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No Smooth Julia Sets for Polynomial Diffeomorphisms of $\mathbb{C}^2$ with Positive Entropy

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Abstract For any polynomial diffeomorphism f of $\mathbb{C}^2$ with positive entropy, the Julia set of f is never C^1 smooth as a manifold-with-boundary.

Keywords Polynomial diffeomorphisms of $\mathbb{C}^2$ · Julia set · Generalized Hénon maps

Mathematics Subject Classification 37F10

1 Introduction

There are several reasons why the polynomial diffeomorphisms of $\mathbb{C}^2$ form an interesting family of dynamical systems. One of these is the fact that there are connections with two other areas of dynamics: polynomial maps of $\mathbb{C}$ and diffeomorphisms of $\mathbb{R}^2$, which have each received a great deal of attention. Among the endomorphisms of $\mathbb{P}^k$, certain ones have more special, and regular, geometric structure.

The question arises whether, among the polynomial diffeomorphisms of $\mathbb{C}^2$, are there analogous special maps with special geometry? The Julia set of such a special map would be expected to have some smoothness. Here we show that this does not happen.

More generally, we consider a holomorphic mapping $f : X \to X$ of a complex manifold X. The Fatou set of f is defined as the set of points $x \in X$ where the iterates $f^n := f \circ \cdots \circ f$ are locally equicontinuous. If X is not compact, then in

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the definition of equicontinuity, we consider the one point compactification of X; in this case, a sequence which diverges uniformly to infinity is equicontinuous. By the nature of equicontinuity, the dynamics of f is regular on the Fatou set. The Julia set is defined as the complement of the Fatou set, and this is where any chaotic dynamics of f will take place. The first nontrivial case is where $X = \mathbb{P}^1$ is the Riemann sphere, and in this case Fatou (see [17]) showed that if the Julia set J is a smooth curve, then either J is the unit circle or J is a real interval. If J is the circle, then f is equivalent to $z \mapsto z^d$, where d is an integer with $|d| \geq 2$; if J is the interval, then f is equivalent to a Chebyshev polynomial. These maps with smooth J play special roles, and this sparked our interest to look for smooth Julia sets in other cases. (The higher dimensional case is discussed, for instance, in Nakane [19] and Uchimura [23,24].)

Here we address the case where $X = \mathbb{C}^2$, and f is a polynomial automorphism, which means that f is biholomorphic, and the coordinates are polynomials. Since f is invertible, there are two Julia sets: J^+ for iterates in forward time, and J^- for iterates in backward time. Polynomial automorphisms have been classified by Friedland and Milnor [12]; every such automorphism is conjugate to a map which is either affine or elementary, or it belongs to the family $\mathcal{H}$. The affine and elementary maps have simple dynamics, and $J^{\pm}$ are (possibly empty) algebraic sets (see [12]).

Thus we will restrict our attention to the maps in $\mathcal{H}$, which are finite compositions $f := f_k \circ \cdots \circ f_1$, where each f_j is a generalized Hénon map, which by definition has the form $f_j(x, y) = (y, p_j(y) - \delta_j x)$, where $\delta_j \in \mathbb{C}$ is nonzero, and $p_j(y)$ is a monic polynomial of degree $d_j \geq 2$. The degree of f is $d := d_1 \cdots d_k$, and the complex Jacobian of f is $\delta := \delta_1 \cdots \delta_k$. In [12] and [22], it is shown that the topological entropy of f is $\log d > 0$. The dynamics of such maps is complicated and has received much study, starting with the papers [3,11,14,15].

For maps in $\mathcal{H}$, we can ask whether J^+ can be a manifold. For any saddle point q, the stable manifold $W^s(q)$ is a Riemann surface contained in J^+. Thus J^+ would have to have real dimension at least two. However, J^+ is also the support of a positive, closed current μ^+ with continuous potential, and such potentials cannot be supported on a Riemann surface (see [3,11]). On the other hand, since $J^+ = \partial K^+$ is a boundary, it cannot have interior. Thus dimension 3 is the only possibility for J^+ to be a manifold. In fact, there are examples of f for which J^+ has been shown to be a topological 3-manifold (see [8,11,16,20]). Fornæss and Sibony [11] have shown that J^+ cannot be smooth for a generic element of $\mathcal{H}$.

The purpose of this paper is to prove the following:

Theorem *For any polynomial automorphism of $\mathbb{C}^2$ of positive entropy, neither J^+ nor J^- is smooth of class C^1, in the sense of manifold-with-boundary.*

We may interchange the roles of J^+ and J^- by replacing f by f^{-1}, so there is no loss of generality if we consider only J^+.

In an Appendix, we discuss the nonsmoothness of the related sets J, J^*, and K.

2 No Boundary

Let us start by showing that if J^+ is a C^1 manifold-with-boundary, then the boundary is empty. Recall that if J^+ is C^1, then for each $q_0 \in J^+$ there is a neighborhood $U \ni q_0$

and $r, \rho \in C^1(U)$ with $dr \wedge d\rho \neq 0$ on U, such that $U \cap J^+ = \{r = 0, \rho \leq 0\}$. If J^+ has boundary, it is given locally by $\{r = \rho = 0\}$. For $q \in J^+$, the tangent space $T_q J^+$ consists of the vectors that annihilate dr. This contains the subspace $H_q \subset T_q J^+$ consisting of the vectors that annihilate ∂r. H_q is the unique complex subspace inside $T_q J^+$, so if $M \subset J^+$ is a complex submanifold, then $T_q M = H_q$.

We start by showing that if J^+ is C^1, then it carries a Riemann surface lamination.

Lemma 2.1 *If J^+ is C^1 smooth, then J^+ carries a Riemann surface foliation $\mathcal{R}$ with the property that if $W^s(q)$ is the stable manifold of a saddle point q, then $W^s(q)$ is a leaf of $\mathcal{R}$. If J^+ is a C^1 smooth manifold-with-boundary, then $\mathcal{R}$ extends to a Riemann surface lamination of J^+. In particular, any boundary component is a leaf of $\mathcal{R}$.*

Proof Given $q_0 \in J^+$, let us choose holomorphic coordinates (z, w) such that $dr(q_0) = dw$. We work in a small neighborhood which is a bidisk $\Delta_\eta \times \Delta_\eta$. We may choose η small enough that $|r_z/r_w| < 1$. In the (z, w)-coordinates, the tangent space H_q has slope less than 1 at every point $\{|z|, |w| < \eta\}$. Now let $\hat{q}$ be a saddle point, and let $W^s(\hat{q})$ be the stable manifold, which is a complex submanifold of $\mathbb{C}^2$, contained in J^+. Let M denote a connected component of $W^s(\hat{q}) \cap (\Delta_\eta \times \Delta_{\eta/2})$. Since the slope is <1, it follows that there is an analytic function $\varphi : \Delta_\eta \to \Delta_\eta$ such that $M \subset \Gamma_\varphi := \{(z, \varphi(z)) : z \in \Delta_\eta\}$. Let Φ denote the set of all such functions φ. Since a stable manifold can have no self-intersections, it follows that if $\varphi_1, \varphi_2 \in \Phi$, then either $\Gamma_{\varphi_1} = \Gamma_{\varphi_2}$ or $\Gamma_{\varphi_1} \cap \Gamma_{\varphi_2} = \emptyset$. Now let $\hat{\Phi}$ denote the set of all normal limits (uniform on compact subsets of Δ_η) of elements of Φ. We note that by Hurwitz's Theorem, the graphs $\Gamma_\varphi, \varphi \in \hat{\Phi}$ have the same pairwise disjointness property. Finally, by [4], $W^s(q_0)$ is dense in J^+, so the graphs $\Gamma_\varphi, \varphi \in \hat{\Phi}$ give the local Riemann surface lamination.

If q_1 is another saddle point, we may follow the same procedure and obtain a Riemann surface lamination whose graphs are given locally by $\varphi \in \hat{\Phi}_1$. However, we have seen that the tangent space to the foliation at a point q is given by H_q. Since these two foliations have the same tangent spaces everywhere, they must coincide.

We have seen that all the graphs are contained in J^+, so if J^+ has boundary, then the boundary must coincide locally with one of the graphs. □

We will use the observation that $K^+ \subset \{(x, y) \in \mathbb{C}^2 : |y| > \max(|x|, R)\}$. Further, we will use the Green function G^- which has many properties, including

(i) G^- is pluriharmonic on $\{G^- > 0\}$,
(ii) $\{G^- = 0\} = K^-$, and
(iii) $G^- \circ f = d^{-1} G^-$.

Further, the restriction of G^- to $\{|y| \leq \max(|x|, R)\}$ is a proper exhaustion.

Lemma 2.2 *Suppose that J^+ is a C^1 smooth manifold-with-boundary, and M is a component of the boundary of J^+. Then M is a closed Riemann surface, and $M \cap K \neq \emptyset$.*

Proof We consider the restriction $g := G^-|_M$. If $M \cap K = \emptyset$, then g is harmonic on M. On the other hand, g is a proper exhaustion of M, which means that $g(z) \to \infty$ as

$z \in M$ leaves every compact subset of M. This means that g must assume a minimum value at some point of M, which would violate the minimum principle for harmonic functions. □

Lemma 2.3 *Suppose that J^+ is a C^1 smooth manifold-with-boundary. Then the boundary is empty.*

Proof Let M be a component of the boundary of J^+. By Lemma 2.2, M must intersect Δ_R^2. Since J^+ is C^1, there can be only finitely many boundary components of $J^+ \cap \Delta_R^2$. Thus there can be only finitely many components M, which must be permuted by f. If we take a sufficiently high iterate f^N, we may assume that M is invariant. Now let $h := f^N|_M$ denote the restriction to M. We see that h is an automorphism of the Riemann surface M, and the iterates of all points of M approach $K \cap M$ in forward time. It follows that M must have a fixed point $q \in M$, and $|h'(q)| < 1$. The other multiplier of Df at q is $\delta/h'(q)$.

We consider three cases. First, if $|\delta/h'(q)| > 1$, then q is a saddle point, and $M = W^s(q)$. On the other hand, by [4], the stable manifold of a saddle points is dense in J^+, which makes it impossible for M to be the boundary of J^+. This contradiction means that there can be no boundary component M.

The second case is $|\delta/h'(q)| < 1$. This case cannot occur because the multipliers are less than 1, so q is a sink, which means that q is contained in the interior of K^+ and not in J^+.

The last case is where $|\delta/h'(q)| = 1$. In this case, we know that f preserves J^+, so Df must preserve $T_q(J^+)$. This means that the outward normal to M inside J^+ is preserved, and thus the second multiplier must be +1. It follows that q is a semi-parabolic/semi-attracting fixed point. It follows that J^+ must have a cusp at q and cannot be C^1 (see Ueda [25] and Hakim [13]). □

3 Maps that Do Not Decrease Volume

We note the following topological result (see Samelson [21] for an elegant proof): If M is a smooth 3-manifold (without boundary) of class C^1 in $\mathbb{R}^4$, then it is orientable. This gives:

Proposition 3.1 *For any $q \in M$, there is a neighborhood U about q so that $U - M$ consists of two components $\mathcal{O}_1$ and $\mathcal{O}_2$, which belong to different components of $\mathbb{R}^4 - M$.*

Proof Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ belong to the same component of $\mathbb{R}^4 - M$. Then we can construct a simple closed curve $\gamma \subset \mathbb{R}^4$ which crosses M transversally at q and has no other intersection with M. It follows that the (oriented) intersection is $\gamma \cdot M = 1$ (modulo 2). But the oriented intersection modulo 2 is a homotopy invariant (see [18]), and γ is contractible in $\mathbb{R}^4$, so we must have $\gamma \cdot M = 0$ (modulo 2). □

Corollary 3.2 *If J^+ is C^1 smooth, then f is an orientation preserving map of J^+.*

Proof $U^+ := \mathbb{C}^2 - K^+$ is a connected (see [15]) and thus it is a component of $\mathbb{C}^2 - J^+$. Since f preserves U^+, it also preserves the orientation of J^+, which is $\pm\partial U^+$. □

We recall the following result of Friedland and Milnor:

Theorem *([12]) If $|\delta| > 1$, then K^+ has zero Lebesgue volume, and thus $J^+ = K^+$. If $|\delta| = 1$, then $\mathrm{int}(K^+) = \mathrm{int}(K^-) = \mathrm{int}(K)$. In particular, there exists R such that $J^+ = K^+$ outside Δ_R^2.*

Proof of Theorem in the case $|\delta| \geq 1$. Let $q \in J^+$ be a point outside Δ_R^2, as in the Theorem above. Then near q there must be a component $\mathcal{O}$, which is distinct from $U^+ = \mathbb{C}^2 - K^+$. Thus $\mathcal{O}$ must belong to the interior of K^+. But by the Theorem above, the interior of K^+ is not near q. □

4 Volume Decreasing Maps

Throughout this section, we continue to suppose that J^+ is C^1 smooth, and in addition we suppose that $|\delta| < 1$. For a point $q \in J^+$, we let $T_q := T_q(J^+)$ denote the real tangent space to J^+. We let $H_q := T_q \cap iT_q$ denote the unique (one-dimensional) complex subspace inside T_q. Since J^+ is invariant under f, so is H_q, and we let α_q denote the multiplier of $D_q f|_{H_q}$.

Lemma 4.1 *Let $q \in J^+$ be a fixed point. There is a $D_q f$-invariant subspace $E_q \subset T_q(\mathbb{C}^2)$ such that H_q and E_q generate T_q. We denote the multiplier of $D_q f|_{E_q}$ by β_q. Thus $D_q f$ is linearly conjugate to the diagonal matrix with diagonal elements α_q and β_q. Further, $\beta_q \in \mathbb{R}$ and $\beta_q > 0$.*

Proof We have identified an eigenvalue α_q of $D_q f$. If $D_q f$ is not diagonalizable, then it must have a Jordan canonical form $\begin{pmatrix} \alpha_q & 1 \\ 0 & \alpha_q \end{pmatrix}$. The determinant is $\alpha_q^2 = \delta$, which has modulus less than 1. Thus $|\alpha_q| < 1$, which means that q is an attracting fixed point and thus in the interior of K^+, not in J^+. Thus $D_q f$ must be diagonalizable, which means that H_q has a complementary invariant subspace E_q. Since E_q and T_q are invariant under $D_q f$, the real subspace $E_q \cap T_q \subset E_q$ is invariant, too. Thus $\beta_q \in \mathbb{R}$. By Corollary 3.2, $D_q f$ will preserve the orientation of T_q, and so $\beta_q > 0$. □

Let us recall the Riemann surface foliation of J^+ which was obtained in Lemma 2.1. For $q \in J^+$, we let R_q denote the leaf of $\mathcal{R}$ containing q. If q is a fixed point, then f defines an automorphism $g := f|_{R_q}$ of the Riemann surface R_q. Since $R_q \subset K^+$, we know that the iterates of g^n are bounded in a complex disk $q \in \Delta_q \subset R_q$. Thus the derivatives $(Dg)^n = D(g^n)$ are bounded at q. We conclude that $|\alpha_q| = |D_q(g)| \leq 1$. If $|\alpha_q| = 1$, then α_q is not a root of unity. Otherwise g is an automorphism of R_q fixing q, and $Dg^n(q) = 1$ for some n. It follows that g^n must be the identity on R_q. This means that Rq would be a curve of fixed points for f^n, but by [FM] all periodic points of f are isolated, so this cannot happen.

Lemma 4.2 *If $q \in J^+$ is a fixed point, then q is a saddle point, and $\alpha_q = \delta/d$, and $\beta_q = d$.*

Proof First we claim that $|\alpha_p| < 1$. Otherwise, we have $|\alpha_q| = 1$, and by the discussion above, this means that α_q is not a root of unity. Thus the restriction $g = f|_{R_q}$ is an irrational rotation. Let $\Delta \subset R_q$ denote a g-invariant disk containing q. Since $|\delta| = |\alpha_q \beta_q| = |\beta_q|$ has modulus less than 1, we conclude that f is normally attracting to Δ, and thus q must be in the interior of K^+, which contradicts the assumption that $q \in J^+$.

Now we have $|\alpha_q| < 1$, so if $|\beta_q| = 1$, we have $\beta_q = 1$, since β_q is real and positive. This means that q is a semi-parabolic, semi-attracting fixed point for f. We conclude by Ueda [25] and Hakim [13] that J^+ has a cusp at q and thus is not smooth. Thus we conclude that $|\beta_q| > 1$, which means that q is a saddle point.

Now since E_q is transverse to H_q, it follows that $W^u(q)$ intersects J^+ transversally, and thus $J^+ \cap W^u(q)$ is C^1 smooth. Let us consider the uniformization

$$\phi : \mathbb{C} \to W^u(q) \subset \mathbb{C}^2, \quad \phi(0) = q, \quad f \circ \phi(\zeta) = \phi(\lambda^u \zeta).$$

The pre-image $\tau := \phi^{-1}(W^u(q) \cap J^+) \subset \mathbb{C}$ is a C^1 curve passing through the origin and invariant under $\zeta \mapsto \lambda^u \zeta$. It follows that $\lambda^u \in \mathbb{R}$, and τ is a straight line containing the origin. Further, $g^+ := G^+ \circ \phi$ is harmonic on $\mathbb{C} - \tau$, vanishing on τ, and satisfying $g^+(\lambda^u \zeta) = d \cdot g^+(\zeta)$. Since τ is a line, it follows that g^+ is piecewise linear, so we must have $\lambda^u = \pm d$. Finally, since f preserves orientation, we have $\lambda^u = d$.

Lemma 4.3 *There can be at most one fixed point in the interior of K^+. There are at least $d-1$ fixed points contained in J^+, and at each of these fixed points, the differential Df has multiplier of d.*

Proof Suppose that q is a fixed point in the interior of K^+. Then q is contained in a recurrent Fatou domain Ω, and by [4], $\partial\Omega = J^+$. If there is more than one fixed point in the interior of K^+, we would have J^+ simultaneously being the boundary of more than one domain, in addition to being the boundary of $U^+ = \mathbb{C}^2 - K^+$. This is not possible if J^+ is a topological submanifold of $\mathbb{C}^2$.

By [FM] there are exactly d fixed points, counted with multiplicity. By Lemma 4.3, the fixed points in J^+ are of saddle type, so they have multiplicity 1. Thus there are at least $d-1$ of them. $\square$

5 Fixed Points with Given Multipliers

If $q = (x, y)$ is a fixed point for $f = f_n \circ \cdots \circ f_1$, then we may represent it as a finite sequence (x_j, y_j) with $j \in \mathbb{Z}/n\mathbb{Z}$, subject to the conditions $(x, y) = (x_1, y_1) = (x_{n+1}, y_{n+1})$ and $f_j(x_j, y_j) = (x_{j+1}, y_{j+1})$. Given the form of f_j, we have $x_{j+1} = y_j$, so we may drop the x_j's from our notation and write $q = (y_n, y_1)$. We identify this point with the sequence $\hat{q} = (y_1, \ldots, y_n) \in \mathbb{C}^n$, and we define the polynomials

$$\begin{aligned} \varphi_1 &:= p_1(y_1) - \delta_1 y_n - y_2 \\ \varphi_2 &:= p_2(y_2) - \delta_2 y_1 - y_3 \\ &\cdots\cdots\cdots \\ \varphi_n &:= p_n(y_n) - \delta_n y_{n-1} - y_1. \end{aligned}$$

The condition to be a fixed point is that $\hat{q} = (y_1, \ldots, y_n)$ belongs to the zero locus $Z(\varphi_1, \ldots, \varphi_n)$ of the φ_i's. We define $q_i(y_i) := p_i(y_i) - y_i^{d_i}$ and $Q_i := q_i(y_i) - y_{i+1} - \delta_i y_{i-1}$, so

$$\varphi_i = y_i^{d_i} + q_i(y_i) - y_{i+1} - \delta_i y_{i-1} = y_i^{d_1} + Q_i \tag{*}$$

Since p_j is monic, the degrees of q_i and Q_i are $\leq d_i - 1$.

By the Chain Rule, the differential of f at $q = (y_n, y_1)$ is given by

$$Df(q) = \begin{pmatrix} 0 & 1 \\ -\delta_n & p_n'(y_n) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -\delta_1 & p_n'(y_1) \end{pmatrix}$$

We will denote this by $M_n = M_n(y_1, \ldots, y_n) := \begin{pmatrix} m_{11}^{(n)} & m_{12}^{(n)} \\ m_{21}^{(n)} & m_{22}^{(n)} \end{pmatrix}$.

We consider special monomials in $p_j' = p_j'(y_j)$ which have the form $(p')^L := p_{\ell_1}' \cdots p_{\ell_s}'$, with $L = \{\ell_1, \ldots, \ell_s\} \subset \{1, \ldots, n\}$. Note that the factors p_{ℓ_i}' in $(p')^L$ are distinct. Let us use the notation $|L|$ for the number of elements in L and $H_{\mathbf{m}}$ for the linear span of $\{(p')^L : |L| = m - 2k, 0 \leq k \leq n/2\}$. With this notation, $\mathbf{m}$ indicates the maximum number of factors of p_j' in any monomial, and in every case the number of factors differs from $\mathbf{m}$ by an even number.

Lemma 5.1 *The entries of M_n:*

(1) $m_{11}^{(n)}$ *and* $m_{22}^{(n)} - p_1'(y_1) \cdots p_n'(y_n)$ *both belong to* $H_{\mathbf{n-2}}$.
(2) $m_{12}^{(n)}, m_{21}^{(n)} \in H_{\mathbf{n-1}}$.

Proof We proceed by induction. The case $n = 1$ is clear. If $n = 2$,

$$M_2 = \begin{pmatrix} 0 & 1 \\ -\delta_2 & p_2' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\delta_1 & p_1' \end{pmatrix} = \begin{pmatrix} -\delta_1 & p_1' \\ -\delta_1 p_2' & p_1' p_2' - \delta_2 \end{pmatrix}$$

which satisfies (1) and (2). For $n > 2$, we have

$$M_n = \begin{pmatrix} 0 & 1 \\ -\delta_n & p_n' \end{pmatrix} M_{n-1} = \begin{pmatrix} m_{21}^{(n-1)} & m_{22}^{(n-1)} \\ -\delta_n m_{11}^{(n-1)} + m_{21}^{(n-1)} p_n' & -\delta_n m_{12}^{(n-1)} + p_n' m_{22}^{(n-1)} \end{pmatrix}$$

which gives (1) and (2) for all n. $\square$

The condition for Df to have a multiplier λ at q is $\Phi(\hat{q}) = 0$, where

$$\Phi = \det \left(M_n - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right)$$

Lemma 5.2 $\Phi - p_1'(y_1) \cdots p_n'(y_n) \in H_{\mathbf{n-2}}$.

Proof The formula for the determinant gives

$$\Phi = \lambda^2 - \lambda \mathrm{Tr}(M_n) + \det(M_n) = \lambda^2 - \lambda(m_{11}^{(n)} + m_{22}^{(n)}) + \delta$$

since δ is the Jacobian determinant of Df. The Lemma now follows from Lemma 5.1. □

The degree of the monomial $y^a := y_1^{a_1} \cdots y_n^{a_n}$ is $\deg(y^a) = a_1 + \cdots + a_n$. We will use the *graded lexicographical order* on the monomials in $\{y_1, \dots, y_n\}$. That is, $y^a > y^b$ if either $\deg(y^a) > \deg(y^b)$, or if $\deg(y^a) = \deg(y^b)$ and $a_i > b_i$, where $i = \min\{1 \le j \le n : a_j \ne b_j\}$. If $f \in \mathbb{C}[y_1, \dots, y_n]$, we denote $\mathrm{LT}(f)$ for the leading term of f, $\mathrm{LC}(f)$ for the leading coefficient, and $\mathrm{LM}(f)$ for the leading monomial.

Lemma 5.3 *With the graded lexicographical order, $G := \{\varphi_1, \dots, \varphi_n\}$ is a Gröbner basis.*

Proof We will use Buchberger's Algorithm (see [10, Chapter 2]). For each $i = 1, \dots, n$, $\mathrm{LT}(\varphi_i) = \mathrm{LM}(\varphi_i) = y_i^{d_i}$, so for $i \ne j$, the least common multiple of the leading terms is $\mathrm{L.C.M.} = y_i^{d_i} y_j^{d_j}$. The S-polynomial is

$$S(\varphi_i, \varphi_j) := \frac{\mathrm{L.C.M.}}{\mathrm{LM}(\varphi_j)}\varphi_i - \frac{\mathrm{L.C.M.}}{\mathrm{LM}(\varphi_i)}\varphi_j = y_j^{d_j} Q_i - y_i^{d_i} Q_j = \varphi_j Q_i - Q_j \varphi_i$$

where we use the Q_j from (4.1) and cancel terms. Now let $\mu_i := \deg(Q_i)$. Since $\mu_i < d_i$ for all i, the monomials $\mathrm{LM}(\varphi_j Q_i) = y_j^{d_j} y_i^{\mu_i}$ and $\mathrm{LM}(\varphi_i Q_j) = y_i^{d_i} y_j^{\mu_j}$ are not equal in our monomial ordering. Thus $\mathrm{LM}(S(\varphi_i, \varphi_j) \ge \max(\mathrm{LM}(\varphi_j Q_i), \mathrm{LM}(\varphi_i Q_j))$. It follows from Buchberger's Algorithm that $\{\varphi_1, \dots, \varphi_n\}$ is a Gröbner basis. □

We will use the Multivariable Division Algorithm, by which any polynomial $g \in \mathbb{C}[y_1, \dots, y_n]$ may be written as $g = A_1\varphi_1 + \cdots + A_n\varphi_n + R$ where $\mathrm{LM}(g) \ge \mathrm{LM}(A_j\varphi_j)$ for all $1 \le j \le n$, and R contains no terms divisible by any $\mathrm{LM}(\varphi_j)$. An important property of a Gröbner basis is that g belongs to the ideal $\langle \varphi_1, \dots, \varphi_n \rangle$ if and only if $R = 0$ (see, for instance, [10] or [1]).

If all fixed points have the same value of λ as multiplier, then it follows that Φ must vanish on the whole zero set $Z(\varphi_1, \dots, \varphi_n)$. Since we have a Gröbner basis, we easily determine the following:

Corollary 5.4 $\Phi \notin \langle \varphi_1, \dots, \varphi_n \rangle$.

Proof The leading monomial of Φ is $y_1^{d_1-1} \cdots y_n^{d_n-1}$, but this is not divisible by any of the leading monomials $\mathrm{LM}(\varphi_j) = y_j^{d_j}$. Since $\{\varphi_1, \dots, \varphi_n\}$ is a Gröbner basis, it follows that Φ does not belong to the ideal $\langle \varphi_1, \dots, \varphi_n \rangle$. □

6 Proof of the Theorem

In this section we prove the Theorem, which will follow from 4.3, in combination with:

Proposition 6.1 *Suppose $F = f_n \circ \cdots \circ f_1$, $n \geq 3$, is a composition of generalized Hénon maps with $|\delta| < 1$. Suppose that F has $d = d_1 \cdots d_n$ distinct fixed points. It is not possible that $d - 1$ of these points have the same multipliers.*

Proof that Proposition 6.1 implies the Theorem To prove the Theorem, it remains to deal with the case $|\delta| < 1$. If $f = f_1$ is a single generalized Hénon map, we consider $F = f_1 \circ f_1 \circ f_1$ with $n = 3$ and the same Julia set. Lemma 3.4 asserts that if J^+ is C^1, there are $d - 1$ saddle points with unstable multiplier $\lambda = d$. So by Proposition 4.1, we conclude that J^+ cannot be C^1 smooth. □

We give the proof of Proposition 6.1 at the end of this section. For $J \subset \{1, \ldots, n\}$, we write

$$\Lambda_J := \{(p')^L : L \subset J, |L| = |J| - 2k, \text{ for some, } 1 \leq k \leq |J|/2\},$$

We let H_J denote the linear span of Λ_J. To compare with our earlier notation, we note that $H_J \subset H_{|\mathbf{J}|-\mathbf{2}}$ and that $(p')^J \notin H_J$. The elements of H_J depend only on the variables y_j for $j \in J$. Now we formulate a result for dividing certain terms by φ_j:

Lemma 6.2 *Suppose that $J \subset \{1, \ldots, n\}$ and $h \in H_J$. Then for each $j \in J$ and $\alpha \in \mathbb{C}$, we have*

$$(y_j - \alpha)\left((p')^J + h\right) = A(y)\varphi_j + B(y)\left((p')^{J-\{j\}} + \rho_1\right) + (y_j - \alpha) \cdot \rho_2, \qquad (\dagger)$$

where $\rho_1, \rho_2 \in H_{J-\{j\}}$, and $B = \eta_j(y_j) + d_j y_{j+1} + d_j \delta_j y_{j-1}$ with

$$\eta_j(y_j) = y_j q_j'(y_j) - \alpha p_j'(y_j) - d_j q_j(y_j). \qquad (\ddagger)$$

The leading monomials satisfy

$$LM\left((y_j - \alpha)\left((p')^J + h\right)\right) = LM(A(y)\varphi_j)$$

Proof Let us start with the case $J = \{1, \ldots, m\}$, $m \leq n$, and $j = 1$, so $J - \{j\} = J_{\hat{1}} = \{2, \ldots, n\}$. We divide by p_1' and remove any factor of p_1' in h. This gives

$$(p')^J + h = p_1'(y_1)\mu_1 + \rho_2$$

where $\mu_1 = (p')^{J_{\hat{1}}} + \rho_1$, $\rho_1, \rho_2 \in H_{\{2,\ldots,m\}}$, and μ_1, ρ_1, ρ_2 are independent of the variable y_1. Thus

$$\begin{aligned}(y_1 - \alpha)\left((p')^J + h\right) &= (y_1 - \alpha)(d_1 y_1^{d_1 - 1} + q_1'(y_1))\mu_1 + (y_1 - \alpha)\rho_2 \\ &= d_1 y_1^{d_1}\mu_1 + (y_1 q_1'(y_1) - \alpha p_1'(y_1))\mu_1 + (y_1 - \alpha)\rho_2 \\ &= (d_1\mu_1)\varphi_1 + (\eta_1(y_1) + d_1 y_2 + d_1\delta_1 y_n)\mu_1 + (y_1 - \alpha)\rho_2\end{aligned}$$

where in the last line we substitute η_1 defined by (‡). Using (∗), we see that this gives (†).

It remains to look at the leading terms of $T_1 := (y_1 - \alpha)\left((p')^J + h\right)$ and $T_2 := d_1\mu_1\varphi_1$. We see that T_1 and T_2 both contain nonzero multiples of $y_j \prod_{i=1}^m y_i^{d_i-1}$, and all other monomials in T_1 and T_2 have lower degree. Thus we have $LM(T_1) = LM(T_2)$ for the graded ordering, independent of any ordering on the variables $y_1, \ldots, y_n$. The choices of $J = \{1, \ldots, m\}$ and $j = 1$ just correspond to a permutation of variables, and this does not affect the conclusion that $LM(T_1) = LM(T_2)$. □

Lemma 6.3 *For any $\alpha \in \mathbb{C}$, $(y_1 - \alpha)\Phi \notin \langle \varphi_1, \ldots, \varphi_n \rangle$.*

Proof By [FM], we may assume that $p_j(y_j) = y_j^{d_j} + q_j(y_j)$ and $\deg(q_j) \le d_j - 2$. We consider two cases. The first case is that there is at least one j such that η_j is not the zero polynomial. If we conjugate by $f_{j-1} \circ \cdots \circ f_1$, we may "rotate" the maps in f so that the factor f_j becomes the first factor. If there exists a j for which $\eta_j(y_j)$ is nonconstant, we choose this for f_1. Otherwise, if all the η_j are constant, we choose f_1 to be any factor such that $\eta_1 \ne 0$.

We will apply the Multivariate Division Algorithm on $(y_1 - \alpha)\Phi$ with respect to the set $\{\varphi_1, \ldots, \varphi_n\}$. We will find that there is a nonzero remainder, and since $\{\varphi_1, \ldots, \varphi_n\}$ is a Gröbner basis, it will follow that $(y_1 - \alpha)\Phi$ does not belong to the ideal $\langle \varphi_1, \ldots, \varphi_n \rangle$.

We start with Lemma 5.2, according to which $\Phi = p'_1 \cdots p'_n + h$, where $h \in H_{\mathbf{n-2}} = H_{\{1,\ldots,n\}}$. The leading monomial of $(y_1 - \alpha)\Phi$ is $y_1^{d_1} \prod_{i=2}^n y_i^{d_i-1}$, and φ_1 is the only element of the basis whose leading monomial divides this. Thus we apply Lemma 6.2, with $J = \{1, \ldots, n\}$, $j = 1$, and $J_{\hat{1}} := J - \{j\} = \{2, \ldots, n\}$. This gives

$$
\begin{aligned}
(y_1 - \alpha)\Phi &= A_1\varphi_1 + (\eta_1(y_1) + d_1 y_2 + d_1\delta_1 y_n)\left(\prod_{i=2}^n p'_i(y_i) + \rho_1\right) + (y_1 - \alpha)\rho_2 \\
&= A_1\varphi_1 + \left[d_1 y_2\left((p')^{J_{\hat{1}}} + \rho_1\right)\right] \\
&\quad + \left[d_1\delta_1 y_n\left((p')^{J_{\hat{1}}} + \rho_1\right)\right] + \left[\eta_1\left((p')^{J_{\hat{1}}} + \rho_1\right)\right] + \ell.o.t \\
&= A_1\varphi_1 + T_2 + T_n + R_1 + \ell.o.t
\end{aligned}
$$

where $\rho_1, \rho_2 \in H_{\{2,\ldots,n\}}$. In particular, T_2 and T_n depend on $y_2, \ldots, y_n$ but not on y_1. We note that T_2 (respectively, T_n) contains a term divisible by $\mathrm{LM}(\varphi_2)$ (respectively, $\mathrm{LM}(\varphi_n)$). We view R_1 as a remainder term, and note that $\mathrm{LM}(R_1)$ is divisible by $y_2^{d_2-1} \cdots y_n^{d_n-1}$, as well as the largest power of y_1 in $\eta_1(y_1)$. By "$\ell.o.t.$," we mean that none of its monomials is divisible by $\mathrm{LM}(R_1)$ or by any of the $\mathrm{LM}(\varphi_j)$.

Now we apply Lemma 6.2 to T_2, this time with $J = \{2, \ldots, n\}$ and $j = 2$, with $J - \{2\} = J_{\hat{1}\hat{2}} = \{3, \ldots, n\}$. We have

$$
\begin{aligned}
T_2 =& A_2\varphi_2 + d_2 y_3((p')^{J_{\hat{1}\hat{2}}} + \rho_1^{(2)}) + d_2\delta_2 y_1\left((p')^{J_{\hat{1}\hat{2}}} + \rho_1^{(2)}\right) + \eta_2(y_2)(p')^{J_{\hat{1}\hat{2}}} + \ell.o.t. \\
=& A_2\varphi_2 + T_2^{(2)} + R_1^{(2)} + R_2^{(2)} + \ell.o.t.
\end{aligned}
$$

We see that $T_2^{(2)}$ contains terms that are divisible by $\mathrm{LM}(\varphi_3)$, but the monomials in $R_1^{(2)}$ and $R_2^{(2)}$ are not divisible by $\mathrm{LM}(\varphi_i)$ for any i. The remainder term here is $R_1^{(2)} + R_2^{(2)}$, and we observe that this cannot cancel the largest term in R_1. This is because $\mathrm{LM}(R_1^{(2)})$ lacks a factor of y_2, and $\mathrm{LM}(R_2^{(2)})$ is equal to $y_3^{d_3-1} \cdots y_n^{d_n-1}$ times the largest power of y_2 in $\eta_2(y_2)$, and by (‡), this power is no bigger than $d_2 - 1$. If η_1 is not constant, then we see that $\mathrm{LM}(R_1) > \mathrm{LM}(R_2^{(2)})$. If η_1 is constant, then η_2 must be constant, too, and again we have $\mathrm{LM}(R_1) > \mathrm{LM}(R_2^{(2)})$. Thus, with our earlier notation, $R_1^{(2)} + R_2^{(2)} = \ell.o.t.$

We do a similar procedure with T_n, $T_2^{(2)}$, etc., and again find that the remainder term does not contain a multiple of the leading monomial of R_1. We see that each time we do this process, the size of the exponent L decreases in the term $(p')^L$. When we have $L = \emptyset$, there are no terms that can be divided by any $\mathrm{LM}(\varphi_j)$. Thus we end up with

$$(y_1 - \alpha)\Phi = A_1\varphi_1 + \cdots + A_n\varphi_n + R_1 + \ell.o.t.$$

and $\mathrm{LT}((y_1 - \alpha)\Phi) \geq \mathrm{LT}(A_j\varphi_j)$ for all $1 \leq j \leq n$, and none of the remaining terms is divisible by any of the leading monomials of φ_j. Thus we have now finished the Multivariate Division Algorithm, and we have a nonzero remainder. Thus $(y_1 - \alpha)\Phi$ does not belong to the ideal of the φ_j's.

Now we turn to the second case, in which $\eta_j = 0$ for all j. By [12], we may assume that $\deg(q_j) \leq d_j - 2$. It follows that $\alpha = 0$ and $q_j = 0$. Thus $p_j = y_j^{d_j}$ for all $1 \leq j \leq n$, so $p'_j = d_j y_j^{d_j-1}$, and H_J consists of linear combinations of products $(p')^I = y_{i_1}^{d_{i_1}-1} \cdots y_{i_k}^{d_{i_k}-1}$ for $I = \{i_1, \ldots, i_k\} \subset J$, for even $k \leq |J| - 2$. We will go through the Multivariate Division Algorithm again. The principle is the same as before, but the details are different; in the first case we needed $n \geq 2$, and now we will need $n \geq 3$.

Again, it is only φ_1 which has a leading monomial which can divide some terms in $(y_1 - \alpha)\Phi$. As before, we apply Lemma 6.2 with $J = \{1, \ldots, n\}$, $j = 1$, and $J - \{1\} = J_{\hat{1}} = \{2, \ldots, n\}$. The polynomial in (‡) becomes $B = d_j y_{j+1} + d_j \delta_j y_{j-1}$, and we have

$$\begin{aligned} y_1\Phi &= A_1\varphi_1 + d_1 y_2 \left((p')^{J_{\hat{1}}} + \rho_1\right) + d_1\delta_1 y_n \left((p')^{J_{\hat{1}}} + \rho_1\right) + y_1\rho_2 \\ &= A_1\varphi_1 + T_2 + T_n + \ell.o.t. \end{aligned}$$

where $\rho_1, \rho_2 \in H_{\{2,\ldots,n\}}$. Now we apply Lemma 6.2 to divide T_2 (respectively, T_n) by φ_2 (respectively, φ_n). This yields

$$y_1\Phi = A_1\varphi_1 + A_2\varphi_2 + A_n\varphi_n + T_3 + T_n + R + \ell.o.t.,$$

where

$$T_3 = d_1 d_2 y_3 \left((p')^{J_{\hat{1}\hat{2}}} + \tilde{\rho}_3\right), \quad T_n = d_1 d_n \delta_1 \delta_n y_{n-1} \left((p')^{J_{\hat{1}\hat{n}}} + \tilde{\rho}_n\right)$$

with $\tilde{\rho}_3 \in H_{\{3,\ldots,n\}}$ and $\tilde{\rho}_n \in H_{\{2,\ldots,n-1\}}$, and

$$R = \left(d_1 d_2 \delta_2 y_1 y_n^{d_n-1} + d_1 d_n \delta_1 y_1 y_2^{d_2-1}\right) \prod_{i=3}^{n-1} y_i^{d_i-1}$$

Since $n > 2$, R is not the zero polynomial. We will continue the Multivariate Division Algorithm by dividing T_3 by φ_3 and T_n by φ_n, but we see that any terms created cannot cancel R. Thus when we finish the Multivariable Division Algorithm, we will have a nonzero remainder. As in the previous case, we conclude that $y_1\Phi$ is not in the ideal $\langle\varphi_1, \ldots, \varphi_n\rangle$.

Proof of Proposition 6.1. The fixed points of f coincide with the elements of $Z(\varphi_1, \ldots, \varphi_n)$, which is a variety of pure dimension zero. Saddle points have multiplicity 1, and since there are $d-1$ of these, and since the total multiplicity is d, there must be one more fixed point, also of multiplicity 1. It follows that the ideal $I := \langle\varphi_1, \ldots, \varphi_n\rangle$ is equal to its radical (see [1]). Since the saddle points all have multiplier λ, Φ must vanish at all the saddle points. If (α, β) is the other fixed point, we conclude that $(y_1 - \alpha)\Phi$ vanishes at all the fixed points. Thus $(y_1 - \alpha)\Phi$ belongs to the radical of I, and thus I itself. This contradicts Lemma 6.3, which completes the proof of Proposition 6.1. $\square$

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Appendix: Nonsmoothness of J, J^*, and K

Let us turn our attention to other dynamical sets for polynomial diffeomorphisms of positive entropy. These are $J := J^+ \cap J^-$, $K := K^+ \cap K^-$, and the set J^*, which coincides with the closure of the set of periodic points of saddle type. (See [3,5], and [2] for other characterizations of J^*.) We have $J^* \subset J \subset K$. We note that none of these sets can be a smooth 3-manifold: otherwise, for any saddle point p, it would be a bounded set containing $W^s(p)$ or $W^u(p)$, which is the holomorphic image of $\mathbb{C}$. The following was suggested by Remark 5.9 of Cantat in [9]; we sketch his proof:

Proposition 6.1 *If $J = J^*$, then it is not a smooth 2-manifold.*

Proof Let p be a saddle point, and let $W^u(p)$ be the unstable manifold. The slice $J \cap W^u(p)$ is smooth and invariant under multiplication by the multiplier of Df. This means that in fact, the multiplier must be real, and the restriction of G^+ to the slice must be linear on each (half-space) component of $W^u(p) - J$.

The identity $G^+ \circ f = d \cdot G^+$ means that the canonical metric (defined in [6]) is multiplied by d. Thus f is quasi-expanding on J^*. Now, applying this argument to f^{-1} we get that f is quasi-hyperbolic. Further, $J^* = J$, so it is quasi-hyperbolic on J. If f fails to be hyperbolic, then by [7] there will be a one-sided saddle point, which cannot happen since J is smooth.

Now that f is hyperbolic on J, there is a splitting $E^s \oplus E^u$ of the tangent bundle, so we conclude that J is a 2-torus. The dynamical degree must be the spectral radius of an invertible 2-by-2 integer matrix, but this means it is not an integer, which contradicts the fact the dynamical degree of a Hénon map is its algebraic degree.

Proposition 6.2 *Suppose that the complex Jacobian is not equal to* ± 1. *Then for each saddle (periodic) point* p *and each neighborhood* U *of* p, *neither* $J \cap U$ *nor* $J^* \cap U$ *nor* $K \cap U$ *is a* C^1 *smooth 2-manifold.*

Proof Let us write $M := J \cap U$ and $g := f|_M$. (The following argument works, too, if we take $M = J^* \cap U$ or $M = K \cap U$.) The tangent space $T_p M$ is invariant under Df. The stable/unstable spaces $E^{s/u} \subset T_p\mathbb{C}^2$ are invariant under $D_p f$. The space E^s (or E^u) cannot coincide with $T_p M$, for otherwise the complex stable manifold $W^s(p)$ (or $W^u(p)$) would be locally contained in M, and thus globally contained in J. But the $W^{s/u}$ are uniformized by $\mathbb{C}$, whereas J is bounded. We conclude that p is a saddle point for g, and thus the local stable manifold $W^s_{\mathrm{loc}}(p; g)$ is a C^1-curve inside the complex stable manifold $W^s(p)$. As in Lemma 4.3, we conclude that the multiplier for $D_p f|_{E^u_p}$ is $\pm d$ and the multiplier for $D_p f|_{E^s_p}$ is $\pm 1/d$. Thus the complex Jacobian is $\delta = \pm 1$.

Solenoids The two results above concern smoothness, but no example is known where J, J^*, or K is even a topological 2-manifold. In the cases where J^+ has been shown to be a topological 3-manifold (see [8,11,16,20]) it also happens that J is a (topological) real solenoid, and in these cases it is also the case that $J = J^*$. Further, for every saddle (periodic) point p, there is a real arc $\gamma_p = W^u_{\mathrm{loc}}(p) \cap J$. If we apply the argument of Proposition 6.2 to this case, we conclude that γ_p is not C^1 smooth.

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