MATH 545—HOMEWORK 9

1. Riemann surfaces. Let M be a compact Riemann surface.

(a) Define the degree of a holomorphic line bundle L as
$$\deg L = \int_M c_1(L) \in \mathbb{Z}.$$
Prove that L is positive if and only if $\deg L > 0$.

(b) Let L be an arbitrary positive line bundle. By imitating the proof of Kodaira's embedding theorem, show directly that the line bundle $\omega_M \otimes L^3$ gives an embedding of M into projective space.

(c) Would $\omega_M \otimes L^2$ also work?

2. Meromorphic functions. Recall that a meromorphic function on a complex manifold M is a holomorphic function f on a dense open subset of M, with the following property: for every point $p \in M$, there exists an open neighborhood U and a holomorphic function $0 \neq h \in \mathscr{O}_M(U)$ such that $hf \in \mathscr{O}_M(U)$. Show that on $\mathbb{P}^n$, every meromorphic function is the quotient of two homogeneous polynomials of equal degree.

3. Positive line bundles. Let M be a compact complex manifold of dimension n, and let P be a positive line bundle on M.

(a) Let L be any other holomorphic line bundle. Show that there exists an integer $k \gg 0$ such that $H^j(M, L \otimes P^k) = 0$ for every $j > 0$.

(b) Show that there also exists an integer $k \gg 0$ such that $L \otimes P^k$ has a nontrivial holomorphic section on M.

(c) Prove that one can find complex submanifolds Z_1, Z_2 of dimension $n-1$ such that $L \simeq \mathscr{O}_M(Z_1) \otimes \mathscr{O}_M(-Z_2)$.

4. Picard variety. Let M be a compact Kähler manifold of dimension $n \geq 1$.

(a) Prove that the composition $H^1(M, \mathbb{R}) \to H^1(M, \mathbb{C}) \to H^1(M, \mathscr{O})$ is an isomorphism of real vector spaces.

(b) Show that the quotient $H^1(M, \mathscr{O})/H^1(M, \mathbb{Z})$ is a complex torus.

(c) Show that if M is a projective manifold, then $H^1(M, \mathscr{O})/H^1(M, \mathbb{Z})$ is also projective, and therefore an abelian variety.

Due on Thursday, December 5.