Recall: • Taylor Series of $f(x)$

Centered at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a)$$
$$+ \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

• Maclaurin Series of $f(x)$ is the Taylor series centered at $x = 0$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$
$$+ \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

• $\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$ $|x| < 1$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

Ex: $f(x) = \cos x$. Since we know the Maclaurin series for $\sin x$, and $\frac{d}{dx} \sin x = \cos x$, we can differentiate the Maclaurin series of $\sin x$:

$$\frac{d}{dx} \sin x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \frac{d}{dx} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

for all x.

__Ex__ $f(x) = \arctan(x)$.

Note $f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$

$$= \sum_{n=0}^{\infty} (-x^2)^n \qquad |-x^2| < 1$$
$$\Leftrightarrow |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Find f by integrating!

$$\int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan(x).$$

for $|x| < 1$.

We know $\arctan(0) = 0$ so plugging in $x = 0$ gives

$C + 0 = 0$.

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x| < 1$$

EX Find the Maclaurin series for $x\cos x$. We already know

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

So it suffices to multiply both sides by x.

$$x\cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

Binomial Series

Let k be some real number and try to find the Maclaurin series for $f(x) = (1+x)^k$.

Derivatives & values at $x=0$:

$f(x) = (1+x)^k$, $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$, $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$, $f''(0) = k(k-1)$

$$\vdots$$

$f^{(N)}(x) = k \cdot (k-1) \cdots (k-N+1)(1+x)^{k-N}$

$f^{(N)}(0) = k \cdot (k-1) \cdots (k-N+1)$

So the Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k \cdot (k-1) \cdot \cdots \cdot (k-n+1)}{n!} x^n.$$

BINOMIAL SERIES

The coefficients are called binomial coefficients

$$\binom{k}{n} = \begin{cases} \dfrac{k(k-1) \cdots (k-n+1)}{n!} \\[2em] 0 \qquad\qquad\qquad\qquad n > k \end{cases}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \qquad |x| < 1$$