Problem 1. Find the volume of a hemisphere two ways:

(a) Use the disc method to find the volume of the "eastern" hemisphere formed by rotating the region under $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$ around the x axis.

(b) Use the shell method to find the volume of the "northern" hemisphere formed by rotating the region under $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$ around the y axis.

It may be helpful to recall that the curve $y = \sqrt{1 - x^2}$ is a semicircle of radius 1 centered at the origin.

Problem 2. Compute the arclength of curve $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$.

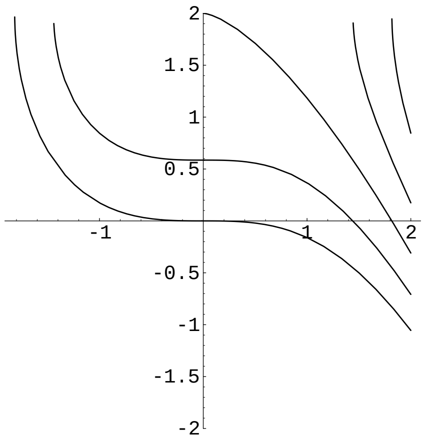
Problem 3. It is easy to check that $\dfrac{1}{x^2+x} = \dfrac{1}{x} - \dfrac{1}{x+1}$. Use this fact to

(a) Compute $\displaystyle\int_1^\infty \frac{dx}{x^2+x}$

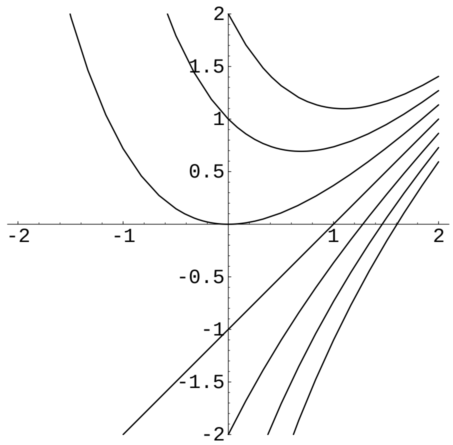
(b) Compute $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n}$

Problem 4.

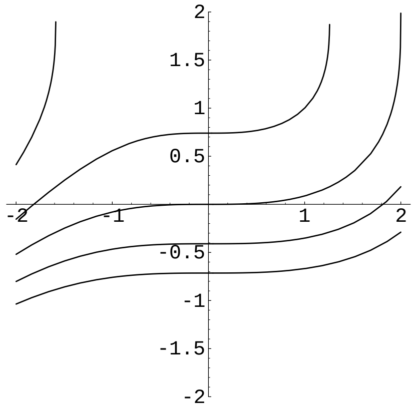
(a) Which plot shows solution curves for the differential equation $y' = x - y$?



(i)



(iii)



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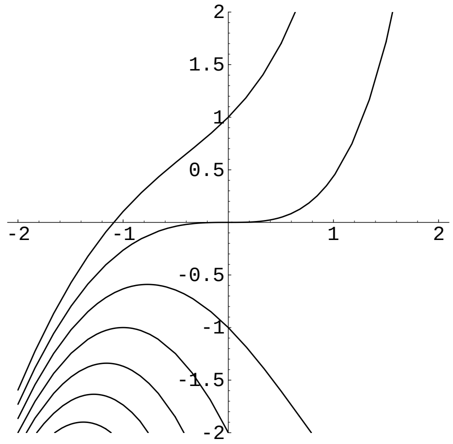

(iv)

(b) Which is a solution to the differential equation $y' = x - y$?

(i) $y = x + \dfrac{1}{e^x} - 1$

(ii) $y = \dfrac{x^2}{2} - x + 1$

(iii) $y = \sin(x)$

(iv) $y = e^x \cos(x)$

Problem 5. Use separation of variables to find the solution to

$$\frac{dy}{dx} = xe^y, \quad y(1) = 0.$$

Problem 6. Essay Question. Compare the exponential and logistic models for population growth. A full analysis will include a discussion of direction fields, sensitivity to initial conditions, asymptotic behavior, and the analytic solutions.

Problem 7. Determine whether the following converge or diverge. Justify your answers completely.

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

(b) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{3n}{n^2+1}$

(g) $\displaystyle\int_1^\infty \frac{x}{e^x}\,dx$

Problem 8. Use Euler's method with a step size of $\frac{1}{3}$ to approximate $y\left(\frac{2}{3}\right)$ if y satisfies the differential equation

$$y' = y\left(2 - \frac{1}{2}y^2\right), \quad y(0) = 1.$$

Problem 9. True or False?

(a) $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$.

(b) For any constant c, $p = \dfrac{1}{1+(c-1)\,e^{-t}}$ is a solution to $p' = p(1-p)$.

(c) $\dfrac{1}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - + \cdots$

(d) If $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$ then $\displaystyle\sum_{n=1}^\infty a_n$ diverges but $\displaystyle\sum_{n=1}^\infty \frac{a_n}{3^n}$ converges.

(e) If $\displaystyle\sum_{k=1}^\infty |a_k|$ diverges then $\displaystyle\sum_{k=1}^\infty a_k$ diverges also.

(f) Suppose $a_n > 0$ for all n and $\displaystyle\lim_{n\to\infty} n a_n \to 3$. Then the series $\sum_{n=1}^\infty a_n$ converges.

(g) $\displaystyle\int f(x)g(x)dx = \left(\int f(x)dx\right)\left(\int g(x)dx\right)$.

Problem 10. Sometimes it is possible to find the sum of a convergent series precisely by comparing it to a familiar power series specialized to a particular value of x. Find the sum:

(a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$

(b) $\frac{\pi}{2} - \frac{\pi^3}{3! \cdot 2^3} + \frac{\pi^5}{5! \cdot 2^5} - + \cdots$

(c) $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

(d) $1 + 6\left(-\frac{1}{2}\right) + 15\left(-\frac{1}{2}\right)^2 + 20\left(-\frac{1}{2}\right)^3 + 15\left(-\frac{1}{2}\right)^4 + 6\left(-\frac{1}{2}\right)^5 + \left(-\frac{1}{2}\right)^6$

Problem 11. Use power series to approximate

$$\int_0^1 x^2 \cos\left(x^{\frac{3}{2}}\right)\, dx$$

with an error less than $\dfrac{1}{(12)(720)}$.

Problem 12. Let $f(x) = \dfrac{x^2}{e^{2x}}$. Use power series to find $f^{(5)}(0)$, the fifth derivative of f at $x = 0$.

EXAM

Practice Final

Math 132

May 10, 2004