MATH 554—HOMEWORK 7

1. Harmonic forms. Let M be a Kähler manifold with Kähler form ω, and denote by L and Λ the Lefschetz operator and its adjoint. Prove that a form $\alpha \in A^k(M)$ is harmonic iff in the Lefschetz decomposition

$$\alpha = \sum_{j \geq \max(0,k)} \frac{L^j}{j!} \alpha_j,$$

all the forms $\alpha_j \in A^{k-2j}(M)$ satisfy $\partial \alpha_j = 0$ and $\bar{\partial} \alpha_j = 0$.

2. Betti numbers of Kähler manifolds. We have seen that there are certain restrictions on the Betti numbers $b^k = \dim H^k(M, \mathbb{C})$ of a compact Kähler manifold. Here is another result of this type: Suppose that $\dim M \geq 4m+2$ and $b^{2m+1} \geq 1$. Prove that we must have $b^{4m+2} \geq 2$. (*Hint:* Show that there exist harmonic forms $\alpha, \beta \in A^{4m+2}(M)$ with $\alpha \wedge \alpha = 0$ and $\beta \wedge \beta \neq 0$.)

3. Hodge index theorem. Let M be a compact Kähler surface with Kähler form ω, and let $H^{1,1} = H_0^{1,1} \oplus \mathbb{C}\omega$, where $H_0^{1,1}$ is the subspace of primitive classes. Recall that the bilinear form $\alpha \cdot \beta = \int_M \alpha \wedge \beta$ is positive definite on the subspace $\mathbb{C}\omega$, and negative definite on the subspace $H_0^{1,1}$.

(a) Prove that $H_0^{1,1} = \{ \alpha \in H^{1,1} \mid \omega \cdot \alpha = 0 \}$.

(b) Prove that for any $\alpha \in H^{1,1}$, we have $(\omega \cdot \omega)(\alpha \cdot \alpha) \leq (\omega \cdot \alpha)^2$, with equality iff α is a multiple of ω.

4. The Kähler condition. Let (M, h) be a complex manifold with a Hermitian metric h. The Riemannian metric $g = \operatorname{Re} h$ determines the Levi-Cività connection ∇^g on the real tangent bundle $T_{\mathbb{R}}M$; likewise, h is a Hermitian metric on the holomorphic tangent bundle $T'M$, and we denote by ∇^h the associated Chern connection. Lastly, recall that the map $T_{\mathbb{R}}M \to T'M$ is an isomorphism of real vector bundles. Show that the following two statements are equivalent:

(1) The metric h is a Kähler metric.

(2) We have $\nabla^g = \nabla^h$ under the isomorphism $T_{\mathbb{R}}M \simeq T'M$.

(*Hint:* Prove that (1) implies (2) by reducing to the case of $\mathbb{C}^n$; prove that (2) implies that $\nabla^g \circ J = J \circ \nabla^g$, and deduce (1) by appealing to a result from class.)

5. Surfaces in $\mathbb{P}^3$. Let $F \in \mathbb{C}[z_0, z_1, z_2, z_3]$ be a homogeneous polynomial of degree d whose zero set is a complex submanifold $X \subseteq \mathbb{P}^3$.

(a) Compute the Hodge numbers $h^{p,q}$ of the surface X.

(b) Conclude that $h^{p,q}$ is independent of F, as asserted in class.

6. Representation theory. Prove the identity $wHw^{-1} = -H$ without looking at 2×2-matrices. (*Hint:* Use $w = e^E e^{-F} e^E$ and the fact that $e^A B e^{-A} = e^{\operatorname{ad} A}(B)$, where $(\operatorname{ad} A)(B) = [A, B]$.)

Due on Tuesday, November 12.