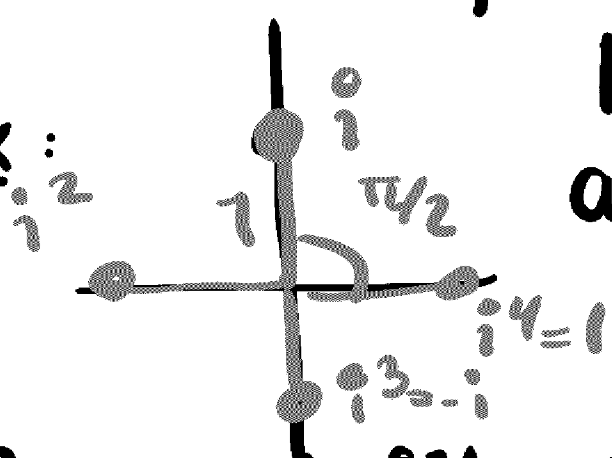
multiplication is easier:

$z = re^{i\theta}$, $w = se^{i\varphi}$ then

$$zw = (re^{i\theta})(se^{i\varphi}) = rse^{i(\theta+\varphi)}$$



Ex:



$|i| = 1$

$\arg i = \frac{\pi}{2}$

$i^2 = -1$, $|i^2| = 1$, $\arg i^2 = \pi$

$i^3 = i \cdot i^2 = i \cdot (-1) = -i$

$\arg i^3 = \frac{3\pi}{2}$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$\arg i^4 = 0$ (or 2π)

$$\frac{1}{i} = \frac{-i}{i(-i)} = \frac{-i}{-i^2} = \frac{-i}{1} = -i$$

Euler's formula

Remember

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Let's compute e^{ix}.

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!}$$

$$= 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!}$$

$$+\frac{i^5x^5}{5!}+\cdots$$

$$=1+ix-\frac{x^2}{2!}-\frac{ix^3}{3!}+\frac{x^4}{4!}$$

$$+\frac{ix^5}{5!}+\cdots$$

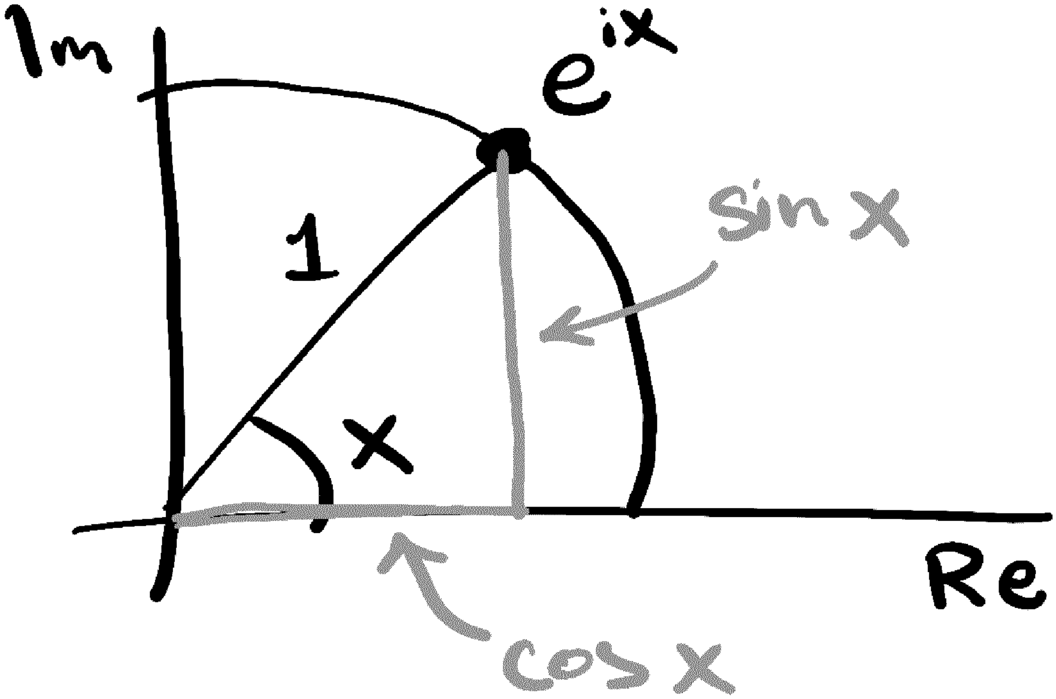
$$=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)$$

$$+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)$$

$$=\boxed{\cos x+i\sin x=e^{ix}}$$

Geometric explanation

$$z=e^{ix},\quad |z|=1,\quad \arg z=x$$



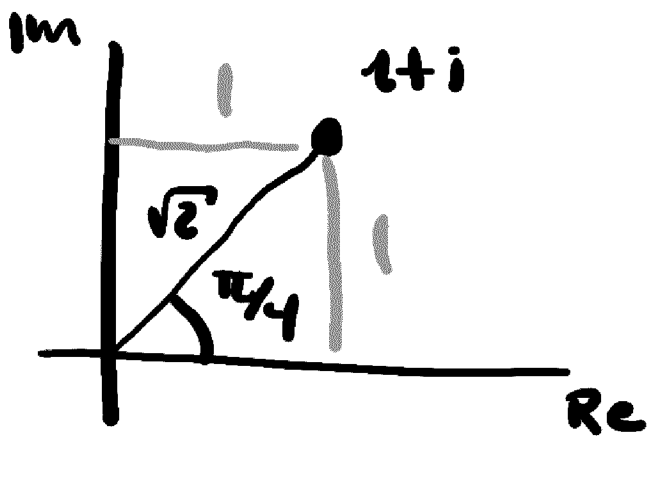
$$e^{ix} = a + bi, \text{ then } \left| \begin{array}{l} a = \cos x \\ b = \sin x \end{array} \right.$$

<u>Ex</u>: To compute $(1+i)^{10}$ we first write $1+i$ on the form $re^{i\theta}$.

$r = |1+i| = \sqrt{1+1} = \sqrt{2}$

$\arg(1+i) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$, so

$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

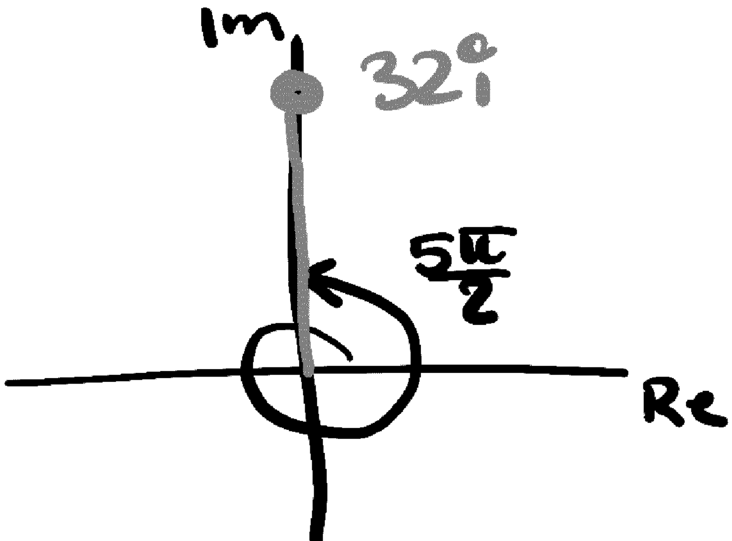


Then $(1+i)^{10} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{10}$

$= \sqrt{2}^{10}\, e^{i\frac{\pi}{4}\cdot 10}$

$= \left(\sqrt{2}^{2}\right)^{5} e^{i\frac{5\pi}{2}} = 2^{5} e^{i\left(2\pi + \frac{\pi}{2}\right)}$

$= 32 e^{i\frac{\pi}{2}} = 32i$



If $z = re^{i\theta}$ then $z^2 = \left(re^{i\theta}\right)^2$

$= r^2 e^{2i\theta} = r^2\left(\cos(2\theta) + i\sin(2\theta)\right)$

can also multiply out:

$\left(re^{i\theta}\right)^2 = \left(r(\cos\theta + i\sin\theta)\right)^2$

$= r^2(\cos\theta + i\sin\theta)^2$

$= r^2\left((\cos^2\theta - \sin^2\theta) + 2\cos\theta\sin\theta\, i\right)$

so this derives the identities

$$\begin{cases} \cos(2\theta) = \cos^2\theta - \sin^2\theta \\ \sin(2\theta) = 2\cos\theta\sin\theta \end{cases}.$$

In general we have de Moivre's theorem:

If $z = re^{i\theta}$, then

$$\boxed{z^n = r^n(\cos(n\theta) + i\sin(n\theta))}.$$

Note $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos(-x) + i\sin(-x)$

$= \cos x - i\sin x$

$\boxed{\begin{matrix}\cos(-x) = \cos x \\ \sin(-x) = -\sin x\end{matrix}}$

So

$$e^{ix}+e^{-ix}=(\cos x+i\sin x)$$
$$+(\cos x-i\sin x)$$
$$=2\cos x$$

$$\boxed{\cos x=\frac{e^{ix}+e^{-ix}}{2}}$$

Similarly: $e^{ix}-e^{-ix}=2i\sin x$

$$\boxed{\sin x=\frac{e^{ix}-e^{-ix}}{2i}}$$