

Math 589
Problem Set 3

due Wednesday, March 4, 2026

These problems are from Gathmann, *Algebraic Geometry*; from Shafarevich, *Basic Algebraic Geometry I*; and from Artin, *Notes for a Course in Algebraic Geometry*.

1. Associate with any affine variety $X \subseteq \mathbb{A}_n^0$ its closure in \mathbb{P}^n . Prove that this defines a one-to-one correspondence between affine subvarieties of \mathbb{A}_n^0 and projective subvarieties of \mathbb{P}^n with no components contained in the hyperplane $x_0 = 0$.
2. (a) Show that a graded ring R is a domain if and only if $fg = 0$ implies $f = 0$ or $g = 0$ for all homogeneous elements $f, g \in R$.
(b) Show that a projective variety X is irreducible if and only if the graded ring $S(X) = k[x_0, \dots, x_n]/I(X)$ is a domain.
3. Let X be the affine surface in \mathbb{A}^3 defined by the equation $x_1^3 + x_1x_2x_3 + x_1x_3 + x_2^2 + x_3 = 0$, and let \bar{X} be its closure in \mathbb{P}^3 . Describe the intersection of \bar{X} with the plan at infinity in \mathbb{P}^3 .
4. Show by example that not every hypersurface Y in a projective variety X is of the form $V(f)$ for a homogeneous polynomial $f \in S(X)$. (*Hint*: Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into \mathbb{P}^3 .)
5. Find equations for the image of the Veronese embedding

$$F: \mathbb{P}^1 \rightarrow \mathbb{P}^n, \quad [x_0, x_1] \mapsto [x_0^n, x_0^{n-1}x_1, \dots, x_1^n].$$

6. Let $G(2, 4) \subseteq \mathbb{P}^5$ be the Grassmannian whose points correspond to 2-dimensional linear subspaces of k^4 , or equivalently, to lines in \mathbb{P}^3 . Let $L \subseteq \mathbb{P}^3$ be any line. Show that the set of lines that intersect L , viewed as a subset of $G(2, 4)$, is the zero locus of a homogeneous linear polynomial.
7. A conic is a curve of degree 2 in \mathbb{P}^2 . The set of all conics is in bijection with \mathbb{P}^5 , by associating to the point $[a_0, a_1, \dots, a_5] \in \mathbb{P}^5$ the conic defined by the equation $a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 = 0$. Show that the set of reducible conics (= products of two linear forms) is a hypersurface in \mathbb{P}^5 . What is its degree?