Class 2. The Weierstrass theorems (September 3)

Germs of holomorphic functions. In one complex variable, the local behavior of holomorphic functions is very simple. Consider a holomorphic function $f(z)$ that is defined on some open neighborhood of $0 \in \mathbb{C}$. Then we can uniquely write $f(z) = u(z) \cdot z^k$, where $k \in \mathbb{N}$ and $u(z)$ is a unit, meaning that $u(0) \neq 0$, or equivalently that $1/u(z)$ is holomorphic near the origin. In several variables, the situation is much more complicated.

Fix an integer $n \geq 0$. We begin the local study of holomorphic functions in n variables by recalling the notion of a germ. Consider holomorphic functions $f \in \mathscr{O}(U)$ that are defined in some neighborhood U of the origin in $\mathbb{C}^n$. We say that $f \in \mathscr{O}(U)$ and $g \in \mathscr{O}(V)$ are equivalent if there is an open set $W \subseteq U \cap V$, containing the origin, such that $f|_W = g|_W$. The equivalence class of $f \in \mathscr{O}(U)$ is called the *germ* of f at $0 \in \mathbb{C}^n$. We denote the set of all germs of holomorphic functions by $\mathscr{O}_n$. Obviously, germs of holomorphic functions can be added and multiplied, and so $\mathscr{O}_n$ is a (commutative) ring. We have $\mathbb{C} \subseteq \mathscr{O}_n$ through the germs of constant functions.

The ring $\mathscr{O}_n$ can be described more formally as the direct limit

$$\mathscr{O}_n = \lim_{U \ni 0} \mathscr{O}(U),$$

where U ranges over all open neighborhoods of $0 \in \mathbb{C}^n$, ordered by inclusion. For $V \subseteq U$, we have the restriction map $\mathscr{O}(U) \to \mathscr{O}(V)$, and the limit is taken with respect to this family of maps.

Either way, we think of $f \in \mathscr{O}_n$ as saying that f is a holomorphic function on some (unspecified) neighborhood of the origin in $\mathbb{C}^n$. Note that the value $f(0) \in \mathbb{C}$ is well-defined for germs, but the same is not true at other points of $\mathbb{C}^n$. By definition, a function $f \in C(U)$ is holomorphic at $0 \in \mathbb{C}^n$ if it can be expanded into a convergent power series $\sum c_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}$. It follows immediately that

$$\mathscr{O}_n \simeq \mathbb{C}\{z_1, \dots, z_n\}$$

is isomorphic to the ring of convergent power series in the variables $z_1, \dots, z_n$.

Example 2.1. For $n = 0$, we have $\mathscr{O}_0 \simeq \mathbb{C}$. For $n = 1$, we have $\mathscr{O}_1 \simeq \mathbb{C}\{z\}$. The simple local form of holomorphic functions in one variable corresponds to the simple algebraic structure of the ring $\mathbb{C}\{z\}$: it is a discrete valuation ring, meaning that all of its ideals are of the form (z^k) for $k \in \mathbb{N}$.

As in the example, the philosophy behind the local study of holomorphic functions is to relate local properties of holomorphic functions to algebraic properties of the ring $\mathscr{O}_n$. This is the purpose of the next few lectures.

The first observation is that $\mathscr{O}_n$ is a semi-local[1] ring. Recall that a ring A is called *semi-local* if it has a unique maximal ideal $\mathfrak{m}$, and every element $a \in A$ is either a unit (meaning that it has an inverse a^{-1} in A), or belongs to $\mathfrak{m}$. The unique maximal ideal is

$$\mathfrak{m}_n = \{ f \in \mathscr{O}_n \mid f(0) = 0 \};$$

indeed, if $f \in \mathscr{O}_n$ satisfies $f(0) \neq 0$, then $1/f$ is holomorphic in a neighborhood of the origin, and therefore $1/f \in \mathscr{O}_n$. Note that the residue field $\mathscr{O}_n/\mathfrak{m}_n$ is isomorphic

[1] $\mathscr{O}_n$ is even a local ring, since it is also Noetherian (meaning that every ideal is finitely generated); however, it will takes us some time to prove this.

to $\mathbb{C}$. We digress to point out that integer n (the dimension of $\mathbb{C}^n$) can be recovered from the ring $\mathscr{O}_n$, because of the following lemma.

Lemma 2.2. *The quotient $\mathfrak{m}_n/\mathfrak{m}_n^2$ is a complex vector space of dimension n.*

Proof. Since $\mathscr{O}_n \simeq \mathbb{C}\{z_1, \dots, z_n\}$, the maximal ideal is generated by $z_1, \dots, z_n$, and their images give a basis for the quotient $\mathfrak{m}_n/\mathfrak{m}_n^2$. □

Here is another basic property of the ring $\mathscr{O}_n$.

Proposition 2.3. *The ring $\mathscr{O}_n$ is a domain.*

Proof. We have to show that there are no nontrivial zero-divisors in $\mathscr{O}_n$. So suppose that we have $f, g \in \mathscr{O}_n$ with $fg = 0$ and $g \neq 0$. Let $\Delta(0; r)$ be a polydisk on which both f and g are holomorphic functions. Since $g \neq 0$, there is some point $a \in \Delta(0; r)$ with $g(a) \neq 0$; then g is nonzero, and therefore f is identically zero, in some neighborhood of a. By Theorem 1.8, it follows that $f(z) = 0$ for every $z \in \Delta(0; r)$; in particular, $f = 0$ in $\mathscr{O}_n$. □

Weierstraß polynomials. To get at the deeper properties of the ring $\mathscr{O}_n$, we have to study the local structure of holomorphic functions more carefully. We will proceed by induction on $n \geq 0$, by using the inclusions of rings

$$\mathscr{O}_{n-1} \subseteq \mathscr{O}_{n-1}[z_n] \subseteq \mathscr{O}_n. \tag{2.4}$$

Elements of the intermediate ring are polynomials of the form $z_n^k + a_1 z_n^{k-1} + \cdots + a_k$ with coefficients $a_1, \dots, a_k \in \mathscr{O}_{n-1}$; they are obviously holomorphic germs. The first inclusion in (2.4) is a simple algebraic extension; we will see that the second one is of a more analytic nature.

Throughout this section, we write the coordinates on $\mathbb{C}^n$ in the form $z = (w, z_n)$, so that $w = (z_1, \dots, z_{n-1})$. To understand the second inclusion in (2.4), we make the following definition.

Definition 2.5. An element $h = z_n^d + a_1 z_n^{d-1} + \cdots + a_d \in \mathscr{O}_{n-1}[z_n]$ with $d \geq 1$ is called a *Weierstraß polynomial* if $a_1, \dots, a_d \in \mathfrak{m}_{n-1}$.

In analogy with the one-variable case, we will show that essentially every $f \in \mathscr{O}_n$ can be written in the form $f = uh$ with h a Weierstraß polynomial and u a unit. This statement has to be qualified, however, because we have $h(0, z_n) = z_n^d$, which means that if $f = uh$, then the restriction of f to the line $w = 0$ cannot be identically zero.

Definition 2.6. Let U be an open neighborhood of $0 \in \mathbb{C}^n$, and $f \in \mathscr{O}(U)$. We say that f is *regular* (in z_n) if the holomorphic function $f(0, z_n)$ is not identically equal to zero.

If f is regular, we can write $f(0, z_n) = u(z_n) z_n^d$, where d is the order of vanishing of $f(0, z_n)$ at the origin, and $u(0) \neq 0$. We may summarize this by saying that f is *regular of order d*. The notion of regularity also makes sense for elements of $\mathscr{O}_n$, since it only depends on the behavior of f in arbitrarily small neighborhoods of the origin.

The preparation theorem. We continue to write the coordinates on $\mathbb{C}^n$ in the form $z = (w, z_n)$.

Recall that a function $f \in \mathscr{O}_n$ is said to be regular in z_n if $f(0, z_n)$ is not identically equal to zero. Of course, not every holomorphic function is regular (for instance, z_j for $j < n$ is not), but if $f \neq 0$, then we can always make it regular by changing the coordinate system.

Lemma 2.7. *Given finitely many nonzero elements of $\mathscr{O}_n$, there is a linear change of coordinates that makes all of them regular in the variable z_n.*

Proof. By taking the product of the finitely many germs, we reduce to the case of a single $f \in \mathscr{O}_n$. Since $f \neq 0$, there is some vector $a \in \mathbb{C}^n$ such that the holomorphic function $f(t \cdot a)$ is not identically zero for $t \in \mathbb{C}$ sufficiently close to 0. After making a change of basis in the vector space $\mathbb{C}^n$, we can assume that $a = (0, \dots, 0, 1)$; but then $f(0, z_n)$ is not identically zero, proving that f is regular in z_n. □

The following fundamental result is known as the Weierstraß preparation theorem; it is the key to understanding the second inclusion in (2.4).

Theorem 2.8 (Weierstrass Preparation Theorem)**.** *If $f \in \mathscr{O}_n$ is regular of order d in the variable z_n, then there exists a unique Weierstraß polynomial $h \in \mathscr{O}_{n-1}[z_n]$ of degree d such that $f = uh$ for some unit $u \in \mathscr{O}_n$.*

The idea of the proof is quite simple: Fix $w \in \mathbb{C}^n$ sufficiently close to 0, and consider $f_w(z_n) = f(w, z_n)$ as a holomorphic function of z_n. Since $f_0(z_n)$ vanishes to order d when z_n, each $f_w(z_n)$ will have exactly d zeros (counted with multiplicities) close to the origin; call them $\zeta_1(w), \dots, \zeta_d(w)$. Now if $f = uh$ for a unit u and a monic polynomial h, then we should have $h_w(z_n) = \big(z_n - \zeta_1(w)\big) \cdots \big(z_n - \zeta_d(w)\big)$. The main point is to show that, after expanding this into a polynomial, the coefficients are holomorphic functions of w. Here is the rigorous proof.

Proof. The germ $f \in \mathscr{O}_n$ can be represented by a holomorphic function on some neighborhood of $0 \in \mathbb{C}^n$. We begin by constructing the required Weierstraß polynomial h. Since f is regular in the variable z_n, we have $f(0, z_n) \neq 0$ for sufficiently small $z_n \neq 0$. We can therefore find $r > 0$ and $\delta > 0$ with the property that $|f(0, z_n)| \geq \delta$ for $|z_n| = r$; because f is continuous, we can then choose $\varepsilon > 0$ such that $|f(w, z_n)| \geq \delta/2$ as long as $|z_n| = r$ and $|w| \leq \varepsilon$.

For any fixed $w \in \mathbb{C}^{n-1}$ with $|w| \leq \varepsilon$, consider the integral

$$N(w) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{(\partial f/\partial z_n)(w, \zeta)}{f(w, \zeta)} d\zeta;$$

by the residue theorem, it counts the zeros of the holomorphic function $f(w, \zeta)$ inside the disk $|\zeta| < r$ (with multiplicities). We clearly have $N(0) = d$, and so by continuity, $N(w) = d$ whenever $|w| \leq \varepsilon$. We can therefore define $\zeta_1(w), \dots, \zeta_d(w)$ to be those zeros (in any order). We also set

$$h(w, z_n) = \prod_{j=1}^{d} \big(z_n - \zeta_j(w)\big) = z_n^d - \sigma_1(w) z_n^{d-1} + \cdots + (-1)^d \sigma_d(w),$$

where $\sigma_1(w), \dots, \sigma_d(w)$ are the elementary symmetric polynomials in the roots $\zeta_1(w), \dots, \zeta_d(w)$: that is to say,

$$\sigma_1(w) = \sum_{j=1}^{d} \zeta_j(w), \quad \dots, \quad \sigma_d(w) = \prod_{j=1}^{d} \zeta_j(w).$$

Of course, each $\zeta_j(w)$ by itself is probably not holomorphic (or even continuous) in w. But by invoking the residue theorem one more time, we see that

$$\zeta_1(w)^k + \dots + \zeta_d(w)^k = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\zeta^k \cdot (\partial f/\partial z_n)(w,\zeta)}{f(w,\zeta)} d\zeta,$$

which is holomorphic in w (this can be seen by differentiating under the integral sign). By Newton's formulas, the elementary symmetric polynomials $\sigma_j(w)$ are therefore holomorphic functions in w as well; it follows that h is holomorphic for $|w| < \varepsilon$ and $|z_n| < r$. The regularity of f implies that $\sigma_j(0) = 0$ for all j, and therefore h is a Weierstraß polynomial of degree d.

For $|w| < \varepsilon$ and $|z_n| < r$, we consider the quotient

$$u(w, z_n) = \frac{f(w, z_n)}{h(w, z_n)},$$

which is a holomorphic function outside the zero set of h. For fixed w, the singularities of the function $u(w, z_n)$ inside the disk $|z_n| < r$ are removable by construction, and so $u(w, z_n)$ is holomorphic in z_n. But by the Cauchy integral formula, we then have

$$u(w, z_n) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{u(w,\zeta)}{\zeta - z_n} d\zeta,$$

and so u is actually a holomorphic function of (w, z_n). To conclude that u is a unit, note that $u(0, z_n) = f(0, z_n)/h(0, z_n) = f(0, z_n)/z_n^d$, whose value at 0 is nonzero by assumption. We now have the desired representation $f = uh$ where h is a Weierstraß polynomial and u a unit.

The uniqueness of the Weierstraß polynomial for given f is clear: indeed, since u is a unit, $h(w, z_n)$ necessarily has the same zeros as $f(w, z_n)$ for every $w \in \mathbb{C}^{n-1}$ near the origin, and so its coefficients have to be given by the $\sigma_j(w)$, which are uniquely determined by f. □

The preparation theorem allows us to deduce one important property of the ring $\mathscr{O}_n$, namely that it has unique factorization. Recall that in a domain A, an element $a \in A$ is called *irreducible* if in any factorization $a = bc$, either b or c has to be a unit. Moreover, A is called a *unique factorization domain* (UFD) if every nonzero element $a \in A$ can be uniquely factored into a product of finitely many irreducible elements, each unique up to units.

Theorem 2.9. *The ring $\mathscr{O}_n$ is a unique factorization domain.*

Proof. We argue by induction on $n \geq 0$; the case $n = 0$ is trivial since $\mathscr{O}_0 \simeq \mathbb{C}$ is a field. We may suppose that $\mathscr{O}_{n-1}$ is a UFD; by Gauß' lemma, the polynomial ring $\mathscr{O}_{n-1}[z_n]$ is then also a UFD. Let $f \in \mathscr{O}_n$ be any nonzero element; without loss of generality, we may assume that it is regular in z_n. According to Theorem 2.8, we have $f = uh$ for a unique Weierstraß polynomial $h \in \mathscr{O}_{n-1}[z_n]$.

Now suppose that we have a factorization $f = f_1 f_2$ in $\mathscr{O}_n$. Then each f_j is necessarily regular in z_n, and can therefore be written as $f_j = u_j h_j$ with h_j a Weierstraß polynomial and u_j a unit. Then

$$uh = f = (u_1 u_2) \cdot (h_1 h_2),$$

and the uniqueness part of Theorem 2.8 shows that $h = h_1 h_2$. Existence and uniqueness of a factorization for f are thus reduced to the corresponding problems for h in the ring $\mathscr{O}_{n-1}[z_n]$; but $\mathscr{O}_{n-1}[z_n]$ is already known to be a UFD. □

The division theorem. The next result is the so-called Weierstraß division theorem; it shows that one can do long division with Weierstraß polynomials, in the same way as in the ring $\mathbb{C}[z]$. We continue to write the coordinates on $\mathbb{C}^n$ in the form $z = (w, z_n)$, in order to do to induction on $n \geq 0$.

Theorem 2.10 (Weierstrass Division Theorem)**.** *Let $h \in \mathscr{O}_{n-1}[z_n]$ be a Weierstraß polynomial of degree d. Then any $f \in \mathscr{O}_n$ can be uniquely written in the form $f = qh + r$, where $q \in \mathscr{O}_n$, and $r \in \mathscr{O}_{n-1}[z_n]$ is a polynomial of degree $< d$. Moreover, if $f \in \mathscr{O}_{n-1}[z_n]$, then also $q \in \mathscr{O}_{n-1}[z_n]$.*

Proof. As in the proof of Theorem 2.8, we can choose $\rho, \varepsilon > 0$ sufficiently small, to insure that for each fixed $w \in \mathbb{C}^{n-1}$ with $|w| < \varepsilon$, the polynomial $h(w, z_n)$ has exactly d zeros in the disk $|z_n| < \rho$. For $|z_n| < \rho$ and $|w| < \varepsilon$, we may then define

$$q(w, z_n) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(w, \zeta)}{h(w, \zeta)} \frac{d\zeta}{\zeta - z_n}.$$

The idea is that if f was actually divisible by h – so that f/h was holomorphic – then by the Cauchy integral formula, the above integral would give us exactly f/h. As usual, differentiation under the integral sign shows that q is holomorphic; hence $r = f - qh$ is holomorphic as well. The function r can also be written as an integral,

$$\begin{aligned} r(w, z_n) &= f(w, z_n) - q(w, z_n) h(w, z_n) \\ &= \frac{1}{2\pi i} \int_{|\zeta| = \rho} \left(f(w, \zeta) - h(w, z_n) \frac{f(w, \zeta)}{h(w, \zeta)} \right) \frac{d\zeta}{\zeta - z_n} \\ &= \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(w, \zeta)}{h(w, \zeta)} \cdot p(z, \zeta, z_n) d\zeta, \end{aligned}$$

where we have introduced the new function

$$p(w, \zeta, z_n) = \frac{h(w, \zeta) - h(w, z_n)}{\zeta - z_n}.$$

Now $h \in \mathscr{O}_{n-1}[z_n]$ is a monic polynomial of degree d, and so $\zeta - z_n$ divides the numerator; therefore $p \in \mathscr{O}_{n-1}[z_n]$ is monic of degree $d - 1$. This means that we can write

$$p(w, \zeta, z_n) = a_0(w, \zeta) z_n^{d-1} + a_1(w, \zeta) z_n^{d-2} + \cdots + a_{d-1}(w, \zeta),$$

where $a_0(w, \zeta) \equiv 1$ (to simplify the notation later on). We then have $r(w, z_n) = b_0(w) z_n^{d-1} + b_1(w) z_n^{d-2} + \cdots + b_{d-1}(w)$, where the coefficients are given by the integrals

$$b_j(w) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(w, \zeta)}{h(w, \zeta)} \cdot a_j(w, \zeta) d\zeta.$$

This proves that $r \in \mathscr{O}_{n-1}[z_n]$ is a polynomial of degree $< d$, and completes the main part of the proof.

To prove the uniqueness of q and r, it suffices to consider the case $f = 0$. Suppose then that we have $0 = qh + r$, where $r \in \mathscr{O}_{n-1}[z_n]$ has degree $< d$. For fixed w with $|w| < \varepsilon$, the function $r(w, z_n) = -q(w, z_n)h(w, z_n)$ has at least d zeros in the disk $|z_n| < \varepsilon$; but since it is a polynomial in z_n of degree $< d$, this can only happen if $r = 0$, and hence $q = 0$.

Finally, suppose that $f \in \mathscr{O}_{n-1}[z_n]$. Because h is monic, we can apply the division algorithm for polynomials to obtain $f = q'h + r'$ with $q', r' \in \mathscr{O}_{n-1}[z_n]$. By uniqueness, $q' = q$ and $r' = r$, and so q is a polynomial in that case. $\square$