Class 22. The Kodaira vanishing theorem (November 14)

Harmonic forms. Let L be a holomorphic line bundle on a compact complex manifold M. Since the $\bar\partial$-operator $\bar\partial_L \colon A^{p,q}(M,L) \to A^{p,q+1}(M,L)$ satisfies $\bar\partial_L \circ \bar\partial_L = 0$, we can define the Dolbeault cohomology groups of L as

$$H^{p,q}(M,L) = \frac{\ker\bigl(\bar\partial_L \colon A^{p,q}(M,L) \to A^{p,q+1}(M,L)\bigr)}{\operatorname{im}\bigl(\bar\partial_L \colon A^{p,q-1}(M,L) \to A^{p,q}(M,L)\bigr)}.$$

Note that the usual Dolbeault cohomology is the special case of the trivial bundle $M \times \mathbb{C}$. As before, the Dolbeault complex

$$0 \longrightarrow \mathscr{A}^{p,0}(L) \xrightarrow{\bar\partial_L} \mathscr{A}^{p,1}(L) \xrightarrow{\bar\partial_L} \mathscr{A}^{p,2}(L) \longrightarrow \cdots \longrightarrow \mathscr{A}^{p,n}(L) \longrightarrow 0$$

resolves the sheaf $\Omega_M^p \otimes L$ of holomorphic p-forms with coefficients in L, and since each $\mathscr{A}^{p,q}(L)$ is a fine sheaf, we find that

$$H^{p,q}(M,L) \simeq H^q\bigl(M, \Omega_M^p \otimes L\bigr)$$

computes the sheaf cohomology groups of $\Omega_M^p \otimes L$. We would now like to generalize the Hodge theorem to this setting, and show that cohomology classes in $H^{p,q}(M,L)$ can be represented by harmonic forms.

We proceed along the same lines as before, and so the first step is to define Hermitian inner products on the spaces $A^{p,q}(M,L)$. To do that, choose a Hermitian metric h on the complex manifold M, and let g be the corresponding Riemannian metric and ω the associated $(1,1)$-form. (For the time being, it is not necessary to assume that h is a Kähler metric.) We also choose a Hermitian metric h_L on the holomorphic line bundle L. With both metrics in hand, we can define a Hermitian inner product on $A^{p,q}(M,L)$; writing a typical element as $\alpha \otimes s$, with $\alpha \in A^{p,q}(M)$ and $s \in A(M,L)$ smooth, we set

$$(\alpha_1 \otimes s_1, \alpha_2 \otimes s_2)_M = \int_M h(\alpha_1, \alpha_2) h_L(s_1, s_2) vol(g).$$

We then let $\bar\partial_L^* \colon A^{p,q}(M,L) \to A^{p,q-1}(M,L)$ be the adjoint of $\bar\partial_L$ with respect to the inner product, and define the Laplace operator

$$\overline{\Box}_L = \bar\partial_\circ \bar\partial_L^* + \bar\partial_L^* \circ \bar\partial_L \colon A^{p,q}(M,L) \to A^{p,q}(M,L).$$

A local calculation (made easy by the fact that L is locally trivial) shows that $\overline{\Box}_L$ is an elliptic operator of order two. Thus if we let $\mathcal{H}^{p,q}(M,L) = \ker \overline{\Box}_L$ denote the space of $\bar\partial$-*harmonic* forms with coefficients in L, we get

$$\mathcal{H}^{p,q}(M,L) \simeq H^{p,q}(M,L)$$

by applying the general theorem about elliptic operators (Theorem 12.9).

The Kodaira vanishing theorem. The most important consequence of being able to represent classes in $H^{p,q}(M,L)$ by $\bar\partial$-harmonic forms is the famous Kodaira vanishing theorem; roughly speaking, it says that if L is "positive" (in the way that the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ on projective space is positive), then its Dolbeault cohomology vanishes for $p + q > n$.

To see in what sense the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ is positive, recall that its curvature form satisfies $\frac{i}{2\pi}\Theta = \omega_{FS}$, which is the Kähler form of the Fubini-Study metric. The following definition generalizes this situation.

Definition 22.1. A holomorphic line bundle $L \to M$ is called *positive* if its first Chern class $c_1(L)$ can be represented by a closed $(1,1)$-form Ω whose associated Hermitian form is positive definite.

More concretely, what this means is that if we write $\Omega = \frac{i}{2}\sum_{j,k} f_{j,k} dz_j \wedge d\bar{z}_k$ in local coordinates $z_1, \dots, z_n$, then the Hermitian matrix with entries $f_{j,k}$ should be positive definite. We express this more concisely by saying that Ω is a *positive* form. Of course, such a form Ω is the associated $(1,1)$-form of a Hermitian metric on M, and since $d\Omega = 0$, this metric is Kähler. In particular, if there exists a positive line bundle on M, then M is necessarily a Kähler manifold.

Here is the precise statement of Kodaira's vanishing theorem.

Theorem 22.2. *Let L be a positive line bundle on an n-dimensional compact complex manifold M. Then $H^{p,q}(M,L) = 0$ whenever $p+q > n$.*

Generalized Kähler identities and the proof. Throughout, we fix a compact complex manifold M and a positive line bundle L on it. As mentioned above, there is a Kähler metric h on M whose associated $(1,1)$-form ω represents $c_1(L)$, and we assume from now on that M has been given that metric. The following lemma allows us to choose a compatible Hermitian metric h_L on the line bundle L, with the property that $\frac{i}{2\pi}\Theta_L = \omega$ is the Kähler form.

Lemma 22.3. *Let L be a positive line bundle on a compact Kähler manifold M, and suppose that ω is a closed $(1,1)$-form that represents $c_1(L)$. Then there is a (essentially unique) Hermitian metric on L whose curvature satisfies $\frac{i}{2\pi}\Theta = \omega$.*

Proof. Choose an arbitrary Hermitian metric h_0 on L, and let $\Theta_0 \in A^{1,1}(M)$ be the associated curvature form. Then both $\frac{i}{2\pi}\Theta_0$ and ω represent the first Chern class of L, and so their difference is a $(1,1)$-form that is both closed and $\bar{\partial}$-exact. By the $\partial\bar{\partial}$-Lemma (see Proposition 18.12), there exists a smooth real-valued function $\psi \in A(M)$ such that

$$\omega = \frac{i}{2\pi}\Theta_0 + \frac{i}{2\pi}\partial\bar{\partial}\psi.$$

Now define a new Hermitian metric on L by setting $h_L = e^{-\psi} h_0$. We then have

$$\Theta_L = -\partial\bar{\partial}\log h = \Theta_0 + \partial\bar{\partial}\psi,$$

and hence $\frac{i}{2\pi}\Theta_L = \omega$ as asserted. □

The Hermitian metric h_L on the line bundle L also gives rise to the Chern connection $\nabla\colon A(M,L) \to A^1(M,L)$. We have $\nabla = \nabla' + \nabla''$, and by definition of the Chern connection, $\nabla'' = \bar{\partial}_L$. To emphasize the analogy with the case of usual forms, we shall write ∂_L instead of ∇' throughout this section. We then get operators

$$\bar{\partial}_L\colon A^{p,q}(M,L) \to A^{p,q+1}(M,L) \quad \text{and} \quad \partial_L\colon A^{p,q}(M,L) \to A^{p+1,q}(M,L)$$

by enforcing the Leibniz rule. Note that we have $\bar{\partial}_L \circ \bar{\partial}_L = 0$ and $\partial_L \circ \partial_L = 0$; on the other hand, $\partial_L \circ \bar{\partial}_L + \bar{\partial}_L \circ \partial_L$ is not usually zero, but is related to the curvature of L. (In the case of the trivial bundle $L = M \times \mathbb{C}$, the curvature is zero, which explains why we have $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$.)

Lemma 22.4. *If $\Theta_L \in A^{1,1}(M)$ denotes the curvature form of the metric h_L, then we have $\partial_L\bar{\partial}_L + \bar{\partial}_L\partial_L = \Theta_L$.*

Proof. From the definition of the curvature form, we have $\Theta_L = \nabla^2 = (\partial_L + \bar{\partial}_L) \circ (\partial_L + \bar{\partial}_L)$, and so the identity follows. To illustrate what is going on, here is a more concrete proof. Let $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}$ be a local trivialization of L, and let s be the corresponding holomorphic section of L over U. As usual, write $\nabla s = \theta \otimes s$, and since we are dealing with the Chern connection, we have $\theta = \partial \log h_L(s,s)$.

For any $\alpha \in A^{p,q}(U)$, we have $\bar{\partial}_L(\alpha \otimes s) = (\bar{\partial}\alpha) \otimes s$ by definition of the $\bar{\partial}$-operator; on the other hand,

$$\begin{aligned}\partial_L(\alpha \otimes s) &= \nabla'(\alpha \otimes s) = (\partial\alpha) \otimes s + (-1)^{p+q} \alpha \wedge \nabla' s \\ &= (\partial\alpha) \otimes s + (-1)^{p+q} (\alpha \wedge \theta) \otimes s = (\partial\alpha + \theta \wedge \alpha) \otimes s.\end{aligned}$$

Consequently,

$$\begin{aligned}(\partial_L \bar{\partial}_L + \bar{\partial}_L \partial_L)(\alpha \otimes s) &= \partial_L\big(\bar{\partial}\alpha \otimes s\big) + \bar{\partial}_L\big((\partial\alpha + \theta \wedge \alpha) \otimes s\big) \\ &= (\partial\bar{\partial}\alpha + \theta \wedge \bar{\partial}\alpha) \otimes s + (\bar{\partial}\partial\alpha + \bar{\partial}\theta \wedge \alpha - \theta \wedge \bar{\partial}\alpha) \otimes s \\ &= (\bar{\partial}\theta \wedge \alpha) \otimes s = (\Theta_L \wedge \alpha) \otimes s.\end{aligned}$$

□

As usual, we let ∂_L^* and $\bar{\partial}_L^*$ denote the adjoint operators of ∂_L and $\bar{\partial}_L$, with respect to the inner product introduced above. To make this more explicit, let us describe the operator ∂_L^* in a local trivialization $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}$. If s denotes the corresponding holomorphic section, then any element of $A^{p,q}(U, L)$ can be written as $\alpha \otimes s$ for a unique $\alpha \in A^{p,q}(U)$. Fix $\beta \in A^{p,q}(U)$ with compact support. By definition of the adjoint, we have

$$\big(\partial_L^*(\alpha \otimes s), \beta \otimes s\big)_M = \big(\alpha \otimes s, \partial_L(\beta \otimes s)\big)_M.$$

We already computed that $\partial_L(\beta \otimes s) = (\partial\beta + \theta \wedge \beta) \otimes s$, where $\theta = \partial \log f = f^{-1}\partial f$ and $f = h_L(s,s)$ is smooth and positive real-valued. Consequently,

$$\begin{aligned}\big(\alpha \otimes s, \partial_L(\beta \otimes s)\big)_L &= \int_M h\big(\alpha, \partial\beta + \theta \wedge \beta\big) h_L(s,s) vol(g) \\ &= \int_M h\big(\alpha, f\partial\beta + \partial f \wedge \beta\big) vol(g) = \int_M h\big(\alpha, \partial(f\beta)\big) vol(g).\end{aligned}$$

This latter is the usual inner product between α and $\partial(f\beta)$, and therefore equals

$$\int_M h\big(\partial^*\alpha, f\beta)\big) vol(g) = \int_M h(\partial^*\alpha, \beta) h_L(s,s) vol(g) = \big((\partial^*\alpha) \otimes s, \beta\big)_L.$$

The conclusion is that $\partial_L^*(\alpha \otimes s) = (\partial^*\alpha) \otimes s$.

Lastly, we extend the usual Lefschetz operator $L_\omega(\alpha) = \omega \wedge \alpha$ to forms with coefficients in the line bundle by the rule

$$L_\omega(\alpha \otimes s) = (\omega \wedge \alpha) \otimes s.$$

Likewise, we define $\Lambda(\alpha \otimes s) = (\Lambda\alpha) \otimes s$. It is not hard to see that the operator $\Lambda\colon A^{p,q}(M,L) \to A^{p-1,q-1}(M,L)$ is the adjoint of $L_\omega\colon A^{p,q}(M,L) \to A^{p+1,q+1}(M,L)$ with respect to the inner product introduced above.

For the proof, we only need two identities between all these operators, and we have already proved both of them. Firstly, note that $\Theta_L = -2\pi i\omega$, and so we can restate the formula in Lemma 22.4 as $\partial_L \bar{\partial}_L + \bar{\partial}_L \partial_L = -2\pi i L_\omega$. After taking adjoints, we obtain the first important identity:

$$\partial_L^* \bar{\partial}_L^* + \bar{\partial}_L^* \partial_L^* = 2\pi i \Lambda. \tag{22.5}$$

Moreover, the fact that ∂_L^* is locally given by $\partial_L^*(\alpha \otimes s) = (\partial^* \alpha) \otimes s$ shows that the Kähler identity $[\Lambda, \bar{\partial}] = -i\partial^*$ generalizes to this setting of L-valued forms, giving us the second important identity

$$\Lambda \bar{\partial}_L - \bar{\partial} \Lambda_L = -i\partial_L^* \tag{22.6}$$

as operators on the space $A^{p,q}(M, L)$. We are now ready to prove Theorem 22.2

Proof. Since M is compact, we can represent classes in $H^{p,q}(M, L)$ by $\bar{\partial}$-harmonic forms, and so it suffices to prove that any $\alpha \in \mathcal{H}^{p,q}(M, L)$ with $p + q > n$ has to be zero. As α is $\bar{\partial}$-harmonic, we have $\bar{\partial}_L \alpha = 0$ and $\bar{\partial}_L^* \alpha = 0$. Now we use the two identities (22.5) and (22.6) to compute the norm of $\Lambda\alpha$. This goes as follows:

$$\begin{aligned}
(\Lambda\alpha, \Lambda\alpha)_M &= \frac{i}{2\pi} \big(\Lambda\alpha, (\bar{\partial}_L^* \partial_L^* + \partial_L^* \bar{\partial}_L^*)\alpha\big)_M = \frac{i}{2\pi} \big(\Lambda\alpha, \bar{\partial}_L^* \partial_L^* \alpha\big)_M \\
&= \frac{i}{2\pi} \big(\bar{\partial}_L \Lambda\alpha, \partial_L^* \alpha\big)_M = \frac{i}{2\pi} \big((\bar{\partial}_L \Lambda - \Lambda \bar{\partial}_L)\alpha, \partial_L^* \alpha\big)_M \\
&= \frac{i}{2\pi} (i\partial_L^* \alpha, \partial_L^* \alpha)_M = -\frac{1}{2\pi} (\partial_L^* \alpha, \partial_L^* \alpha)_M.
\end{aligned}$$

Because we are dealing with an inner product, it follows that both sides have to be zero; in particular, $\Lambda\alpha = 0$, and so α is primitive. But we have already seen that there are no nonzero primitive forms in degree above n, and so if $p + q > n$, we get that $\alpha = 0$, as claimed. □

Note. Note that the proof depends on the identity $\partial_L \bar{\partial}_L + \bar{\partial}_L \partial_L = -2\pi i\omega$, which holds because the first Chern class of L is representable by a Kähler form. It is in this way that the positivity of the line bundle gives us the additional minus sign, which is crucial to the proof.

Since $H^{p,q}(M, L)$ computes the sheaf cohomology groups of $\Omega_M^p \otimes L$, we can also conclude the following.

Corollary 22.7. *If L is a positive line bundle on a compact complex manifold M, then $H^q(M, \Omega_M^p \otimes L) = 0$ for $p + q > n$. In particular, we have $H^q(M, \Omega_M^n \otimes L) = 0$ for every $q > 0$.*