Class 9. Riemannian and hermitian manifolds (September 26)

Riemannian manifolds. Let M be a smooth manifold of dimension n. Recall that a Riemannian metric on M is a collection of positive definite symmetric bilinear forms $g_p \colon T_{\mathbb{R},p}M \otimes T_{\mathbb{R},p}M \to \mathbb{R}$ that vary smoothly with $p \in M$. In other words, for any pair of smooth vector fields $\xi, \eta \in \Gamma(U, T_{\mathbb{R}}M)$ on an open subset $U \subseteq M$, the real-valued function $g(\xi, \eta)$ is required to be smooth on U. In local coordinates $x_1, \dots, x_n$, we define the smooth functions

$$g_{i,j}(x) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right),$$

and then the $n \times n$-matrix $G(x)$ with those entries is symmetric and positive definite at each point of U.

Example 9.1. On $\mathbb{R}^n$, we have the Euclidean metric for which $g_{i,j} = 1$ if $i = j$, and 0 otherwise. Since the n-sphere $\mathbb{S}^n$ is contained in $\mathbb{R}^{n+1}$, it inherits a Riemannian metric (by noting that $T_{\mathbb{R},p}\mathbb{S}^n \subseteq T_{\mathbb{R},p}\mathbb{R}^{n+1}$ at each point). It is a good exercise to compute the coefficients $g_{i,j}$ for $\mathbb{S}^2$, in the two coordinate charts given by stereographic projection.

On an oriented manifold, the Riemannian metric also determines a differential form in $A^n(M)$, called the volume form. Let us first consider the case of a real vector space V of dimension n. Recall that the vector space $\bigwedge^n V$ is one-dimensional, and that an orientation of V consists in choosing one of the two connected components of $\bigwedge^n V \setminus \{0\}$ and calling it the positive one. (We may then say that a basis $v_1, \dots, v_n$ is positive if $v_1 \wedge \dots \wedge v_n$ lies in that component.) Now suppose that V is endowed with an inner product $g \colon V \otimes V \to \mathbb{R}$. It induces inner products on each of the spaces $\bigwedge^k V$, with the property that

$$g\big(v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_k\big) = \det\big(g(v_j, w_k)\big)_{1 \leq j,k \leq n}.$$

This allows us to choose a distinguished generator for $\bigwedge^n V$, namely the unique positive element φ with the property that $g(\varphi, \varphi) = 1$; it is usually called the *fundamental element.* To describe it directly, let $e_1, \dots, e_n$ be a positively oriented orthonormal basis for V; then $\varphi = e_1 \wedge \dots \wedge e_n$.

Through the isomorphism $V \to V^*$ given by $v \mapsto g(v, -)$, the dual space $V^* = \operatorname{Hom}(V, \mathbb{R})$ also inherits an orientation and an inner product, and we have a fundamental element in $\bigwedge^n V^*$. In fact, it is not hard to see that the latter is given by the formula $g(\varphi, -)$.

Let M be an oriented Riemannian manifold of dimension n. Then the *volume form* is the unique smooth form $vol(g) \in A^n(M)$ whose value at any point $p \in M$ is the fundamental element in $\bigwedge^n T^*_{\mathbb{R},p}M$. If $x_1, \dots, x_n$ are local coordinates on an open subset $U \subseteq M$, such that $\partial/\partial x_1, \dots, \partial/\partial x_n$ is a positive basis at each point, then we have

$$vol(g)|_U = \sqrt{\det G(x)} \cdot dx_1 \wedge \dots \wedge dx_n. \tag{9.2}$$

We define the *volume* of M to be $\operatorname{vol}(M) = \int_M vol(g)$; note that this integral may be infinite if M is noncompact.

Example 9.3. If we let M be the sphere of radius r in $\mathbb{R}^3$, with the induced Riemannian metric, then $\operatorname{vol}(M) = 4\pi r^2$.

Hermitian linear algebra. We begin by looking at some linear algebra on a complex vector space V. To begin with, V is also a real vector space (of twice the dimension); when considering V as a real vector space, we use the symbol J to denote multiplication by i for clarity. $J \in \operatorname{End}_{\mathbb{R}}(V)$ satisfies $J \circ J = -\operatorname{id}$, and contains the information about the original complex structure on V.

A *Hermitian form* on V is a map $h \colon V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in its first argument, and such that $h(v_2, v_1) = \overline{h(v_1, v_2)}$ for all $v_1, v_2 \in V$. It follows that h is $\mathbb{C}$-antilinear in its second argument. We say that h is *positive definite* if $h(v,v) > 0$ for every nonzero $v \in V$; note that $h(v,v) \in \mathbb{R}$.

It is not hard to verify that if h is positive definite, then its real part

$$g(v_1, v_2) = \operatorname{Re} h(v_1, v_2) = \frac{1}{2}\big(h(v_1, v_2) + h(v_2, v_1)\big)$$

defines an inner product on the underlying real vector space; h is uniquely determined by g, as a brief calculation shows that $h(v_1, v_2) = g(v_1, v_2) + ig(v_1, Jv_2)$. (In fact, this formula defines a Hermitian form iff g is compatible with J, in the sense that $g(Jv_1, Jv_2) = g(v_1, v_2)$ for all $v_1, v_2 \in V$.)

Consider next the imaginary part of h, or

$$\omega(v_1, v_2) = -\operatorname{Im} h(v_1, v_2) = \frac{i}{2}\big(h(v_1, v_2) - h(v_2, v_1)\big).$$

It follows from the properties of h that ω is a real bilinear form that is alternating, meaning that $\omega(v_2, v_1) = -\omega(v_1, v_2)$. One easily sees that $\omega(v_1, Jv_2) = g(v_1, v_2)$; consequently, an alternating real-valued form ω comes from a Hermitian form iff $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ for all $v_1, v_2 \in V$; moreover, ω uniquely determines h.

Hermitian manifolds. We now generalize this to complex manifolds. Recall that if $p \in M$ is a point on a complex manifold, then the composition $T_{\mathbb{R},p}M \hookrightarrow T_{\mathbb{C},p}M \twoheadrightarrow T'_pM$ is an isomorphism of real vector spaces. We use this isomorphism to identify the underlying real vector space of T'_pM with $T_{\mathbb{R},p}M$; we continue to denote by J the endomorphism of $T_{\mathbb{R},p}M$ induced from multiplication by i.

Definition 9.4. A *Hermitian metric* h on a complex manifold M is a collection of positive definite Hermitian forms h_p on the holomorphic tangent spaces $T_{\mathbb{C},p}M$, whose real parts $g_p = \operatorname{Re} h_p$ induce a Riemannian metric on the underlying smooth manifold.

On a Hermitian manifold (M, h), we thus have a Riemannian metric $g = \operatorname{Re} h$ and a real-valued differential 2-form $\omega = -\operatorname{Im} h$.

We make the definition more concrete by writing down formulas in local holomorphic coordinates $z_1, \dots, z_n$. First off, we let H be the $n \times n$-matrix with entries the smooth functions

$$h_{j,k} = h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right);$$

at each point, H is Hermitian-symmetric and positive definite.

To find the Riemannian metric, let $z_j = x_j + iy_j$, and recall that, under our identification of $T_{\mathbb{R},p}M$ with T'_pM, the vector field $\partial/\partial x_j$ corresponds to $\partial/\partial z_j$, and $\partial/\partial y_j$ to $i \cdot \partial/\partial z_j$. (From this, we also see that $J\partial/\partial x_j = \partial/\partial y_j$.) Thus we have for instance that

$$g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \operatorname{Re} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \operatorname{Re} h_{j,k},$$

while

$$g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) = g\left(\frac{\partial}{\partial x_j}, J\frac{\partial}{\partial x_k}\right) = \operatorname{Re} h\left(\frac{\partial}{\partial z_j}, i\frac{\partial}{\partial z_k}\right) = \operatorname{Im} h_{j,k}.$$

In the basis $\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial y_1, \dots, \partial/\partial y_n$, the Riemannian metric g is therefore given by the $2n \times 2n$-matrix

$$G = \begin{pmatrix} \operatorname{Re} H & \operatorname{Im} H \\ -\operatorname{Im} H & \operatorname{Re} H \end{pmatrix},$$

Note that G is a symmetric matrix, as expected: H being Hermitian symmetric, it follows that $\operatorname{Re} H$ is symmetric, while $-\operatorname{Im} H = H^T$.

Finally, consider the 2-form ω; we compute that

$$\omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = -\operatorname{Im} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = -\operatorname{Im} h_{j,k},$$

while

$$\omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) = -\operatorname{Im} h\left(\frac{\partial}{\partial z_j}, i\frac{\partial}{\partial z_k}\right) = \operatorname{Re} h_{j,k}.$$

To make sense of these formulas, let us view ω as a complex-valued 2-form by extending it bilinearly to the complexified tangent spaces $T_{\mathbb{C},p}M$; here we have to be careful to distinguish multiplication by i and the effect of the operator J. We would now like express ω in terms of $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$. We compute that

$$\begin{aligned} 4\omega\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) &= \omega\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right) \\ &= \omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) - i\omega\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k}\right) + i\omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) + \omega\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right), \end{aligned}$$

which, by the above formulas, equals $-\operatorname{Im} h_{j,k} + i\operatorname{Re} h_{j,k} + i\operatorname{Re} h_{j,k} - \operatorname{Im} h_{j,k} = 2ih_{j,k}$. Similarly, one proves that $\omega(\partial/\partial z_j, \partial/\partial z_k) = \omega(\partial/\partial \bar{z}_j, \partial/\partial \bar{z}_k) = 0$, and so

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k. \tag{9.5}$$

It follows that ω is of type $(1,1)$; this justifies calling it the *associated* $(1,1)$*-form* of the metric h.

Example 9.6. Consider $\mathbb{C}^n$ with the metric in which the $\partial/\partial z_j$ form a unitary basis; in the notation from above, $H = \mathrm{id}_n$. Then g is the standard Euclidean metric on $\mathbb{R}^{2n}$, and

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

This is one of the reasons for defining $\omega = -\operatorname{Im} h$.

Example 9.7. Let $N \subseteq M$ be a submanifold. For every $p \in N$, we have $T'_pN \subseteq T'_pM$, and so a Hermitian metric h_M on M naturally induces one on N. If we denote the latter by h_N, then a brief computation in local coordinates shows that $\omega_N = i^*\omega_M$, where $i\colon N \to M$ is the inclusion map.

The Fubini-Study metric. We now come to an important example: on $\mathbb{P}^n$, there is a natural Hermitian metric called the Fubini-Study metric. It will be easiest to describe the metric through its associated $(1,1)$-form ω_{FS}. Recall that $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\mathbb{C}^*$, and that the quotient map

$$q \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$$

is holomorphic. Then ω_{FS} is the unique $(1,1)$-form on $\mathbb{P}^n$ whose pullback via the map q to $\mathbb{C}^{n+1} \setminus \{0\}$ is given by the formula

$$q^* \omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log\bigl(|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2\bigr). \tag{9.8}$$

One readily derives formulas in local coordinates: for example, in the chart $U_0 \subseteq \mathbb{P}^n$ with coordinates $[1, z_1, \dots, z_n]$, we have

$$\begin{aligned} \omega_{FS}|_{U_0} &= \frac{i}{2\pi} \partial\bar{\partial} \log\bigl(1 + |z|^2\bigr) \\ &= \frac{i}{2\pi} \left(\frac{1}{1+|z|^2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j - \frac{1}{(1+|z|^2)^2} \sum_{j,k=1}^n \bar{z}_j z_k dz_j \wedge d\bar{z}_k \right), \end{aligned}$$

where we have set $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. One can also show in this way that ω_{FS} really is a well-defined $(1,1)$-form on $\mathbb{P}^n$. To see this, consider the function

$$f_0(z) = \log\bigl(1 + |z|^2\bigr) = \log\bigl(1 + |z_1|^2 + \cdots + |z_n|^2\bigr)$$

on the open subset $U_0 \cong \mathbb{C}^n$. The coordinate transformation to one of the other open subsets $U_k \cong \mathbb{C}^n$ is given by the rule

$$(z_1, \dots, z_n) \mapsto \left(\frac{1}{z_k}, \frac{z_1}{z_k}, \dots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \dots, \frac{z_n}{z_k} \right).$$

This transforms the function above into

$$\log\bigl(1 + |z_1|^2 + \cdots + |z_n|^2\bigr) - \log|z_k|^2 = f_k(z) - \log|z_k|^2,$$

and because $\partial\bar{\partial}$ kills the second summand, the individual forms $\partial\bar{\partial} f_k(z)$ glue together into an element of $A^{1,1}(\mathbb{P}^n)$. One can read off the coefficients of the associated metric using (9.5), and this shows that we have really defined a metric on $\mathbb{P}^n$.

We note two useful properties of the Fubini-Study metric. The first is its invariance under unitary automorphisms of $\mathbb{P}^n$. Suppose that $A \in U(n+1)$ is a unitary matrix; it defines an automorphism f_A of $\mathbb{P}^n$ by the formula $[z] \mapsto [Az]$. Since $|Az| = |z|$ for every $z \in \mathbb{C}^{n+1}$, we clearly have $f_A^* \omega_{FS} = \omega_{FS}$. The unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$, and so the Fubini-Study metric is homogeneous under this action. The second is that ω_{FS} is both d-closed and $\bar{\partial}$-closed, and therefore defines cohomology classes in the de Rham cohomology group $H^2(\mathbb{P}^n, \mathbb{R})$ and in the Dolbeault cohomology group $H^{1,1}(\mathbb{P}^n)$. Both cohomology groups are one-dimensional, and the class of ω_{FS} is the natural generator. The reason for the normalizing factor $1/2\pi$ in the definition of the Fubini-Study metric can be found in one of the exercises: on $\mathbb{P}^1$, we have $\int_{\mathbb{P}^1} \omega_{FS} = 1$.