Class 26. Chow's theorem (December 3)

We now want to show that complex submanifolds (and, more generally, analytic subsets) of projective space are algebraic varieties. As usual, we use homogeneous coordinates $[z_0, z_1, \ldots, z_n]$ on projective space. For a homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \ldots, z_n]$, the condition $F(z)$ is invariant under scaling (since $F(\lambda z) = \lambda^{\deg F} F(z)$ holds); this means that any collection $F_1, \ldots, F_k$ of homogeneous polynomials defines a closed subset $Z = Z(F_1, \ldots, F_k)$ of $\mathbb{P}^n$. It is clearly analytic; in fact, its intersection with each of the standard open subsets $U_0, U_1, \ldots, U_n$ is defined by polynomial functions.

Definition 26.1. An analytic subset $Z \subseteq \mathbb{P}^n$ is said to be a *projective algebraic variety* if it is of the form $Z(F_1, \ldots, F_k)$ for some collection of homogeneous polynomials.

The following result, known as Chow's theorem, shows that any analytic subset of $\mathbb{P}^n$ is actually a projective algebraic variety.

Theorem 26.2. *If $Z \subseteq \mathbb{P}^n$ is an analytic set, then there exist homogeneous polynomials $F_1, \ldots, F_k \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ such that $Z = Z(F_1, \ldots, F_k)$.*

Proof. The proof consists in a simple, but very clever, application of the Levi extension theorem. If $Z = \emptyset$, then we may take $F_1 = 1$; to exclude this trivial case, we assume from now on that $Z \neq \emptyset$. Recall that by definition of $\mathbb{P}^n$ as a quotient, we have the holomorphic quotient map $q \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. The preimage $V = q^{-1}(Z)$ is therefore an analytic subset of $\mathbb{C}^{n+1} \setminus \{0\}$. Note that each component of V has dimension at least 1, since it has to be closed under rescaling the coordinates by $\mathbb{C}^*$. This means that the codimension of V is at most n; on the other hand, the codimension of the origin in $\mathbb{C}^{n+1}$ is $n + 1$. We may thus apply Theorem 25.5 and conclude that the closure $\overline{V}$ is an analytic subset of $\mathbb{C}^{n+1}$. Observe (and this is important) that $\overline{V}$ is a *cone*: for $z \in \overline{V}$ and $\lambda \in \mathbb{C}$, we also have $\lambda z \in \overline{V}$.

It remains to produce polynomial equations that define Z. Let $\mathscr{O}_{n+1}$ be the local ring at the origin in $\mathbb{C}^{n+1}$ (its elements are germs of holomorphic functions, or equivalently, convergent power series), and let $I \subseteq \mathscr{O}_{n+1}$ be the ideal of germs of holomorphic functions that vanish on $\overline{V}$. Any $f \in I$ can be written as a convergent power series in $z = (z_0, z_1, \ldots, z_n)$ in some neighborhood of the origin; thus

$$f(z) = \sum_{j=0}^{\infty} f_j(z),$$

with $f_j \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ homogeneous of degree j. The fact that $\overline{V}$ is a cone now implies that $f_j \in I$. To see why, fix a point $z \in \overline{V}$; for $|\lambda| < 1$, we then have

$$0 = f(\lambda z) = \sum_{j=0}^{\infty} f_j(\lambda z) = \sum_{j=0}^{\infty} \lambda^j f_j(z).$$

Since $f(\lambda z)$ is holomorphic in λ, the identity theorem shows that we have $f_j(z) = 0$ for all $j \geq 0$. Consequently, $f_j \in I$ as claimed. It follows that I is generated by homogeneous polynomials.

By Theorem 3.1, the ring $\mathscr{O}_{n+1}$ is Noetherian, and so I is finitely generated. This means that there are finitely many homogeneous polynomials $F_1, \ldots, F_k \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ such that $I = (F_1, \ldots, F_k)$; it is then obvious that we have $Z = Z(F_1, \ldots, F_k)$. $\qquad\square$

Combining Chow's theorem and the Kodaira embedding theorem, we obtain the following corollary.

Corollary 26.3. *If a compact complex manifold M carries a positive line bundle, then M is isomorphic to a projective algebraic variety.*

As a matter of fact, any globally defined analytic object on projective space is algebraic; this is the content of the so-called *GAGA theorem* of Serre. More precisely, Serre's theorem asserts that there is an equivalence of categories between coherent analytic sheaves and coherent algebraic sheaves on $\mathbb{P}^n$. We will discuss coherent sheaves in more detail later on.

Example 26.4. For a simple example, consider holomorphic line bundles on $\mathbb{P}^n$. The exponential sequence

$$H^1(\mathbb{P}^n, \mathcal{O}) \to H^1(\mathbb{P}^n, \mathcal{O}^*) \to H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathcal{O})$$

shows that the group of line bundles is isomorphic to $\mathbb{Z}$ (the two cohomology groups on the left and right vanish by Lemma 19.2). Thus every holomorphic line bundle is of the form $\mathcal{O}_{\mathbb{P}^n}(d)$ for some $d \in \mathbb{Z}$. These line bundles are actually *algebraic*, because they are locally trivial on the standard open cover of $\mathbb{P}^n$, with transition functions given by polynomials.

The following lemma shows how Hartog's theorem can be used to prove that every holomorphic section of the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ on projective space is given by a homogeneous polynomial of degree d.

Lemma 26.5. *For $d \geq 0$, the space of global sections of the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ is isomorphic to the space of homogeneous polynomials of degree d in $\mathbb{C}[z_0, z_1, \ldots, z_n]$.*

Proof. With respect to the standard open cover $U_0, U_1, \ldots, U_n$, the transition functions of the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ are given by z_j/z_k; hence those of $\mathcal{O}_{\mathbb{P}^n}(d)$ are z_k^d/z_j^d. A global section of the line bundle is a collection of holomorphic functions $f_j \in \mathcal{O}_{\mathbb{P}^n}(U_j)$ such that $f_j = z_k^d/z_j^d f_k$ on $U_j \cap U_k$. As before, let $q \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the quotient map, and put $f_j' = f_j \circ q$, which is defined and holomorphic on the set where $z_j \neq 0$. The relation above becomes

$$z_j^d f_j' = z_k^d f_k',$$

which shows that $z_j^d f_j'$ is the restriction of a holomorphic function on $\mathbb{C}^{n+1} \setminus \{0\}$. By Hartog's theorem, this function extends holomorphically to all of $\mathbb{C}^{n+1}$, and so we obtain $F \in \mathcal{O}(\mathbb{C}^{n+1})$ with the property that $F = z_j^d f_j'$ for $z_j \neq 0$. For $\lambda \in \mathbb{C}^*$ and $z \neq 0$, we now have (for some $0 \leq j \leq n$)

$$F(\lambda z) = (\lambda z_j)^d f_j'(\lambda z) = \lambda^d z_j^d f_j'(z) = \lambda^d F(z),$$

and by expanding F into a convergent power series, we see that F has to be a homogeneous polynomial of degree d. We conclude that $f_j = F/z_j^d$, proving that the original section of $\mathcal{O}_{\mathbb{P}^n}(d)$ is indeed given by a homogeneous polynomial. The converse is obvious. $\qquad\square$

Coherent analytic sheaves. When we studied the local properties of holomorphic functions, we showed that $\mathcal{O}_n$, the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$, is Noetherian. This means that every ideal of $\mathcal{O}_n$ is finitely generated. From this result, we deduced that any analytic set Z containing $0 \in \mathbb{C}^n$ is

locally defined by finitely many holomorphic functions: let $I_Z \subseteq \mathscr{O}_n$ be the ideal of functions vanishing on Z; because I_Z is finitely generated, there is an open neighborhood D of the origin, and holomorphic functions $f_1, \ldots, f_k \in \mathscr{O}(D)$ whose germs generate I_Z, such that $Z \cap D = Z(f_1, \ldots, f_k)$.

The Noetherian property only gives information about Z at the origin, though. For instance, suppose that $g \in \mathscr{O}(D)$ is another holomorphic function that vanishes on Z. The germ of g belongs to the ideal I_Z, and hence we have $g = a_1 f_1 + \cdots + a_k f_k$ in the ring $\mathscr{O}_n$; but since $a_1, \ldots, a_n$ may only be defined on a much smaller open neighborhood of the origin, this relation does not describe g on D.

In fact, this stronger finiteness property is true: there exist holomorphic functions $b_1, \ldots, b_k \in \mathscr{O}(D)$ with the property that $g = a_1 f_1 + \cdots + b_k f_k$. The natural setting for such questions is the theory of analytic sheaves.

Definition 26.6. An *analytic sheaf* $\mathscr{F}$ on a complex manifold M is a sheaf of abelian groups, such that for every open set $U \subseteq M$, the group of sections $\mathscr{F}(U)$ is a module over the ring of holomorphic functions $\mathscr{O}_M(U)$, in a way that is compatible with restriction.

Of course, $\mathscr{O}_M$ itself is an analytic sheaf. Here are two other classes of examples:

Example 26.7. Let $Z \subseteq M$ be an analytic subset. Consider the sheaf $\mathscr{I}_Z$, whose sections over an open set $U \subseteq M$ are those holomorphic functions in $\mathscr{O}_M(U)$ that vanish along the intersection $U \cap Z$. Evidently, $\mathscr{I}_Z$ is an analytic sheaf, known as the *ideal sheaf* of the analytic set Z. Questions about holomorphic functions that vanish along Z are then really questions about this analytic sheaf.

Example 26.8. Let $p \colon E \to M$ be a holomorphic vector bundle of rank r. Consider the sheaf $\mathscr{E}$ of holomorphic sections of E; by definition, $\mathscr{E}(U)$ consists of all holomorphic mappings $s \colon U \to E$ with the property that $p \circ s = \mathrm{id}_U$. Then $\mathscr{E}$ is again an analytic sheaf. As a matter of fact, $\mathscr{E}$ is an example of a *locally free sheaf*: if $p^{-1}(U) \simeq U \times \mathbb{C}^r$, then the restriction of $\mathscr{E}$ to U is isomorphic to $\mathscr{O}_U^{\oplus r}$.

If $\mathscr{F}$ is an analytic sheaf, then at every point $p \in M$, the stalk

$$\mathscr{F}_p = \lim_{U \ni p} \mathscr{F}(U)$$

is a module over the local ring $\mathscr{O}_{M,p}$. For instance, the stalk of the ideal sheaf $\mathscr{I}_Z$ is the ideal in the local ring $\mathscr{O}_{M,p}$ defined by the analytic set Z. The question that we were discussing a few moments ago now leads to the following definition.

Definition 26.9. An analytic sheaf $\mathscr{F}$ is said to be *locally finitely generated* if every point of M has an open neighborhood U with the following property: there are finitely many sections $s_1, \ldots, s_k \in \mathscr{F}(U)$ that generate the stalk $\mathscr{F}_p$ at every point $p \in U$.

The sections $s_1, \ldots, s_k \in \mathscr{F}(U)$ determine a morphism $\mathscr{O}_U^{\oplus k} \to \mathscr{F}|_U$ of analytic sheaves, and the condition on the stalks is equivalent to the surjectivity of that morphism. In general, the kernel of this morphism may not itself be locally finitely generated.

Definition 26.10. An analytic sheaf $\mathscr{F}$ is said to be *coherent* if, in addition to being locally finitely generated, it satisfies the following condition: locally on M, there exists an exact sequence of analytic sheaves

$$\mathscr{O}_U^{\oplus p} \xrightarrow{\ F\ } \mathscr{O}_U^{\oplus q} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

where F is some $q \times p$-matrix of holomorphic functions on U.

The following result, known as *Oka's theorem*, is fundamental in the theory of coherent analytic sheaves.

Theorem 26.11. *If $F \colon \mathscr{O}_M^{\oplus p} \to \mathscr{O}_M^{\oplus q}$ is a morphism of analytic sheaves, then the kernel of F is locally finitely generated.*

To illustrate the statement, suppose that $D \subseteq \mathbb{C}^n$ is an open set, and $f_1, \ldots, f_k \in \mathscr{O}(D)$ are holomorphic functions. The kernel of $f \colon \mathscr{O}(D)^{\oplus k} \to \mathscr{O}(D)$ consists of all the relations between $f_1, \ldots, f_k$, that is, of all k-tuples of holomorphic functions $a_1, \ldots, a_k$ such that $a_1 f_1 + \cdots + a_k f_k = 0$. Oka's theorem is the assertion that finitely many of these k-tuples generate all the relations.

Theorem 26.12. *If $Z \subseteq M$ is an analytic set in a complex manifold M, then the sheaf of ideals $\mathscr{I}_Z$ is coherent.*

The following lemma can be proved directly from the definition of coherence, with some diagram chasing.

Lemma 26.13. *The kernel and cokernel of any morphism between coherent sheaves is again coherent.*

Lemma 26.14. *Let $\mathscr{F}$ be a coherent analytic sheaf. If sections $s_1, \ldots, s_k \in \mathscr{F}(U)$ generate the stalk of $\mathscr{F}$ at some point $p_0 \in M$, then they generate the stalks $\mathscr{F}_p$ at all nearby points.*

Proof. The sections $s_1, \ldots, s_k$ determine a morphism $\phi \colon \mathscr{O}_U^{\oplus k} \to \mathscr{F}$, and by the preceding lemma, $\operatorname{coker} \phi$ is again a coherent analytic sheaf. That the sections generate the stalk of $\mathscr{F}$ at the point p_0 means exactly than $(\operatorname{coker} \phi)_{p_0} \simeq 0$. The problem is thus reduced to the following simpler statement: if $\mathscr{G}$ is a coherent analytic sheaf whose stalk at some point $p_0 \in M$ is isomorphic to zero, then the same is true at all points in a neigborhood of p_0.

The proof is easy. Indeed, $\mathscr{G}$ being coherent, there is an exact sequence

$$\mathscr{O}_U^{\oplus p} \xrightarrow{\ F\ } \mathscr{O}_U^{\oplus q} \longrightarrow \mathscr{G}|_U \longrightarrow 0$$

on some neighborhood U of the point p_0. Now $\mathscr{G}_{p_0} \simeq 0$ means that the matrix F, whose entries are holomorphic functions on U, has maximal rank at the point p_0; in other words, at least one of its $q \times q$-minors does not vanish at the point p_0. But then the same minor is nonzero on some open neighborhood V of p_0, proving that $\mathscr{G}|_V \simeq 0$ as claimed. $\square$