## Class 27. Stein manifolds (December 5)

**Coherent sheaves and Oka's theorem.** The definition of coherent analytic sheaves that I gave last time is correct, but there is a better definition.

**Definition 27.1.** Let $\mathscr{F}$ be an analytic sheaf on a complex manifold $M$. We say that $\mathscr{F}$ is *coherent* if

(1) $\mathscr{F}$ is locally finitely generated.
(2) If $s_1, \dots, s_m \in \mathscr{F}(U)$ are sections of $\mathscr{F}$ on an open subset $U \subseteq M$, then the kernel of the morphism of sheaves

$$\mathscr{O}_U^{\oplus m} \to \mathscr{F}|_U, \quad (f_1, \dots, f_m) \mapsto f_1 s_1 + \dots + f_m s_m,$$

is again locally finitely generated.

With this definition, Oka's theorem becomes the statement that the sheaf of holomorphic functions $\mathscr{O}_M$ is coherent. (The first condition is trivial; the second condition is exactly the statement of Oka's theorem.) The definition from last time is not wrong, but it takes it for granted that $\mathscr{O}_M$ is coherent.

I also need to correct what I said about the ideal sheaf $\mathscr{I}_Z$ of a closed analytic subset being coherent. Oka's theorem shows that the second condition is satisfied, but the first condition is nontrivial. So this is a separate result, instead of just being a corollary of Oka's theorem.

*Example* 27.2. Here is a simple example of an analytic sheaf that is not coherent. On the complex plane $\mathbb{C}$, consider the analytic ideal sheaf $\mathscr{I}$ defined by the rule

$$\mathscr{I}(U) = \begin{cases} \mathscr{O}_{\mathbb{C}}(U) & \text{if } 0 \notin U, \\ 0 & \text{if } 0 \in U. \end{cases}$$

The quotient sheaf $\mathscr{F} = \mathscr{O}_{\mathbb{C}}/\mathscr{I}$ is clearly locally finitely generated (by the morphism $\mathscr{O}_{\mathbb{C}} \to \mathscr{F}$), and we have

$$\mathscr{F}(U) = \begin{cases} 0 & \text{if } 0 \notin U, \\ \mathscr{O}_{\mathbb{C}}(U) & \text{if } 0 \in U. \end{cases}$$

But the kernel of $\mathscr{O}_{\mathbb{C}} \to \mathscr{F}$, which is the ideal sheaf $\mathscr{I}$, is *not* locally finitely generated, because its stalk at the origin is 0, but the stalks at nearby points are nontrivial.

Continuing with additions to the previous lecture, I wanted to talk about the proof of Oka's theorem in a special case. Let $M$ be a complex manifold, and consider a morphism of analytic sheaves

$$\varphi \colon \mathscr{O}_M^{\oplus m} \to \mathscr{O}_M.$$

We are going to use the Weierstrass theorems from Lecture 2 to carry out one of the steps in the proof that $\ker \varphi$ is locally finitely generated. This is a local problem, and so we may assume that $M$ is an open neighborhood of the origin in $\mathbb{C}^n$. We can then work in the local ring $\mathscr{O}_n = \mathscr{O}_{\mathbb{C}^n,0}$, and our morphism of sheaves becomes a morphism of $\mathscr{O}_n$-modules

$$\varphi \colon \mathscr{O}_n^{\oplus m} \to \mathscr{O}_n.$$

It is given by $m$ germs of holomorphic functions $p_1, \dots, p_m \in \mathscr{O}_n$. Using the Weierstrass preparation theorem (in Theorem 2.8), we can make a linear coordinate change and arrange that $p_1, \dots, p_m \in \mathscr{O}_{n-1}[z_n]$ are monic polynomials. Without loss of generality, $p_1$ has the largest degree in $z_n$; call this number $d = \deg p_1$.

The kernel $\ker \varphi$ consists of all $m$-tuples $f_1, \dots, f_m \in \mathscr{O}_n$ such that

$$f_1 p_1 + \dots + f_m p_m = 0.$$

We are going to show that it is generated by relations where $f_1, \dots, f_m \in \mathscr{O}_{n-1}[z_n]$ are polynomials of degree at most $d$. Consider an arbitrary relation $f_1 p_1 + \dots + f_m p_m = 0$. Using the Weierstrass division theorem (in Theorem 2.10), we can uniquely write

$$f_j = q_j p_1 + r_j,$$

where $q_j \in \mathscr{O}_n$ and $r_j \in \mathscr{O}_{n-1}[z_n]$ has degree $\deg r_j \leq d - 1$. We do this for $j = 2, \dots, m$. Then we get

$$\bigl(f_1 + q_2 p_2 + \dots + q_m p_m\bigr) p_1 + r_2 p_2 + \dots + r_m p_m = 0.$$

It follows that $r_1 = f_1 + q_2 p_2 + \dots + q_m p_m \in \mathscr{O}_{n-1}[z_m]$ is also a polynomial, and that $\deg r_1 \leq m - 1$. Because we have

$$\begin{aligned} f_1 &= r_1 - (q_2 p_2 + \dots + q_m p_m) \\ f_2 &= r_2 + q_2 p_1 \\ &\vdots \\ f_m &= r_m + q_m p_1, \end{aligned}$$

we see that all relations are generated by polynomial relations (of degree $\leq d - 1$) together with the "tautological" relations coming from the $p_j$ themselves. This is a finite set, and more importantly, all these relations are defined for arbitrary values of $z_n$, instead of just in a small neighborhood of the origin.

**Grauert's theorem.** We close this brief overview of the theory of coherent sheaves by stating one of the most important results, namely Grauert's theorem. Let $f \colon X \to Y$ be a holomorphic mapping between complex manifolds. For any analytic sheaf $\mathscr{F}$ on $X$, one can define the so-called *direct image sheaf* $f_* \mathscr{F}$; for $U \subseteq Y$, the sections of this sheaf are given by $(f_* \mathscr{F})(U) = \mathscr{F}(f^{-1} U)$.

*Example* 27.3. We always have a morphism of sheaves of rings $\mathscr{O}_Y \to f_* \mathscr{O}_X$. Indeed, for any open set $U \subseteq Y$, composition with the holomorphic mapping $f$ defines a ring homomorphism $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1} U)$.

The morphism $\mathscr{O}_Y \to f_* \mathscr{O}_X$ can be used to give any direct image sheaf $f_* \mathscr{F}$ the structure of an analytic sheaf on $Y$. Grauert's theorem gives the condition for the direct image of a coherent sheaf to be coherent.

**Theorem 27.4.** *If $f \colon X \to Y$ is a proper holomorphic mapping, meaning that the preimage of every compact set is compact, then for every coherent analytic sheaf $\mathscr{F}$ on $X$, the direct image sheaf $f_* \mathscr{F}$ is again coherent.*

On a compact complex manifold $M$, the trivial mapping to a point is proper; a special case of Grauert's theorem is the following finiteness result.

**Corollary 27.5.** *On a compact complex manifold $M$, the space of global sections of any coherent analytic sheaf is a finite-dimensional complex vector space.*

*Proof.* Let $f\colon M \to pt$ map $M$ to a point. Then $f_*\mathscr{F}$ is nothing but the complex vector space $\mathscr{F}(M)$, which is coherent iff it is finite-dimensional. Since $M$ is compact, $f$ is proper, and so the result follows from Grauert's theorem. □

Of course, this need not be true if $M$ is not compact: for instance, on the complex manifold $M = \mathbb{C}$, the sheaf of holomorphic functions $\mathscr{O}$ is coherent; but the space of its global sections is the space of all entire functions, and thus very far from being finite-dimensional.

**Cohomology of analytic sheaves.** Recall that for any sheaf of abelian groups $\mathscr{F}$ on a topological space $X$, we defined cohomology groups $H^i(X, \mathscr{F})$ by the following procedure: $\mathscr{F}$ has a natural resolution

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \mathscr{F}^2 \xrightarrow{d^2} \cdots,$$

the so-called *Godement resolution*, by flabby sheaves. Here $\mathscr{F}^0$ is the sheaf of discontinuous sections of $\mathscr{F}$, then $\mathscr{F}^1$ is the sheaf of discontinuous sections of $\operatorname{coker}\varepsilon$, and so on. By definition, $H^i(X, \mathscr{F})$ is the $i$-th cohomology group of the complex of abelian groups

$$0 \to \mathscr{F}^0(X) \to \mathscr{F}^1(X) \to \mathscr{F}^2(X) \to \cdots.$$

Now suppose that $X$ is a complex manifold, and $\mathscr{F}$ an analytic sheaf. In this case, each sheaf $\mathscr{F}^i$ in the Godement resolution is again an analytic sheaf, and so each abelian group $\mathscr{F}^i(X)$ is a module over the ring $\mathscr{O}_X(X)$ of holomorphic functions on $X$. Consequently, the cohomology groups $H^i(X, \mathscr{F})$ are themselves $\mathscr{O}_X(X)$-modules (and in particular $\mathbb{C}$-vector spaces).

**Theorem 27.6.** *If $X$ is a compact complex manifold, and $\mathscr{F}$ a coherent analytic sheaf, then each cohomology group $H^i(X, \mathscr{F})$ is a finite-dimensional complex vector space.*

When doing Hodge theory on compact Kähler manifolds, we have already seen special cases of this result: for $\mathscr{F} = \mathscr{O}_X$, or $\mathscr{F} = \Omega^1_X$, etc.

**Stein manifolds.** On certain complex manifolds, the higher cohomology groups of every coherent analytic sheaf are trivial.

*Example* 27.7. If $D$ is a polydisk in $\mathbb{C}^n$ (or $\mathbb{C}^n$ itself), then $H^i(D, \mathscr{F}) = 0$ for every coherent analytic sheaf and every $i > 0$. Because of Dolbeault's theorem, we already know that this is true for the sheaf of holomorphic functions $\mathscr{O}_D$. To extend the result to arbitrary coherent analytic sheaves, one proves that $\mathscr{F}$ admits a finite resolution of the form

$$0 \to \mathscr{O}^{\oplus p_r} \to \mathscr{O}^{\oplus p_{r-1}} \to \cdots \to \mathscr{O}^{\oplus p_1} \to \mathscr{O}^{\oplus p_0} \to \mathscr{F} \to 0;$$

the result follows from this by purely formal reasoning.

**Definition 27.8.** A *Stein manifold* is a complex manifold $M$ with the property that $H^i(M, \mathscr{F}) = 0$ for every coherent analytic sheaf $\mathscr{F}$ and every $i > 0$.

*Example* 27.9. Any complex submanifold of a Stein manifold is again a Stein manifold. In particular, any complex submanifold of $\mathbb{C}^n$ is Stein. The proof goes as follows: Let $i\colon N \hookrightarrow M$ denote the inclusion map. Then one can show that for any coherent analytic sheaf $\mathscr{F}$ on $N$, the direct image $i_*\mathscr{F}$ is again coherent. (This is a special case of Grauert's theorem, but much easier to prove.) Moreover, one has $H^i(N, \mathscr{F}) \simeq H^i(M, i_* \mathscr{F})$, and this obviously implies that $N$ is a Stein manifold.

*Example* 27.10. Any non-compact Riemann surface is known to be a Stein manifold by a theorem of Behnke and Stein.

*Example* 27.11. If $M$ is a Stein manifold, then any covering space of $M$ is again a Stein manifold.

A Stein manifold always has a very rich function theory, since the vanishing of higher cohomology groups of coherent sheaves makes it easy to construct holomorphic functions. To illustrate this, let $M$ be a Stein manifold, and let $\mathscr{O}_M(M)$ be the ring of its global holomorphic functions. We shall prove that any holomorphic function on an analytic subset can be extended to all of $M$.

**Lemma 27.12.** *Let $Z \subseteq M$ be an analytic subset of a Stein manifold $M$, and let $f$ be a holomorphic function on $Z$. Then there exist $g \in \mathscr{O}_M(M)$ with the property that $g(z) = f(z)$ for every $z \in Z$.*

*Proof.* Let $\mathscr{I}_Z$ denote the coherent ideal sheaf of the analytic subset $Z$. We have an exact sequence
$$0 \to \mathscr{I}_Z \to \mathscr{O}_M \to i_* \mathscr{O}_Z \to 0,$$
in which $\mathscr{O}_Z$ denotes the sheaf of holomorphic functions on $Z$. Passing to cohomology, we find that $\mathscr{O}_M(M) \to \mathscr{O}_Z(Z) \to H^1(M, \mathscr{I}_Z)$ is exact. Now $\mathscr{I}_Z$ is a coherent sheaf, and therefore $H^1(M, \mathscr{I}_Z) = 0$ because $M$ is a Stein manifold. It follows that the restriction map $\mathscr{O}_M(M) \to \mathscr{O}_Z(Z)$ is surjective. □

In particular, since every pair of points $p, q \in M$ determines an analytic subset $\{p, q\}$, we see that holomorphic functions on a Stein manifold separate points.

**Corollary 27.13.** *In a Stein manifold, every compact analytic subset is finite.*

*Proof.* A holomorphic function on a compact analytic set is locally constant. □

**Lemma 27.14.** *At every point $p \in M$, there exist holomorphic functions $f_1, \dots, f_n \in \mathscr{O}_M(M)$ that define local holomorphic coordinates in a neighborhood of the point.*

*Proof.* Let $z_1, \dots, z_n \in \mathscr{O}_M(U)$ be local holomorphic coordinates, centered at the point $p$. If we denote by $\mathscr{I}$ the ideal sheaf of the point $p$, then we have $z_j \in \mathscr{I}(U)$. The quotient sheaf $\mathscr{I}/\mathscr{I}^2$ is supported at the point $p$, and is in fact an $n$-dimensional complex vector space, spanned by the images of $z_1, \dots, z_n$. Any $f \in \mathscr{I}(M)$ may be expanded on $U$ into a convergent power series of the form
$$f(z) = \sum_{|I| \geq 1} a_I z^I,$$
and the vector determined by $f$ is nothing but the linear part $a_1 z_1 + \cdots + a_n z_n$.

By the same argument as before, the short exact sequence
$$0 \to \mathscr{I}^2 \to \mathscr{I} \to \mathscr{I}/\mathscr{I}^2 \to 0,$$
together with the vanishing of the cohomology group $H^1(M, \mathscr{I}^2)$, proves that the restriction map $\mathscr{I}(M) \to \mathscr{I}/\mathscr{I}^2$ is surjective. We may therefore find a holomorphic function $f_j$ whose image in $\mathscr{I}/\mathscr{I}^2$ equals $z_j$; it follows that the Jacobian matrix $\partial(f_1, \dots, f_n)/\partial(z_1, \dots, z_n)$ is the identity matrix at the point $p$, and so $f_1, \dots, f_n$ define a local holomorphic coordinate system by the implicit mapping theorem. □