MATH 554—HOMEWORK 4

1. Fubini-Study metric. Let ω_{FS} be the associated $(1,1)$-form for the Fubini-Study metric on $\mathbb{P}^1$. Compute the integral $\int_{\mathbb{P}^1} \omega_{FS}$ and show that it equals 1.

2. Cohomology of projective space.

(a) Compute the singular cohomology of projective space. (Hint: Let Z_n be the set of $[z] \in \mathbb{P}^n$ such that $z_n = 0$. Use the Poincaré duality isomorphism $H^k(\mathbb{P}^n, Z_n; \mathbb{Z}) \simeq H_{2n-k}(U_n; \mathbb{Z})$ and the long exact cohomology sequence.)

(b) Let $L \simeq \mathbb{P}^1$ be defined by $z_2 = \cdots = z_n = 0$. Show that the restriction map $H^2(\mathbb{P}^n; \mathbb{Z}) \to H^2(L; \mathbb{Z})$ is an isomorphism.

(c) Conclude from the previous exercise that the cohomology class of ω_{FS} is the natural generator of $H^2(\mathbb{P}^n; \mathbb{R})$.

3. Differential forms.

(a) Prove the Leibniz rule for the exterior derivative: for a smooth k-form $\alpha \in A^k(M)$ and a smooth function $f \in A^0(M)$, we have $d(f\alpha) = df \wedge \alpha + f d\alpha$.

(b) Prove that $d \circ d = 0$, as a map from $A^k(M)$ to $A^{k+2}(M)$.

(c) Deduce that $\partial \circ \partial = 0$, $\bar{\partial} \circ \bar{\partial} = 0$, and $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$.

4. A $\partial\bar{\partial}$-lemma. Let $D \subseteq \mathbb{C}^n$ be a polydisk, and suppose that $\omega \in A^{1,1}(D)$ satisfies $d\omega = 0$ and is real-valued, meaning that $\overline{\omega} = \omega$. Prove that there is a smooth function $f \colon D \to \mathbb{R}$ with the property that $\omega = i\partial\bar{\partial} f$.

5. Hermitian metrics. Let h be a Hermitian metric on an n-dimensional complex manifold M. In local coordinates $z_1, \ldots, z_n$, write $h_{j,k} = h\big(\partial/\partial z_j, \partial/\partial z_k\big)$. Prove very carefully that the associated $(1,1)$-form ω is given by the formula

$$\omega = \frac{i}{2} \cdot \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k.$$

6. Dolbeault cohomology.

(a) Let $g \colon \mathbb{C}^* \to \mathbb{C}$ be a smooth function with compact support. Prove that there always exists a smooth function $f \colon \mathbb{C}^* \to \mathbb{C}$ with $\partial f / \partial \bar{z} = g$.

(b) Conclude that the Dolbeault cohomology groups $H^{0,1}(\mathbb{C}^*)$ and $H^{1,1}(\mathbb{C}^*)$ are both zero.

(c) Compute $H^{0,0}(\mathbb{C}^*)$ and $H^{1,0}(\mathbb{C}^*)$.

Due on Thursday, October 10.