

Recall ODE is separable if it can be written in the form

$$\frac{dy}{dx} = f(y)g(x)$$

We solve it by collecting everything "involving" y in the LHS & everything "involving" x in the RHS.

$$\frac{1}{f(y)} dy = g(x) dx$$

Then integrate:

$$\int \frac{1}{f(y)} dy = \int g(x) dx$$

Ex: $y' = x e^y \Leftrightarrow e^y dy = x dx$

$$\Leftrightarrow \int e^y dy = \int x dx \Leftrightarrow e^y = \frac{x^2}{2} + C$$

$$\Leftrightarrow y = \log\left(\frac{x^2}{2} + C\right)$$

Ex $y' + xy = x$

$$\frac{dy}{dx} = x - xy = x(1-y)$$

$$\Leftrightarrow \frac{1}{1-y} dy = x dx$$

$$\Leftrightarrow -\ln|1-y| = \frac{x^2}{2} + C$$

$$\Leftrightarrow \ln|1-y| = -\frac{x^2}{2} + D$$

$$\Leftrightarrow 1-y = e^{-\frac{x^2}{2} + D} = E e^{-\frac{x^2}{2}}$$

$$\Leftrightarrow \boxed{y = 1 - E \cdot e^{-\frac{x^2}{2}}}$$

Mixing problems

Ex. A tank contains 20 kg of salt dissolved in 5000 L of water.

Brine that contains 0.03 kg salt/L enters the tank at the rate of 25 L/min. The solution is kept thoroughly mixed & it drains with the same rate.

How much salt is in the tank after 30 min?

Let $y(t)$ = amt of salt in kg
at time t (min)

We know $y(0) = 20$ & need to find $y(30)$.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

↑
Salt that
enters
tank per min

↑
Salt that
exits tank
per min

$$\begin{aligned} \text{rate in} &= \left(0.03 \cdot \frac{\text{kg}}{\text{L}}\right) \cdot \left(25 \cdot \frac{\text{L}}{\text{min}}\right) \\ &= 0.75 \frac{\text{kg}}{\text{min}} \end{aligned}$$

To calculate rate out, we need to know the current amt of salt.

$$\text{rate out} = \left(\text{curr salt concentration} \frac{\text{kg}}{\text{L}}\right) \cdot \left(25 \frac{\text{L}}{\text{min}}\right)$$

$$\left[\text{curr salt concentration} = \frac{y(t)}{5000} \frac{\text{kg}}{\text{L}}\right]$$

$$\text{Hence} = \frac{y(t)}{5000} \cdot 25 = \frac{y(t)}{200}$$

$$\boxed{\frac{dy}{dt} = \frac{3}{4} - \frac{y(t)}{200} \quad y(0) = 20}$$

Let's now solve it!

$$200 \frac{dy}{dt} = 150 - y(t)$$

$$\Leftrightarrow \int \frac{200}{150 - y} dy = \int dt$$

$$\Leftrightarrow -200 \ln|150-y| = t + C$$

$$\Leftrightarrow \ln|150-y| = -\frac{t}{200} + D$$

$$\Leftrightarrow 150-y = e^{-\frac{t}{200} + D} = E \cdot e^{-\frac{t}{200}}$$

$$\Leftrightarrow \boxed{y = 150 - E \cdot e^{-\frac{t}{200}}}$$

$$y(0) = 150 - E \cdot e^0 = 20$$

$$\Leftrightarrow E = 130$$

$$y(t) = 150 - 130 e^{-\frac{t}{200}}$$

$$y(30) = 150 - 130 e^{-\frac{30}{200}}$$

$$\approx 38.108 \text{ kg}$$

Ex A vat with 500 gallons of beer contains 4% alcohol

(by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min & the mixture is pumped out at the same rate.

What's the % of alcohol after 1 hour?

Let $P(t)$ = alcohol (gal) at time t in the vat.

$$P(0) = 500 \cdot 4\% = 500 \cdot 0.04 = 20 \text{ gal}$$

$$\frac{dP}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 5 \cdot 0.06 \frac{\text{gal alc}}{\text{min}}$$

$$= 0.3 \text{ gal/min}$$

$$\begin{aligned}\text{rate out} &= 5 \cdot \frac{P(t)}{500} \text{ gal alc/min} \\ &= \frac{P(t)}{100} \text{ gal/min}\end{aligned}$$

$$\frac{dP}{dt} = 0.3 - \frac{P(t)}{100}$$

Therefore want to solve the IVP

$$\boxed{\frac{dP}{dt} = 0.3 - \frac{P(t)}{100}, P(0) = 20}$$

$$\int \frac{1}{0.3 - \frac{P(t)}{100}} dP = \int dt$$

$$\Leftrightarrow \frac{1}{-1/100} \ln \left| 0.3 - \frac{P(t)}{100} \right| = t + C$$

$$\Leftrightarrow \ln \left| 0.3 - \frac{P(t)}{100} \right| = -\frac{t}{100} + D$$

$$\Leftrightarrow 0.3 - \frac{P(t)}{100} = e^{-\frac{t}{100} + D} = E e^{-\frac{t}{100}}$$

$$\Leftrightarrow 30 - P(t) = F e^{-\frac{t}{100}}$$

$$\Leftrightarrow P(t) = 30 - F e^{-\frac{t}{100}}$$

$$P(0) = 20 \Leftrightarrow 30 - F = 20$$

$$\Leftrightarrow F = 10$$

$$P(t) = 30 - 10 e^{-\frac{t}{100}}$$

want to find $\frac{P(60)}{500}$.

$$\frac{P(60)}{500} = \frac{30 - 10 e^{-\frac{6}{10}}}{500} \approx 0.0529$$

$$= 5.29\%$$
