Class 21. Holomorphic line bundles (November 12)

We will be especially interested in the case $r = 1$, that is, in holomorphic *line bundles*. Local trivializations now take the form $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$, and consequently, a holomorphic line bundle can be described by a collection of holomorphic functions $g_{\alpha,\beta} \in \mathscr{O}_M^*(U_\alpha \cap U_\beta)$ that satisfy the cocycle condition $g_{\alpha,\beta} g_{\beta,\gamma} = g_{\alpha,\gamma}$. A line bundle is trivial, meaning isomorphic to $M \times \mathbb{C}$, precisely when it admits a nowhere vanishing section; in that case, we have $g_{\alpha,\beta} = s_\beta / s_\alpha$, with $s_\alpha \in \mathscr{O}_M^*(U_\alpha)$. If we only consider line bundles that are trivial on a fixed open cover $\mathbf{U}$, then the set of isomorphism classes of such line bundles is naturally in bijection with the Čech cohomology group $H^1(\mathbf{U}, \mathscr{O}_M^*)$. Likewise, the set of isomorphism classes of arbitrary line bundles is in bijection with the group $H^1(M, \mathscr{O}_M^*)$.

Example 21.1. The *tensor product* of two holomorphic line bundles L' and L'' is the holomorphic line bundle $L = L' \otimes L''$ with transition functions $g_{\alpha,\beta} = g'_{\alpha,\beta} g''_{\alpha,\beta}$. This operation corresponds to multiplication in the group $H^1(M, \mathscr{O}_M^*)$.

Example 21.2. The *dual* of a holomorphic line bundle L is the holomorphic line bundle L^{-1} with transition functions $g_{\alpha,\beta}^{-1}$. Since $L \otimes L^{-1}$ is isomorphic to the trivial bundle $M \times \mathbb{C}$, we see that L^{-1} is the inverse of L in the group $H^1(M, \mathscr{O}_M^*)$.

Example 21.3. Let $D \subseteq M$ be a hypersurface in M, that is, an analytic subset of dimension $n-1$ that is locally defined by the vanishing of a single holomorphic function. Then there is a holomorphic line bundle $\mathscr{O}_X(-D)$, whose sections over an open set U are all holomorphic functions $f \in \mathscr{O}_X(U)$ that vanish along $U \cap D$. To compute the transition functions, suppose that we have $U_\alpha \cap D = Z(f_\alpha)$, where each f_α is not divisible by the square of any nonunit, and hence unique up to multiplication by units. It follows that the ratios $g_{\alpha,\beta} = f_\beta / f_\alpha$ are nowhere vanishing holomorphic functions on $U_\alpha \cap U_\beta$.

Let us describe Hermitian metrics and the Chern connection in the case of a holomorphic line bundle. A local trivialization $\phi \colon \pi^{-1}(U) \to U \times \mathbb{C}$ of the line bundle is the same as a nonvanishing holomorphic section $s \in A(U, L)$, and so a Hermitian metric h on L is locally described by a single smooth function $h = h(s,s)$ with values in the positive real numbers.

The Chern connection is determined by its action on s, and if we put $\nabla s = \theta \otimes s$, then we have seen that $\theta \in A^{0,1}(U)$ because $\nabla'' s = \bar{\partial} s = 0$. The other defining property of the Chern connection,

$$\partial h + \bar{\partial} h = dh(s,s) = h(\nabla s, s) + h(s, \nabla s) = h\theta + h\bar{\theta}$$

shows that we have $\theta = h^{-1} \partial h = \partial \log h$. The $(1,1)$-form $\Theta = \bar{\partial}\theta = -\partial\bar{\partial} \log h$ is called the *curvature* of the connection.

Lemma 21.4. *Let h be a Hermitian metric on a holomorphic line bundle $L \to M$.*

1. *The curvature form $\Theta_L \in A^{1,1}(M)$ is globally well-defined.*
2. *$\partial \Theta_L = \bar{\partial} \Theta_L = 0$, and the class of Θ_L in $H^{1,1}(M)$ does not depend on h.*
3. *With the induced metric on L^{-1}, we have $\Theta_{L^{-1}} = -\Theta_L$.*
4. *With the induced metric on $L_1 \otimes L_2$, we have $\Theta_{L_1 \otimes L_2} = \Theta_{L_1} + \Theta_{L_2}$.*

Proof. In a local trivialization $\phi \colon \pi^{-1}(U) \to U \times \mathbb{C}$, we have $\Theta = -\partial\bar{\partial} \log h(s,s)$, where s is the distinguished holomorphic section determined by ϕ. For a second trivialization ϕ', we have $s' = fs$ for some $f \in \mathscr{O}_M^*(U)$, and hence $h(s', s') = |f|^2 h(s, s)$. It follows that

$$\Theta' = -\partial\bar{\partial}\big(|f|^2 + h(s,s)\big) = -\partial\bar{\partial}(f\overline{f}) + \Theta = \Theta,$$

and so Θ is independent of the local trivializations. Moreover, the local formula clearly shows that $\partial\Theta = \bar{\partial}\Theta = 0$, and so Θ defines a class in the Dolbeault cohomology group $H^{1,1}(M)$.

To prove that $[\Theta]$ does not depend on h, note that since the fibers of L are one-dimensional, any other choice of Hermitian metric has to differ from h by multiplication by a positive real-valued function $\psi \in A(M)$. We then have $-\partial\bar{\partial}\log(\psi h) = \Theta + \bar{\partial}\partial \log\psi$, and so both forms differ by a $\bar{\partial}$-exact form, and hence define the same cohomology class. The remaining two assertions are left as an exercise. □

Note. More generally, suppose that we have a connection $\nabla\colon \mathscr{A}(E) \to \mathscr{A}^1(E)$ on a holomorphic vector bundle E. It induces a mapping $\nabla\colon \mathscr{A}^1(E) \to \mathscr{A}^2(E)$, by requiring that the product rule

$$\nabla(\alpha \otimes s) = d\alpha \otimes s - \alpha \wedge \nabla s$$

be satisfied for smooth forms $\alpha \in A^1(M)$ and smooth sections $s \in A(M, E)$. The composition $\nabla^2\colon \mathscr{A}(E) \to \mathscr{A}^2(E)$ is called the *curvature* of the connection. From

$$\nabla^2(fs) = \nabla\big(df \otimes s + f\nabla s\big) = -df \wedge \nabla s + df \wedge \nabla s + f \cdot \nabla^2 s,$$

we see that ∇^2 is an $A(M)$-linear operator, and hence described in local trivializations by an $r \times r$-matrix of 2-forms. Now suppose that (E, h) is a holomorphic vector bundle with a Hermitian metric, and let ∇ be its Chern connection. As we have seen, the connection is locally given by the formula

$$\nabla s_j = \sum_{k=1}^r \theta_{j,k} \otimes s_k$$

and so we have

$$\begin{aligned}\nabla^2 s_j &= \sum_{k=1}^r \big(d\theta_{j,k} \otimes s_k - \theta_{j,k} \wedge \nabla s_k\big) = \sum_{k=1}^r d\theta_{j,k} \otimes s_k - \sum_{k,l=1}^r \theta_{j,k} \wedge \theta_{k,l} \otimes s_l \\ &= \sum_{k=1}^r \left(d\theta_{j,k} - \sum_{l=1}^r \theta_{j,l} \wedge \theta_{l,k} \right) \otimes s_k = \sum_{k=1}^r \Theta_{j,k} \otimes s_k.\end{aligned}$$

To describe more concretely the forms $\Theta_{j,k} \in A^2(U)$, we note that the Chern connection satisfies $\partial h_{j,k} = \sum h_{l,k}\theta_{j,l}$; from this, we obtain

$$0 = \partial^2 h_{j,k} = \sum_{l=1}^r \big(\partial h_{l,k} \wedge \theta_{j,l} + h_{l,k}\partial\theta_{j,l}\big) = \sum_{l,m=1}^r h_{m,k}\theta_{l,m} \wedge \theta_{j,l} + \sum_{l=1}^r h_{l,k}\partial\theta_{j,l}.$$

It follows that we have $\Theta_{j,k} = \bar{\partial}\theta_{j,k}$, which are therefore $(1,1)$-forms. Just as in the case of line bundles, one can show that the curvature is a globally defined $(1,1)$-form with coefficients in the bundle $\mathrm{Hom}(E, E)$.

It follows from Lemma 21.4 that we have a group homomorphism

$$H^1(M, \mathscr{O}_M^*) \to H^{1,1}(M)$$

that associates to a holomorphic line bundle the cohomology class of its curvature form Θ_L (with respect to an arbitrary Hermitian metric). On the other hand, the exponential sequence $0 \to \mathbb{Z}_M \to \mathscr{O}_M \to \mathscr{O}_M^* \to 0$ gives us a long exact sequence, part of which reads

$$H^1(M, \mathscr{O}_M) \to H^1(M, \mathscr{O}_M^*) \to H^2(M, \mathbb{Z}_M) \to H^2(M, \mathscr{O}_M)$$

As mentioned earlier, the sheaf cohomology group $H^2(M, \mathbb{Z}_M)$ is isomorphic to the singular cohomology group $H^2(M, \mathbb{Z})$, which in turn maps to the de Rham cohomology group $H^2(M, \mathbb{C})$. The class in $H^2(M, \mathbb{Z})$ associated to (the isomorphism class of) a holomorphic line bundle L is called the first *Chern class* of L, and is denoted by $c_1(L)$. The following lemma shows that $c_1(L)$ can also be computed from Θ_L.

Lemma 21.5. *We have* $c_1(L) = \frac{i}{2\pi}[\Theta_L]$, *as elements of* $H^2(M, \mathbb{C})$.

Proof. The main issue is to transform the element $c_1(L)$ from a class in Čech cohomology to a class in de Rham cohomology. Since we did not prove Theorem 11.6, we will just go through the procedure here without justifying it. We begin by covering M by open sets U_α over which L is trivial. Choose holomorphic functions $f_{\alpha,\beta} \in \mathscr{O}_M(U_\alpha \cap U_\beta)$ lifting the transition functions $g_{\alpha,\beta}$ under the map exp, meaning that $g_{\alpha,\beta} = e^{2\pi i f_{\alpha,\beta}}$; then $c_1(L)$ is the class of the 2-cocycle

$$c_{\alpha,\beta,\gamma} = f_{\beta,\gamma} - f_{\alpha,\gamma} + f_{\alpha,\beta}.$$

To turn this cocycle into a class in de Rham cohomology, let $1 = \sum \rho_\alpha$ be a partition of unity subordinate to the cover. Since $c_{\alpha,\beta,\gamma}$ are locally constant, $df_{\alpha,\beta}$ is a 1-cocycle for the sheaf $\mathscr{A}_M^1$. Using the partition of unity, we define

$$\varphi_\alpha = \sum_\gamma \rho_\gamma \cdot df_{\gamma,\alpha} \in A^1(U_\alpha)$$

which is easily seen to satisfy $df_{\alpha,\beta} = \varphi_\beta - \varphi_\alpha$. Thus the forms $d\varphi_\alpha \in A^2(U_\alpha)$ agree on the overlaps between open sets, and thus define a global 2-form that is closed and represents the image of $c_1(L)$ in $H^2(M, \mathbb{C})$.

Now choose a Hermitian metric h on L, and let $h_\alpha = h(s_\alpha, s_\alpha)$ be the resulting local functions. From the relation $s_\beta = g_{\alpha,\beta} s_\alpha$, we find that $h_\beta = |g_{\alpha,\beta}|^2 h_\alpha$, and hence

$$\theta_\beta - \theta_\alpha = \partial \log h_\beta - \partial \log h_\alpha = \partial \log\bigl(|g_{\alpha,\beta}|^2\bigr) = \frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}} = 2\pi i \cdot df_{\alpha,\beta}.$$

This means that we have

$$\varphi_\alpha = \frac{i}{2\pi} \sum_\gamma \rho_\gamma(\theta_\alpha - \theta_\gamma) = \frac{i}{2\pi}\theta_\alpha - \psi,$$

where $\psi = \frac{i}{2\pi} \sum \rho_\gamma \theta_\gamma \in A^1(M)$. Remembering that $\Theta_\alpha = \bar{\partial}\theta_\alpha = d\theta_\alpha$, we now get

$$d\varphi_\alpha = \frac{i}{2\pi}\Theta_\alpha + d\psi,$$

which shows that $\frac{i}{2\pi}\Theta_L$ represents the same cohomology class as $c_1(L)$. $\square$

Note. Since Θ_L is a form of type $(1,1)$, the first Chern class $c_1(L)$ is an example of a *Hodge class*: a class in $H^{2p}(M, \mathbb{Z})$ whose image in $H^{2p}(M, \mathbb{C})$ belongs to the subspace $H^{p,p}$ in the Hodge decomposition. In fact, the kind of argument just given proves that any Hodge class in $H^2(M, \mathbb{Z})$ is the first Chern class of a holomorphic line bundle on M, a fact that is known as the Lefschetz $(1,1)$-theorem. To see how this works, consider the diagram

$$\begin{array}{ccccc} H^1(M, \mathscr{O}_M^*) & \xrightarrow{c_1} & H^2(M, \mathbb{Z}) & \longrightarrow & H^2(M, \mathscr{O}_M) \\ & & \downarrow & \nearrow & \\ & & H^2(M, \mathbb{C}) & & \end{array}$$

in which the first row is exact (as part of a long exact sequence). Under the assumption that M is a Kähler manifold, we have the Hodge decomposition $H^2(M,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$; moreover, $H^2(M, \mathscr{O}_M) \simeq H^{0,2}$, and under this identification, the diagonal map in the diagram is the projection map. Note that any $\alpha \in H^2(M,\mathbb{C})$ in the image of $H^2(M,\mathbb{Z})$ is real, and hence satisfies $\alpha^{2,0} = \overline{\alpha^{0,2}}$. Thus a class in $H^2(M,\mathbb{Z})$ is the first Chern class of a holomorphic line bundle iff its image in $H^2(M,\mathbb{C})$ is of type $(1,1)$.

Examples. On $\mathbb{P}^n$, we have the tautological line bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$, described as follows: by definition, each point of $\mathbb{P}^n$ corresponds to a line in $\mathbb{C}^{n+1}$, which we take to be the fiber of $\mathscr{O}_{\mathbb{P}^n}(-1)$ over that point. In other words, the fiber of $\mathscr{O}_{\mathbb{P}^n}(-1)$ over the point $[z_0, z_1, \dots, z_n]$ is the line $\mathbb{C} \cdot (z_0, z_1, \dots, z_n)$. This makes $\mathscr{O}_{\mathbb{P}^n}(-1)$ a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$, and gives it a natural Hermitian metric (induced from the standard metric on the trivial bundle).

To compute the associated curvature form $\Theta \in A^{1,1}(\mathbb{P}^n)$, let $U_0 \simeq \mathbb{C}^n$ be one of the standard open sets; then $[1, z_1, \dots, z_n] \mapsto (1, z_0, \dots, z_n)$ defines a holomorphic section s_0 of our line bundle on U_0, with norm

$$h_0 = h(s_0, s_0) = 1 + |z_1|^2 + \cdots + |z_n|^2.$$

Consequently,

$$\frac{i}{2\pi}\Theta_0 = -\frac{i}{2\pi}\partial\bar{\partial}\log h_0 = -\frac{i}{2\pi}\partial\bar{\partial}\log\bigl(1 + |z_1|^2 + \cdots + |z_n|^2\bigr) = -\omega_{FS}|_{U_0}$$

is the negative of the Fubini-Study form. In particular, this shows that ω_{FS} equals the first Chern class of the dual line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$.

For another example, let M be a complex manifold of dimension n with a Hermitian metric h; in other words, h is a Hermitian metric on the holomorphic tangent bundle $T'M$. Now consider the so-called *canonical bundle* Ω_M^n, whose sections over an open set U are the holomorphic n-forms on U. Locally, any such section can be written in the form $f(z) dz_1 \wedge \cdots \wedge dz_n$, with f holomorphic. Since Ω_M^n can be viewed as the n-th wedge power of the dual of $T'M$, it inherits a Hermitian metric. In local coordinates $z_1, \dots, z_n$, we define as usual a matrix H with entries

$$h_{j,k} = h\bigl(\partial/\partial z_j, \partial/\partial z_k\bigr),$$

and then $h^{j,k} = h(dz_j, dz_k)$ are the entries of the inverse matrix H^{-1}. The induced Hermitian metric on Ω_M^n satisfies

$$h(dz_1 \wedge \cdots \wedge dz_n, dz_1 \wedge \cdots \wedge dz_n) = \det H^{-1} = -\det H,$$

and therefore its curvature form is given by

$$\Theta = -\partial\bar{\partial}\log h(dz_1 \wedge \cdots \wedge dz_n, dz_1 \wedge \cdots \wedge dz_n) = \partial\bar{\partial}\log(\det H).$$