Class 5. More examples of complex manifolds (September 12)

Blowing up a point. Let M be a complex manifold and $p \in M$ a point with $\dim_p M = n$. The blow-up construction produces another complex manifold $\operatorname{Bl}_p M$, in which the point p is replaced by a copy of $\mathbb{P}^{n-1}$ that parametrizes all possible directions from p into M.

We first consider the case of the origin in $\mathbb{C}^n$. Each point $z \in \mathbb{C}^n$ determines a unique line through the origin, and hence a point in $\mathbb{P}^{n-1}$, except when $z = 0$. Thus if we define

$$\operatorname{Bl}_0 \mathbb{C}^n = \left\{ (z, L) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \;\middle|\; z \text{ lies on the line } L \subseteq \mathbb{C}^n \right\},$$

then the projection map $\pi\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ is bijective for $z \neq 0$, but contains an extra copy of $\mathbb{P}^{n-1}$ over the point $z = 0$. We call $\operatorname{Bl}_0 \mathbb{C}^n$ the *blow-up* of $\mathbb{C}^n$ at the origin, and $\pi^{-1}(0)$ the *exceptional set*.

Lemma 5.1. *$\operatorname{Bl}_0 \mathbb{C}^n$ is a complex manifold of dimension n, and the projection map $\pi\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic. Moreover, the exceptional set is a submanifold of dimension $n-1$.*

Proof. On $\mathbb{P}^{n-1}$, we use homogeneous coordinates $[a_1, \dots, a_n]$; then the condition defining the blow-up is that the vectors $(z_1, \dots, z_n)$ and $(a_1, \dots, a_n)$ should be linearly dependent. This translates into the equations $z_i a_j = a_i z_j$ for $1 \leq i, j \leq n$.

Let $q\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{P}^{n-1}$ be the other projection. On $\mathbb{P}^{n-1}$, we have natural coordinate charts U_i defined by the condition $a_i \neq 0$, and

$$U_i = \left\{ [a] \in \mathbb{P}^{n-1} \;\middle|\; a_i \neq 0 \right\} \simeq \mathbb{C}^{n-1}, \quad [a] \mapsto \left(\frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

Consequently, the blow-up is covered by the n open sets $V_i = q^{-1}(U_i)$, and from the equations relating the two vectors z and a, we find that

$$V_i = \left\{ \big(z, [a]\big) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \;\middle|\; a_i \neq 0 \text{ and } z_j = z_i a_j / a_i \text{ for } j \neq i \right\} \simeq \mathbb{C}^n.$$

Explicitly, the isomorphism is given by the formula

$$f_i\colon V_i \to \mathbb{C}^n, \quad \big(z, [a]\big) \mapsto \left(\frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, z_i, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right),$$

and so the inverse mapping takes $b \in \mathbb{C}^n$ to the point with coordinates $\big(z, [a]\big)$, where $a = (b_1, \dots, b_{i-1}, 1, b_{i+1}, \dots, b_n)$, and $z = b_i a$. In this way, we obtain n coordinate charts whose union covers the blow-up.

It is a simple matter to compute the transition functions. For $i \neq j$, the composition $g_{i,j} = f_i \circ f_j^{-1}$ takes the form $g_{i,j}(b_1, \dots, b_n) = (c_1, \dots, c_n)$, where

$$c_k = \begin{cases} b_k / b_i & \text{if } k \neq i, j, \\ b_i b_j & \text{if } k = i, \\ 1/b_i & \text{if } k = j. \end{cases}$$

We observe that $f_j(V_i \cap V_j)$ is the set of points $b \in \mathbb{C}^n$ with $b_i \neq 0$, which means that each $g_{i,j}$ is a holomorphic mapping. Consequently, the n coordinate charts determine a holomorphic atlas, and we can conclude from Proposition 4.6 that $\operatorname{Bl}_0 \mathbb{C}^n$ is an n-dimensional complex manifold.

To prove that the mapping π is holomorphic, note that $\pi \circ f_i^{-1}$ is given in coordinates by the formula

$$\pi\big(f_i^{-1}(b_1, \dots, b_n)\big) = (b_i b_1, \dots, b_i b_{i-1}, b_i, b_i b_{i+1}, \dots, b_i b_n)$$

which is clearly holomorphic on $\mathbb{C}^n$. We see from this description that the intersection $\pi^{-1}(0) \cap U_i$ is mapped, under f_i, to the hyperplane $b_i = 0$. This means that $\pi^{-1}(0)$ is a complex submanifold of dimension $n-1$ (the precise definition of a submanifold will be given later). □

We can also do this construction locally. Suppose that $D \subseteq \mathbb{C}^n$ is an open set containing the origin. We then simply define $\operatorname{Bl}_0 D$ as $\pi^{-1}(D)$, where $\pi \colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ is as above. The restriction of π gives us a holomorphic mapping

$$\pi \colon \operatorname{Bl}_0 D \to D,$$

and we call this the *blowup* of D at the point 0.

We are now in a position to construct the blow-up $\operatorname{Bl}_p M$ of a point on an arbitrary complex manifold. Choose a coordinate chart $f \colon U \to D$ centered at the point p, and let $\tilde{D} = \operatorname{Bl}_0 D$ be the blow-up of D at the origin. Also let $M^* = M - \{p\}$, and $U^* = U \cap M^*$; then U^* is isomorphic to the complement of the exceptional set in $\tilde{D}$, and we can glue M^* and $\tilde{D}$ together along this common open subset. More precisely, define $\operatorname{Bl}_p M$ as the quotient of the disjoint union $M^* \sqcup \tilde{D}$ by the equivalence relation that identifies $q \in M^*$ and $x \in \tilde{D}$ whenever $q \in U^*$ and $f(q) = \pi(x)$. Since f is biholomorphic, and π is biholomorphic outside the origin, it is easy to see that transition functions between coordinate charts on M^* and on $\tilde{D}$ are biholomorphic. Thus $\operatorname{Bl}_p M$ is a complex manifold, and the projection map $\operatorname{Bl}_p M \to M$ is holomorphic.

It remains to show that the construction is independent of the choice of coordinate chart. In order to deal with this technical point, we first prove the following property of $\operatorname{Bl}_0 \mathbb{C}^n$. (The same result is then of course true for the blow-up of a point on any complex manifold.)

Lemma 5.2. *Let $f \colon M \to \mathbb{C}^n$ be a holomorphic mapping from a connected complex manifold. Suppose that $f(M) \neq \{0\}$, and that at every point $p \in M$ with $f(p) = 0$, the ideal generated by $f_1, \dots, f_n$ in the local ring $\mathscr{O}_{M,p}$ is principal. Then there is a unique holomorphic mapping $\tilde{f} \colon M \to \operatorname{Bl}_0 \mathbb{C}^n$ such that $f = \pi \circ \tilde{f}$.*

Proof. Since π is an isomorphism over $\mathbb{C}^n - \{0\}$, the uniqueness of $\tilde{f}$ follows easily from the identity theorem. Because of the uniqueness statement, the existence of $\tilde{f}$ becomes a local problem; we may therefore assume that we are dealing with a holomorphic map $f \colon D \to \mathbb{C}^n$, where D is an open neighborhood of $0 \in \mathbb{C}^m$, and $f(0) = 0$. By assumption, the ideal $(f_1, \dots, f_n) \subseteq \mathscr{O}_m$ is generated by a single element $g \in \mathscr{O}_m$; after possibly shrinking D, we may furthermore assume that $g = a_1 f_1 + \dots + a_n f_n$ and $f_j = b_j g$ for suitable holomorphic functions $a_j, b_j \in \mathscr{O}(D)$. We then have $a_1 b_1 + \dots + a_n b_n = 1$, and so at each point of D, at least one of the functions $b_1, \dots, b_n$ is nonzero. Since, in addition, $[f_1(z), \dots, f_n(z)] = [b_1(z), \dots, b_n(z)] \in \mathbb{P}^{n-1}$, we can now define

$$\tilde{f} \colon D \to \operatorname{Bl}_0 \mathbb{C}^n, \quad \tilde{f}(z) = \big(f_1(z), \dots, f_n(z), [b_1(z), \dots, b_n(z)]\big),$$

which clearly has the required properties. □

Now suppose we have a second coordinate chart centered at $p \in M$; without loss of generality, we may assume that it is of the form $\phi \circ f$, where $\phi \colon D \to E$ is biholomorphic and satisfies $\phi(0) = 0$. To prove that $\operatorname{Bl}_p M$ is independent of the choice of chart, we have to show that ϕ induces an isomorphism $\tilde{\phi} \colon \operatorname{Bl}_0 D \to \operatorname{Bl}_0 E$. By Lemma 5.2, it suffices to show that m coordinate functions of $\phi \circ \pi \colon \operatorname{Bl}_0 D \to E$

generate a principal ideal in the local ring at each point of $\mathrm{Bl}_0 D$. We may consider this question in one of the coordinate charts $f_i \colon V_i \to \mathbb{C}^n$ introduced during the proof of Lemma 5.1. Thus let $\psi = \phi \circ \pi \circ f_i^{-1} \colon \mathbb{C}^n \to E$; we then have

$$\psi(w) = \phi(w_i w_1, \dots, w_i w_{i-1}, w_i, w_i w_{i+1}, \dots, w_i w_n) \tag{5.3}$$

for any $w \in \mathbb{C}^n$.

Now we fix a point $b \in \mathbb{C}^n$, and let $I \subseteq \mathscr{O}_{\mathbb{C}^n,b}$ be the ideal generated by the functions $\psi_1(w), \dots, \psi_n(w)$ in the local ring at b. If $b_i \neq 0$, then since ϕ is bijective, at least one of the values $\psi_j(b)$ has to be nonzero, and so I is the unit ideal. We may therefore assume that $b_i = 0$; we shall argue that $I = (w_i)$. Because $\phi(0) = 0$, we can clearly write

$$\big(\phi_1(z), \dots, \phi_n(z)\big) = (z_1, \dots, z_n) \cdot A(z)$$

for a certain $n \times n$-matrix of holomorphic functions; upon substituting (5.3), we find that every function $\psi_j(w)$ is a multiple of w_i, and therefore $I \subseteq (w_i)$. On the other hand, $A(0) = J(\phi)|_{z=0}$ is invertible, and so $A(z)^{-1}$ is holomorphic on a suitable polydisk $\Delta(0; r) \subseteq D$. If we again substitute (5.3) into the resulting identity

$$(z_1, \dots, z_n) = \big(\phi_1(z), \dots, \phi_n(z)\big) \cdot A(z)^{-1},$$

we see that w_i can itself be expressed as a linear combination of $\psi_1(w), \dots, \psi_n(w)$ in a neighborhood of the point b. (More precisely, we need $|w_i| < r_i$ and $|w_i||b_j| < r_j$ for $j \neq i$.) This proves that $I = (w_i)$ in $\mathscr{O}_{\mathbb{C}^n,b}$, and completes the proof.

Vector bundles. Another useful class of complex manifolds is given by holomorphic vector bundles. Since we will be using vector bundles frequently during the course, we begin by reviewing some general theory. Let $\mathbb{K}$ be one of $\mathbb{R}$ or $\mathbb{C}$. Recall that if M is a topological space, then a $\mathbb{K}$-vector bundle on M is a mapping $\pi \colon E \to M$ of topological spaces, such that all fibers $E_p = \pi^{-1}(p)$ have the structure of $\mathbb{K}$-vector spaces in a compatible way. Informally, we think of a vector bundle as a continuously varying family of vector spaces E_p; here is the precise definition.

Definition 5.4. A $\mathbb{K}$-*vector bundle* of rank k on a topological space M is a continuous mapping $\pi \colon E \to M$, such that the following two conditions are satisfied:

1. For each point $p \in M$, the *fiber* $E_p = \pi^{-1}(p)$ is a $\mathbb{K}$-vector space of dimension k.
2. For every $p \in M$, there is an open neighborhood U and a homeomorphism
$$\phi \colon \pi^{-1}(U) \to U \times \mathbb{K}^k$$
mapping E_p into $\{p\} \times \mathbb{K}^k$, such that the composition $E_p \to \{p\} \times \mathbb{K}^k \to \mathbb{K}^k$ is an isomorphism of $\mathbb{K}$-vector spaces.

The pair (U, ϕ) is called a *local trivialization* of the vector bundle; also, E is called the *total space* and M the *base space*.

Any two local trivializations (U_α, ϕ_α) and (U_β, ϕ_β) can be compared over $U_\alpha \cap U_\beta$. Because of the second condition in the definition, the composition

$$\phi_\alpha \circ \phi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times \mathbb{K}^k \to (U_\alpha \cap U_\beta) \times \mathbb{K}^k$$

is necessarily of the form $(\mathrm{id}, g_{\alpha,\beta})$ for a continuous mapping

$$g_{\alpha,\beta} \colon U_{\alpha,\beta} \to \mathrm{GL}_k(\mathbb{K}).$$

These so-called *transition functions* satisfy the following compatibility conditions:

$$\begin{aligned} g_{\alpha,\beta} \cdot g_{\beta,\gamma} \cdot g_{\gamma,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma; \\ g_{\alpha,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha. \end{aligned} \tag{5.5}$$

When M is a smooth manifold, we say that E is a *smooth vector bundle* if the transition functions $g_{\alpha,\beta}$ are smooth maps. (Note that the group $\mathrm{GL}_k(\mathbb{K})$ has a natural manifold structure, being an open subset of the space of all $k \times k$-matrices over $\mathbb{K}$.) In that case, it is easy to see that E is itself a smooth manifold: indeed, each product $U_\alpha \times \mathbb{K}^k$ is a smooth manifold, and the transition functions $(\mathrm{id}, g_{\alpha,\beta})$ between them are diffeomorphisms. Clearly, the map $\pi \colon E \to M$ and the local trivializations ϕ_α are then smooth maps.

Similarly, when M is a complex manifold, we say that a $\mathbb{C}$-vector bundle E is *holomorphic* if the transition functions $g_{\alpha,\beta}$ are holomorphic maps. (This uses the fact that $\mathrm{GL}_k(\mathbb{C})$ is naturally a complex manifold.) In that case, it follows from Proposition 4.6 that E is itself a complex manifold , and that the map $\pi \colon E \to M$ as well as the local trivializations ϕ_α become holomorphic mappings.

It is possible to describe a vector bundle entirely through its transition functions, because the following result shows that the $g_{\alpha,\beta}$ uniquely determine the bundle.

Proposition 5.6. *Let M be a topological space, covered by open subsets U_α, and let $g_{\alpha,\beta} \colon \mathrm{GL}_k(\mathbb{K})$ be a collection of continuous mappings satisfying the conditions in (5.5). Then the $g_{\alpha,\beta}$ are the transition functions for a (essentially unique) vector bundle E of rank k on M. If M is a smooth (resp., complex) manifold and the $g_{\alpha,\beta}$ are smooth (resp., holomorphic) maps, then E is a smooth (resp., holomorphic) vector bundle.*

Proof. We first define E as a topological space. On the disjoint union

$$\bigsqcup_\alpha U_\alpha \times \mathbb{K}^k,$$

there is a natural equivalence relation: two points $(p, v) \in U_\alpha \times \mathbb{K}^k$ and $(q, w) \in U_\beta \times \mathbb{K}^k$ are equivalent if $p = q$ and $v = g_{\alpha,\beta}(p) \cdot w$. This does define an equivalence relation because of the conditions in (5.5), and so we can let E be the quotient space. The obvious projection map $\pi \colon E \to M$ is then continuous, and it is easy to verify that E is a vector bundle of rank k with transition functions given by $g_{\alpha,\beta}$. The remaining assertion follows from the comments made above. □