MAT 127 — Lec 2, W Jan 24

① Midterms: Tu Feb 27 and M Apr 8
  8:30-9:50 pm

② In-class quizzes:
  returned via <u>Gradescope</u>.

(a) Check stony brook email. You should have received an invitation

(b) If not, contact me!

<u>Recall</u>: • $\{a_n\}$ sequence $\lim\limits_{n \to \infty} a_n = L$

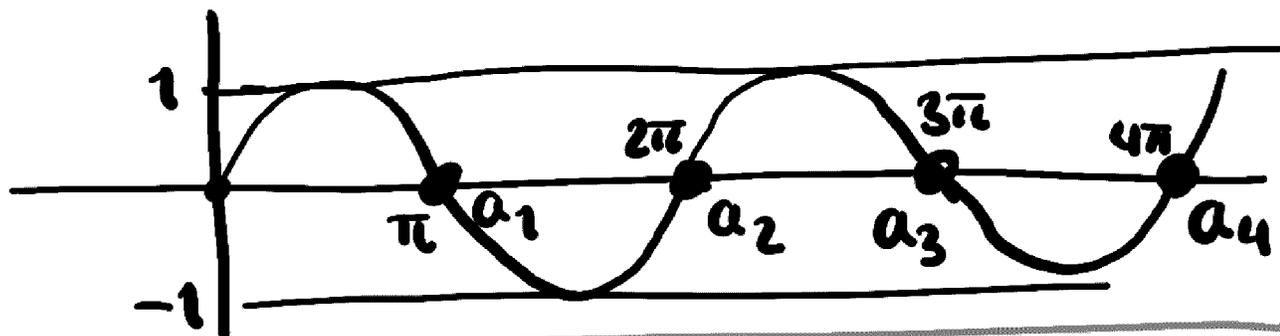if a_n approaches L as n becomes large. It's <u>convergent</u> if L exists, else <u>divergent</u>.

• If $a_n = f(n)$ for some function and $\lim\limits_{x \to \infty} f(x) = L$ then $\lim\limits_{n \to \infty} a_n = L$

<u>Warning</u>: If $a_n = f(n)$ for a function, and $f(x)$ diverges as $x \to \infty$ we <u>DO NOT</u> know whether $\lim\limits_{n \to \infty} a_n$ exists or not!

Ex: $a_n = \sin(\pi n)$, $n = 1, 2, \ldots$

Note $a_n = f(n)$, $f(x) = \sin(\pi x)$
for real x.

$$\lim_{x \to \infty} \sin(\pi x) \text{ does } \underline{\text{not}} \text{ exist}$$
as it oscillates between -1 and 1

However $a_n = \sin(\pi n) = 0$
for all positive integers n.
So $\boxed{\lim_{n \to \infty} \sin(\pi n) = 0}$.

Ex: $a_n = (-1)^n$.

$a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $\ldots$

$\{a_n\}$ does not approach a number as $n \to \infty$ since it oscillates between ± 1.

$\lim\limits_{n \to \infty} (-1)^n$ does not exist,

so $\{(-1)^n\}$ is <u>divergent</u>.

Ex: Find $\lim\limits_{n \to \infty} \dfrac{n^2}{\sqrt{n^3+4n}}$. Divide numerator & denominator w/ highest power of n. It's n^2 because as n is large, denominator is approximated by $\sqrt{n^3} = n^{\frac{3}{2}}$.

$\lim\limits_{n \to \infty} \dfrac{n^2/n^2}{\frac{\sqrt{n^3+4n}}{n^2}} = \lim\limits_{n \to \infty} \dfrac{1}{\sqrt{\frac{n^3+4n}{n^4}}}$

$= \lim\limits_{n \to \infty} \sqrt{\dfrac{1}{\frac{n^3+4n}{n^4}}} = \lim\limits_{n \to \infty} \sqrt{\dfrac{1}{\frac{1}{n} + \frac{4}{n^3}}}$

$$= \sqrt{\frac{1}{\lim\limits_{n \to \infty} \frac{1}{n} + \lim\limits_{n \to \infty} \frac{4}{n^3}}} = \infty.$$

↳ approaches 0

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Thm If $\lim\limits_{n \to \infty} a_n = L$ and the function f is continuous at L,

$$\lim\limits_{n \to \infty} f(a_n) = f(L).$$

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EX: Find $\lim\limits_{n \to \infty} \sin\left(\frac{\pi}{n}\right)$.

$a_n = \frac{\pi}{n}$ and we want to find $\lim\limits_{n \to \infty} f(a_n)$ for $f(x) = \sin x$.

Note $\lim\limits_{n \to \infty} \dfrac{\pi}{n} = 0$.

$f(x) = \sin x$ is continuous at 0 (its continuous at all points)

so $\lim\limits_{n \to \infty} \sin\left(\dfrac{\pi}{n}\right) = \sin\left(\lim\limits_{n \to \infty} \dfrac{\pi}{n}\right)$

$\qquad\qquad = \sin(0) = 0$.

Ex: <u>Geometric sequence</u>

For what values of r is $a_n = r^n$ a convergent sequence?

$a_n = f(n)$, $f(x) = r^x$.

Recall

$\lim\limits_{x \to \infty} r^x = \begin{cases} \infty & \text{fer } r > 1 \\ 0 & \text{fer } 0 \leq r < 1 \end{cases}$

Boundary case: $\lim\limits_{x \to \infty} 1^x = 1$

We will also allow negative values of r, so :

$$\lim_{n \to \infty} r^n = \begin{cases} 1, & r = 1 \\ 0, & -1 < r < 1 \\ \infty, & \text{else} \end{cases}$$

Def A sequence $\{a_n\}$ is

- <u>increasing</u> if $a_n \leq a_{n+1}$ for all $n \geq 1$

- <u>decreasing</u> if $a_n \geq a_{n+1}$ for all $n \geq 1$

- <u>monotonic</u> if it's either increasing or decreasing

Ex: $a_n = \frac{3}{n+5}$ is decreasing:

$$a_{n+1} = \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

$$a_n = \frac{3}{n+5} \geq \frac{3}{n+6} = a_{n+1} \text{ for all } n \geq 1.$$
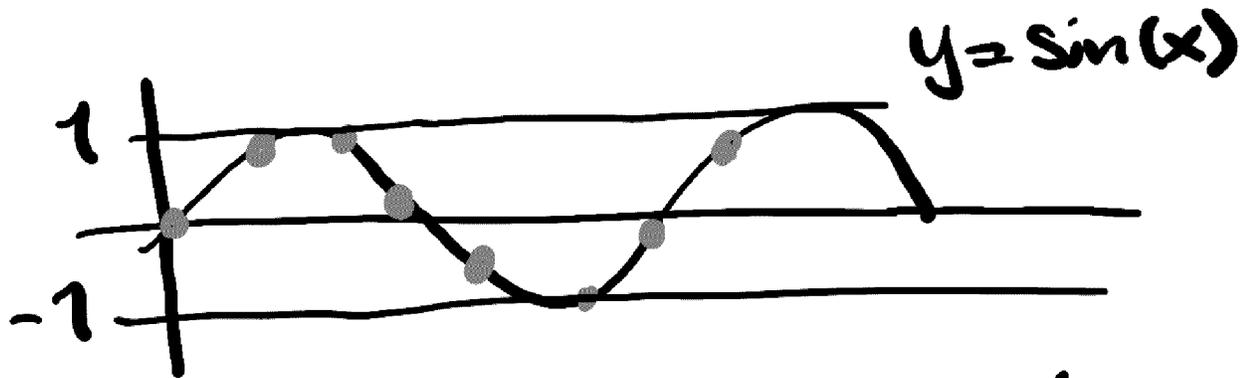
bigger denominator
→ smaller quotient.

Def: A sequence is

- **bounded above** if there's a number M such that $a_n \leq M$ for all $n \geq 1$

- **bounded below** if there's a number m such that $a_n \geq m$ for all $n \geq 1$

- **bounded** if it's both bounded above and below.

Ex $a_n = \sin(n)$ is bounded:
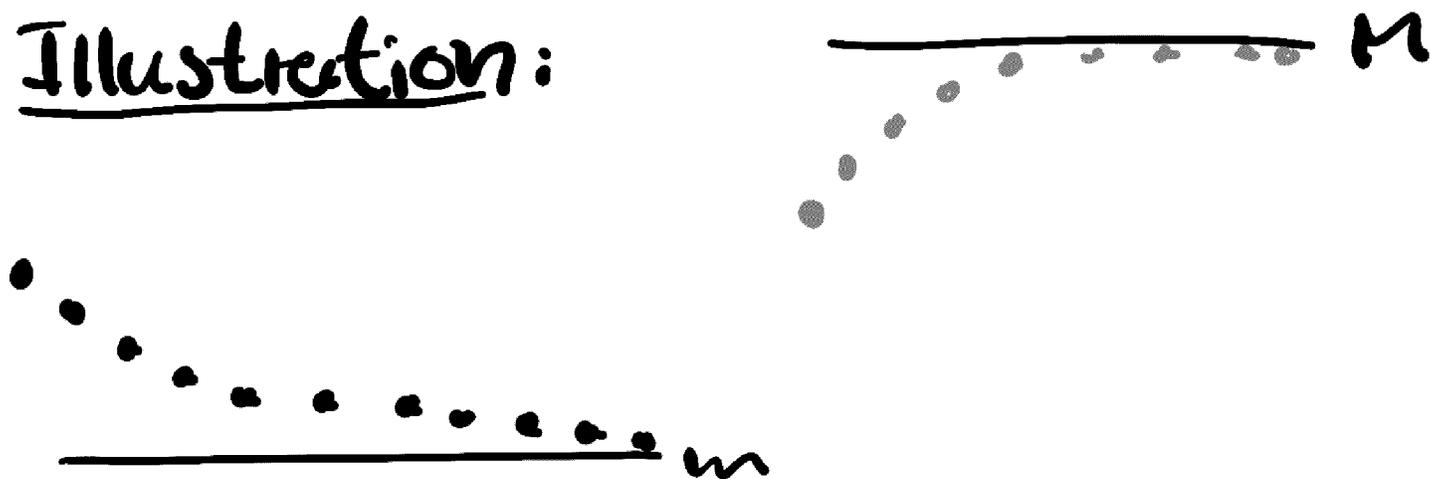
$$-1 \leq \sin(n) \leq 1$$

$y = \sin(x)$

Note $\lim\limits_{n \to \infty} a_n$ does not exist

Thm (Monotone convergence theorem)
  Let $\{a_n\}$ be a sequence.

  • If $\{a_n\}$ is increasing and
     bounded above, then it is
     convergent.

  • If $\{a_n\}$ is decreasing and
     bounded below, then it is
     convergent

Illustration:

Ex: Let a_n be defined by

$$a_1 = 7, \quad a_n = \frac{a_{n-1}}{n} \text{ for } n \geq 2.$$

$$a_2 = \frac{7}{2}, \quad a_3 = \frac{7/2}{3} = \frac{7}{2 \cdot 3}, \quad a_4 = \frac{7/2 \cdot 3}{4}$$

$$= \frac{7}{2 \cdot 3 \cdot 4} = \frac{7}{4!}$$

$$a_5 = \frac{7/2 \cdot 3 \cdot 4}{5} = \frac{7}{5!}$$

$$a_{n+1} = \frac{a_n}{n+1} \leq \frac{a_n}{1} = a_n$$

smaller denominator
→ larger quotient

So sequence is <u>decreasing</u>

Also $a_n \geq 0$ is "obviously" true since we only take quotients of positive integers.

So the monotone convergence thm guarantees that $\lim_{n \to \infty} a_n$ exists.

Ex (alternative sol)
   a_n as above.

Note $a_n = \dfrac{7}{n!}$.

   factorial $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

we then see $\lim_{n \to \infty} \dfrac{7}{n!} = 0$.

Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n \leq b_n$ for all $n \geq 1$.

- If $\lim\limits_{n \to \infty} b_n = B$ exists then

  we have $\lim\limits_{n \to \infty} a_n \leq B$ if the

  limit exists.

- If $\lim\limits_{n \to \infty} a_n = A$ exists, then
  we have $\lim\limits_{n \to \infty} b_n \geq A$ if the
  limit exists.

- If $a_n \longrightarrow +\infty$ then
  $b_n \longrightarrow +\infty$.

Thm (squeeze theorem)

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences such that
$a_n \leq b_n \leq c_n$ for all $n \geq 1$.

If $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$ exists then $\lim_{n \to \infty} b_n$ exists and is equal to L.