Class 7. Submanifolds and analytic sets (September 19)

The submanifold theorem. Examples of submanifolds are given by level sets of holomorphic mappings whose differential has constant rank. A similar result, sometimes called the submanifold theorem, should be familiar from the theory of smooth manifolds.

Let $f\colon M \to N$ be a holomorphic mapping between two complex manifolds M and N; recall that this means that f is continuous, and $g \circ f \in \mathscr{O}_M(f^{-1}(U))$ for every holomorphic function $g \in \mathscr{O}_N(U)$ and every open subset $U \subseteq N$.

Fix a point $p \in M$, and let $z_1, \dots, z_n$ be holomorphic coordinates centered at p; also let $w_1, \dots, w_m$ be holomorphic coordinates centered at $q = f(p)$. We can express f in those coordinates as $w_k = f_k(z_1, \dots, z_n)$, with $f_1, \dots, f_m$ holomorphic functions in a neighborhood of $0 \in \mathbb{C}^n$. In particular, each f_j is then a smooth function, and so f is also a smooth mapping. It therefore induces a linear map

$$f_*\colon T_{\mathbb{R},p}M \to T_{\mathbb{R},q}N$$

between real tangent spaces, and therefore also a map between the complexified tangent spaces. By the complex version of the chain rule, we have

$$f_* \frac{\partial}{\partial z_j} = \sum_{k=1}^m \left(\frac{\partial f_k}{\partial z_j} \frac{\partial}{\partial w_k} + i \cdot \frac{\partial \bar{f}_k}{\partial z_j} \frac{\partial}{\partial \bar{w}_k} \right) = \sum_{k=1}^m \frac{\partial f_k}{\partial z_j} \frac{\partial}{\partial w_k} \tag{7.1}$$

because each f_j is holomorphic; therefore f_* maps $T'_p M$ into $T'_q N$. (In fact, one can show that a smooth map $f\colon M \to N$ is holomorphic iff f_* preserves holomorphic tangent spaces.)

We digress to explain the relationship between $J_{\mathbb{R}}(f)$ and $J(f)$. In the complexified tangent space $T_{\mathbb{C},p}M$, we may use the basis given by $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$, and for $T_{\mathbb{C},q}N$ the basis given by $\partial/\partial w_k$ and $\partial/\partial \bar{w}_k$. According to (7.1), the map f_* is then represented by the $2m \times 2n$-matrix

$$J_{\mathbb{C}}(f) = \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix},$$

where $J(f) = \partial f/\partial z$ is the $m \times n$-matrix with entries $\partial f_k/\partial z_j$. This simple calculation shows that if M and N have the same dimension (i.e., $m = n$), then

$$\det J_{\mathbb{R}}(f) = |\det J(f)|^2. \tag{7.2}$$

This relationship makes it possible to deduce the holomorphic submanifold theorem from its usual version on smooth manifolds (which is a fairly difficult result).

Since we already have the implicit mapping theorem (in Theorem 3.7, whose proof used the Weierstraß theorems), we can give a direct proof.

Theorem 7.3. *Let $f\colon M \to N$ be a holomorphic mapping between complex manifolds, and suppose that the differential $f_*\colon T_p M \to T_{f(p)} N$ has constant rank r at every point $p \in M$. Then for every $q \in N$, the level set $f^{-1}(q)$ is either empty, or a complex submanifold of M. Moreover, if $f(p) = q$, then we have*

$$\dim_p f^{-1}(q) = \dim_p M - r.$$

Proof. We shall suppose that we have a point $p \in M$ with $f(p) = q$. By choosing local coordinates centered at p and q respectively, we reduce to the case where $D \subseteq \mathbb{C}^n$ is an open neighborhood of the origin, and $f\colon D \to \mathbb{C}^m$ is a holomorphic mapping with $f(0) = 0$ and $J(f)$ has rank r throughout D. Moreover, after making linear changes of coordinates and shrinking D, we may clearly assume that the submatrix $\partial(f_1,\dots,f_r)/\partial(z_1,\dots,z_r)$ is nonsingular. The theorem will be proved if we show that, in some neighborhood of the origin, $f^{-1}(0)$ is a complex submanifold of D of dimension $n-r$.

Introduce a holomorphic mapping $g\colon D\to\mathbb{C}^n$ by setting

$$g_j(z)=\begin{cases} f_j(z) & \text{for } j=1,\dots,r,\\ z_j & \text{for } j=r+1,\dots,n.\end{cases}$$

Clearly $J(g)$ is nonsingular for $z=0$, and so by the inverse mapping theorem (from the exercises), g is a biholomorphism between suitable open neighborhoods of $0\in\mathbb{C}^n$; this means that g can be used to define a new coordinate system. After making that change of coordinates (which amounts to replacing f by $f\circ g^{-1}$), we can therefore assume that f has the form

$$f(z)=\bigl(z_1,\dots,z_r,f_{r+1}(z),\dots,f_m(z)\bigr).$$

The remaining functions $f_{r+1}(z),\dots,f_m(z)$ can only depend on $z_1,\dots,z_r$; indeed, since $\operatorname{rk} J(f)=r$, we necessarily have $\partial f_j/\partial z_k=0$ for all $j,k>r$. But this implies that the level set $f^{-1}(0)$ is the intersection of D with the linear subspace $z_1=\dots=z_r=0$, and therefore a complex submanifold of dimension $n-r$. □

Example 7.4. Let $f\colon D\to\mathbb{C}$ be a holomorphic function on an open subset of $\mathbb{C}^n$. Then the level sets $f^{-1}(a)$ are complex submanifolds of D, provided that at each point $z\in D$, at least one of the partial derivatives $\partial f/\partial z_j$ is nonzero. Submanifolds defined by a single holomorphic function are called *hypersurfaces*.

Analytic sets. Definition 3.2 can easily be extended to subsets of arbitrary complex manifolds: a subset $Z\subseteq M$ is said to be *analytic* if it is locally defined by the vanishing of a (finite) collection of holomorphic functions. Proposition 3.6 shows that, at least locally, analytic sets can always be decomposed into finitely many irreducible components.

Sometimes, an analytic subset $Z\subseteq M$ is actually a complex submanifold: if $z_1,\dots,z_n$ are local coordinates centered at a point $p\in Z$, and Z can be defined in a neighborhood U of the point by holomorphic functions $f_1,\dots,f_m$ with the property that $\partial(f_1,\dots,f_m)/\partial(z_1,\dots,z_n)$ has constant rank r, then $Z\cap U$ is a complex submanifold of U of dimension $n-r$. This is exactly the content of Theorem 7.3.

We call a point $p\in Z$ a *smooth point* if Z is a submanifold of M in some neighborhood of p; otherwise, p is said to be *singular*. The set of all singular points of Z is denoted by Z^s, and is called the *singular locus* of Z.

Example 7.5. Let $f(z,w)=z^2+w^3\in\mathscr{O}(\mathbb{C}^2)$. The partial derivatives are $\partial f/\partial z=2z$ and $\partial f/\partial w=3w^2$, and both vanish together exactly at the point $(0,0)$. Thus $Z(f)$ is a submanifold of $\mathbb{C}^2$ at every point except the origin, and $Z^s=\{(0,0)\}$.

Every analytic set Z is a submanifold at most of its points, because of the following lemma (whose proof is contained in the exercises).

Lemma 7.6. *Let $Z\subseteq M$ be an analytic set in a complex manifold M. Then the singular locus Z^s is contained in an analytic subset strictly smaller than Z.*

Note that points where several irreducible components of Z meet are necessarily singular points. In fact, a much stronger statement is true: Z^s is itself an analytic subset of Z. But the proof of this fact requires more theory, and will have to wait until later in the semester.

Differential forms. We now turn to calculus on complex manifolds; just as for smooth manifolds, differential forms are a highly useful tool for this purpose. We briefly recall the definition. Let M be a smooth manifold, with real tangent bundle $T_{\mathbb{R}}M$. A differential k-form ω is a section of the smooth vector bundle $\bigwedge^k T_{\mathbb{R}}^* M$; in other words, it associates to k smooth vector fields $\xi_1, \dots, \xi_k$ a smooth function $\omega(\xi_1, \dots, \xi_k)$, and is multilinear and alternating in its arguments. We denote the space of all differential k-forms on M by $A^k(M)$.

Let $U \subseteq \mathbb{R}^n$ be an open subset, with coordinates $x_1, \dots, x_n$. We then have the basic 1-forms $dx_1, \dots, dx_n$, defined by

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Any 1-form can then be written as $\varphi_1 dx_1 + \dots + \varphi_n dx_n$, for smooth functions $\varphi_j \in A(U)$. Similarly, every $\omega \in A^k(U)$ can be expressed as

$$\omega = \sum_{i_1 < \dots < i_k} \varphi_{i_1, \dots, i_k}(x_1, \dots, x_n) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_k}; \tag{7.7}$$

the coefficient $\varphi_{i_1,\dots,i_k}$ is the smooth function $\omega\big(\partial/\partial x_{i_1}, \dots, \partial/\partial x_{i_k}\big)$. We often use multi-index notation, and write (7.7) more compactly as $\omega = \sum_{|I|=k} \varphi_I dx_I$.

Returning to the case of a smooth manifold M, any $\omega \in A^k(M)$ is locally given by an expression as in (7.7), where $x_1, \dots, x_n$ are now local coordinates. With some patience and the chain rule, one can work out how to transform such expressions from one coordinate system to another, and thereby determine the transition functions for the vector bundle $\bigwedge^k T_{\mathbb{R}}^* M$. For example, for n-forms, they are

$$\det J_{\mathbb{R}}(h_{\alpha,\beta}) \circ f_\beta \tag{7.8}$$

where $f_\alpha \colon U_\alpha \to D_\alpha \subseteq \mathbb{R}^n$ are local charts, and $h_{\alpha,\beta} = f_\alpha \circ f_\beta^{-1}$ as usual.

Given a k-form ω, we can define its *exterior derivative* $d\omega$; it is a $(k+1)$-form, which is given in local coordinates by the rule

$$d\omega = \sum_{j=1}^n \sum_I \frac{\partial \varphi_I}{\partial x_j} \cdot dx_j \wedge dx_I.$$

For instance, if f is a smooth function, then $df = \sum \partial f/\partial x_i \cdot dx_i$. Exterior differentiation gives a map $d \colon A^k(M) \to A^{k+1}(M)$, which satisfies the Leibniz rule $d(f\omega) = df \wedge \omega + f d\omega$. One can easily check that $d \circ d = 0$; this means that

$$0 \longrightarrow A^0(M) \xrightarrow{d} A^1(M) \xrightarrow{d} A^2(M) \longrightarrow \cdots \longrightarrow A^{n-1}(M) \xrightarrow{d} A^n(M) \longrightarrow 0$$

is a complex of vector spaces. According to the de Rham theorem, this complex computes the singular cohomology of the manifold M: for every $k = 0, \dots, n$,

$$H^k(M, \mathbb{R}) \cong \frac{\ker d \colon A^k(M) \to A^{k+1}(M)}{\operatorname{coker} d \colon A^{k-1}(M) \to A^k(M)}.$$

In other words, the space of closed differential forms ($d\omega = 0$) modulo the space of exact differential forms ($\omega = d\psi$) is exactly the singular cohomology of M.

Finally, recall that differential forms can be pulled back along smooth mappings $f \colon M \to N$. For $\omega \in A^k(M)$, the pullback $f^*\omega \in A^k(M)$ is a differential k-form on M; the operation is easily described in local coordinates. Let $x_1, \dots, x_n$ be coordinates centered at $p \in M$, and $y_1, \dots, y_m$ coordinates centered at $f(p) \in N$, and write the components of f as $y_i = f_i(x_1, \dots, x_n)$. Then we have

$$f^* dy_i = df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j,$$

and from this, we can derive a (somewhat complicated) formula for the pullback of any differential form.

Holomorphic differentials and type. Having briefly reviewed the smooth case, suppose now that M is a complex manifold, and let $z_1, \dots, z_n$ be local coordinates. If we set $z_j = x_j + iy_j$, then as noted previously, the smooth functions $x_1, \dots, x_n, y_1, \dots, y_n$ give a local coordinate system for M as a smooth manifold, and we can consequently talk about differential forms on M, and define the spaces $A^k(M)$ as above. But observe that we have

$$dz_j = dx_j + i dy_j \quad \text{and} \quad d\bar{z}_j = dx_j - i dy_j;$$

we can therefore use $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$ instead of $dx_1, \dots, dx_n, dy_1, \dots, dy_n$, and write any k-form locally as

$$\omega = \sum_{|I|+|J|=k} \varphi_{I,J} \cdot dz_I \wedge d\bar{z}_J,$$

where each $\varphi_{I,J}$ is a again a smooth function on U.

Definition 7.9. We say that $\omega \in A^k(M)$ is *of type* (p,q) if it can locally be written in the form

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \varphi_{I,J} \cdot dz_I \wedge d\bar{z}_J.$$

The space of all such (p,q)-forms is denoted by $A^{p,q}(M)$.

Using the chain rule, it is easy to check that this definition is independent of the choice of local coordinate system. We can also decompose the exterior derivative by type as $d = \partial + \bar{\partial}$, where $\partial \colon A^{p,q}(M) \to A^{p+1,q}(M)$ and $\bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)$; in local coordinates, we have

$$\bar{\partial} \left(\sum \varphi_{I,J} dz_I \wedge d\bar{z}_J \right) = \sum \frac{\partial \varphi_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J.$$

As before, we get a complex since $\bar{\partial} \circ \bar{\partial} = 0$; we define the *Dolbeault cohomology* of the complex manifold M as

$$H^{p,q}(M) = \frac{\ker \bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)}{\operatorname{coker} \bar{\partial} \colon A^{p,q-1}(M) \to A^{p,q}(M)}. \tag{7.10}$$