Class 28. The embedding theorem (December 10)

The Oka principle. A basic idea in the theory of Stein manifolds is the following so-called *Oka principle*: On a Stein manifold, any problem that can be formulated in terms of cohomology has only topological obstructions. Said differently, such a problem has a holomorphic solution if and only if it has a continuous solution.

Example 28.1. Consider again the exponential sequence

$$H^1(M, \mathscr{O}_M) \to H^1(M, \mathscr{O}_M^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathscr{O}_M).$$

Since the higher cohomology groups of the sheaf $\mathscr{O}_M$ are zero, it follows that the space of line bundles on M is isomorphic to the group $H^2(M, \mathbb{Z})$. In other words, every integral second cohomology class is the first Chern class of a holomorphic line bundle; unlike the case of compact Kähler manifolds, there are no conditions of type.

A far more powerful theorem along these lines has been proved by Grauert.

Theorem 28.2. *Let M be a Stein manifold, and $E \to M$ a holomorphic vector bundle. If E is topologically trivial, then it is also holomorphically trivial.*

This is a very striking result. Suppose that $E \simeq M \times \mathbb{C}^r$ as topological vector bundles; this means that E admits r continuous sections that are linearly independent at each point $p \in M$. Grauert's theorem says that, in this case, E also has r holomorphic sections with the same property.

The embedding theorem. We have seen that every complex submanifold of $\mathbb{C}^n$ is a Stein manifold. In fact, the converse is also true—this is the content of the famous embedding theorem for Stein manifolds.

Theorem 28.3. *Let M be an n-dimensional Stein manifold. Then there exists a proper holomorphic embedding $i\colon M \hookrightarrow \mathbb{C}^{2n+1}$, and so M is biholomorphic to a complex submanifold of $\mathbb{C}^{2n+1}$.*

The proof works by constructing sufficiently many holomorphic functions on M to give a proper holomorphic embedding into $\mathbb{C}^N$ for some large integer N. As long as $N > 2n+1$, one can show that projection from a generic point outside of M still embeds the manifold into $\mathbb{C}^{N-1}$. In this way, one can reduce the dimension of the ambient space to $2n+1$.

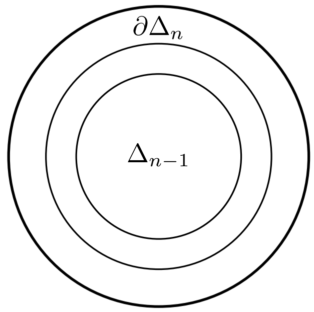
Example 28.4. $\mathbb{C}^*$ is a Stein manifold, and may be embedded into $\mathbb{C}^2$ by the (polynomial) mapping $t \mapsto (t, t^{-1})$.

Example 28.5. Every non-compact Riemann surface is a one-dimensional Stein manifold, and can therefore be embedded into $\mathbb{C}^3$. A famous unsolved problem is whether there always exists an embedding into $\mathbb{C}^2$.

Embedding the unit disk. Rather than describe the proof of the embedding theorem in general, let us focus on a specific example: the unit disk Δ in the complex plane. We already know that Δ is a Stein manifold, and the embedding theorem claims that Δ is isomorphic to a closed submanifold of $\mathbb{C}^3$. To verify this claim, we shall now construct a (more or less explicit) embedding $i\colon \Delta \hookrightarrow \mathbb{C}^3$.

The unit disk is already embedded into the complex plane $\mathbb{C}$, of course, but as an open subset, not as a *closed* complex submanifold. In order for $i(\Delta)$ to be a submanifold of $\mathbb{C}^3$ of that type, it is necessary for the embedding i to be *proper*, which is to say that $i(z)$ should go to infinity as z approaches the boundary of Δ.

For $n \geq 1$, let Δ_n denote the open disk of radius $1 - 1/(n+1)$, centered at the origin (and set $\Delta_0 = \emptyset$).



We shall define two holomorphic functions $f, g \in \mathscr{O}(\Delta)$ in such a way that $|f(z)| + |g(z)| \geq n$ for every $z \in \Delta_n \setminus \Delta_{n-1}$. We may then set

$$i\colon \Delta \to \mathbb{C}^3, \quad i(z) = \big(z, f(z), g(z)\big).$$

This mapping will be a holomorphic embedding (because of the first coordinate), and also proper (because of the second and third coordinate), and its image $i(\Delta)$ is therefore a complex submanifold of $\mathbb{C}^3$, biholomorphic to the unit disk.

We begin by constructing $f \in \mathscr{O}(\Delta)$ with the property that $|f(z)| > n+1$ on $\partial\Delta_n$. To that end, we inductively define a sequence of holomorphic functions $f_1, f_2, \dots$, such that

$$|f_n(z)| \geq \sum_{k=1}^{n-1} |f_k(z)| + n + 2 \qquad \text{for } z \in \partial\Delta_n,$$

while

$$|f_n(z)| \leq 2^{-n} \qquad \text{for } z \in \Delta_{n-1}.$$

Suppose that we already have $f_1, \dots, f_{n-1}$. We may then take f_n to be a monomial of the form

$$f_n(z) = (\alpha_n z)^{\beta_n}.$$

We first choose $(n+1)/n < \alpha_n < n/(n-1)$, to guarantee that $|\alpha_n z|$ is less than 1 on Δ_{n-1}, and greater than 1 on $\partial\Delta_n$, and then take β_n large enough to satisfy both conditions. If we now put

$$f(z) = \sum_{n=1}^{\infty} f_n(z),$$

then $f \in \mathscr{O}(\Delta)$ because the series converges uniformly on compact subsets of Δ. Moreover, for $z \in \partial\Delta_n$, we have

$$|f(z)| \geq |f_n(z)| - \sum_{k=1}^{n-1} |f_k(z)| - \sum_{k=n+1}^{\infty} |f_k(z)| \geq n + 2 - \sum_{k=n+1}^{\infty} 2^{-k} > n + 1,$$

as desired.

Of course, the absolute value of $f(z)$ is large only on the circles $\partial\Delta_n$; on the open annuli between them, there will be other points where $|f(z)|$ is small. In fact, we know from complex analysis that any holomorphic function $f\colon \Delta \to \mathbb{C}$ has open image, and is therefore never proper. To overcome this problem, let

$$E_n = \left\{ z \in \Delta_n \setminus \Delta_{n-1} \;\middle|\; \lvert f(z) \rvert \leq n \right\}.$$

We now construct a second function $g \in \mathscr{O}(\Delta)$, with the property that $\lvert g(z) \rvert \geq n$ on E_n. It will then be the case that $\lvert f(z) \rvert + \lvert g(z) \rvert \geq n$ on $\Delta_n \setminus \Delta_{n-1}$, which is what we need.

Observe that E_n is a compact subset of Δ, due to the fact that $\lvert f(z) \rvert > n+1$ on $\partial \Delta_n$. Moreover, E_n is clearly disjoint from the compact set $\overline{\Delta}_{n-1}$. Proceeding by induction, we shall again define a sequence of holomorphic functions $g_1, g_2, \dotsc$, such that

$$\lvert g_n(z) \rvert \geq \sum_{k=1}^{n-1} \lvert g_k(z) \rvert + n + 1 \qquad \text{for } z \in E_n,$$

while

$$\lvert g_n(z) \rvert \leq 2^{-n} \qquad \text{for } z \in \Delta_{n-1}.$$

Suppose that we already have $g_1, \dotsc, g_{n-1}$. Let M_n denote the supremum of $\sum_{k=1}^{n-1} \lvert g_k(z) \rvert + n + 1$ over the compact set E_n. Define a holomorphic function h_n on a small open neighborhood of $E_n \cup \overline{\Delta}_{n-1}$, by letting h_n be equal to $M_n + 2^{-n}$ near E_n, and equal to 0 near $\overline{\Delta}_{n-1}$. By the Runge approximation theorem, we may find a holomorphic function $g_n \in \mathscr{O}(\Delta)$ that approximates h_n to within 2^{-n} on the compact set $E_n \cup \overline{\Delta}_{n-1}$; this choice of g_n has the desired properties.

We may now set $g(z) = \sum_{n=1}^{\infty} g_n(z)$, which is again holomorphic, and satisfies $\lvert g(z) \rvert > n$ for every $z \in E_n$ by the same reasoning as before. It follows that $\lvert f(z) \rvert + \lvert g(z) \rvert \geq n$ on the annulus $\Delta_n \setminus \Delta_{n-1}$, and this proves that the mapping

$$\Delta \to \mathbb{C}^2, \quad z \mapsto \bigl(f(z), g(z) \bigr)$$

is indeed proper.

Note. It is possible to do better and embed the unit disk into $\mathbb{C}^2$. Alexander (in the paper *Explicit imbedding of the (punctured) disk into* $\mathbb{C}^2$) has constructed an explicit embedding of this kind. It uses the so-called elliptic modular function $\lambda\colon \mathbb{H} \to \mathbb{C} \setminus \{0,1\}$; its basic property is that the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ has Weierstraß form $y^2 = x(x-1)(x - \lambda(\tau))$. Alexander proves that the holomorphic mapping

$$\bigl(\lambda, \lambda' / \lambda(1-\lambda) \bigr) \colon \mathbb{H} \to \mathbb{C}^2$$

factors through the covering space $\mathbb{H} \to \Delta^*$, $z \mapsto e^{2\pi i z}$, and that the resulting holomorphic mapping from the punctured disk Δ^* to $\mathbb{C}^2$ extends holomorphically across the origin and gives a proper holomorphic embedding of Δ into $\mathbb{C}^2$.