Class 16. Representations of $\mathfrak{sl}_2(\mathbb{C})$ (October 24)

We are still working towards the proof of the Kähler identities. Recall that M is an n-dimensional Kähler manifold, and that ω is the Kähler form of a Kähler metric on M. Last time, we introduced the three operators

$$L(\alpha) = \omega \wedge \alpha, \quad \Lambda(\alpha) = (-1)^{\deg \alpha} * L * \alpha, \quad H(\alpha) = \big(\deg \alpha - n\big)\alpha,$$

and showed that they satisfy the three relations $[H, L] = 2L$, $[H, \Lambda] = -2\Lambda$, and $[L, \Lambda] = H$. We concluded that, therefore, they determine a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on the vector space

$$A^{n+*}(M) = \bigoplus_{k=-n}^{n} A^{n+k}(M).$$

The point of writing the space of forms in this way is that $A^{n+k}(M)$ is exactly the k-eigenspace of the operator H. Today, we are mostly going to review general facts about the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$; the Kähler identities will appear again at the end. Recall from last time that $\mathfrak{sl}_2(\mathbb{C})$ is 3-dimensional, and is spanned by the three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The relations among E, F, and H are the following:

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F$$

A *representation* of $\mathfrak{sl}_2(\mathbb{C})$ on a complex vector space V is a linear mapping

$$\rho \colon \mathfrak{sl}_2 \to \operatorname{End}_{\mathbb{C}}(V)$$

that is compatible with taking commutators; the representation is of course determined by the three linear operators $\rho(H)$, $\rho(E)$, and $\rho(F)$. We usually suppress ρ and denote these operators simply by H, E, and F.

A representation is called *irreducible* if it does not contain any nontrivial subspace that is invariant under the action by H, E, and F. A basic fact is that every finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is a direct sum of irreducible representations. The fancy reason is that $\mathfrak{sl}_2(\mathbb{C})$ is the complexification of the real Lie algebra $\mathfrak{su}_2$, and that every finite-dimensional representation of $\mathfrak{su}_2$ lifts to a representation of the Lie group SU_2 (which, as a manifold, is just $\mathbb{S}^3$). Because SU_2 is compact, one can then average over the group to obtain an invariant inner product, which can be used to decompose V into irreducible representations. So much for the general theory; we won't need it, because we can easily prove all these facts by hand.

Because we are really interested in $A^{n+*}(M)$, we are only going to consider representations V with the following property: for every $v \in V$, one has $E^n v = 0$ and $F^n v = 0$ for $n \gg 0$. This happens for example when V is finite-dimensional; our representation is not finite-dimensional, but it still has this property. For the sake of today's discussion, let's call such representations *bounded*.

We first analyze the structure of irreducible representations. Suppose that V is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ that is bounded in the above sense. Choose a nonzero eigenvector $v \in V$ for H, say with $Hv = \lambda v$. For the time being, $\lambda \in \mathbb{C}$ can be any complex number, but we will see in a moment that λ actually has to be an integer. From the two relations $[H,E]=2E$ and $[H,F]=-2F$, we get

$$H(Ev) = [H,E]v + E(Hv) = 2Ev + E(\lambda v) = (\lambda+2)Ev,$$
$$H(Fv) = [H,F]v + F(Hv) = -2Fv + F(\lambda v) = (\lambda-2)Fv,$$

and so E maps the λ-eigenspace $E_\lambda(H)$ into $E_{\lambda+2}(H)$, whereas F maps $E_\lambda(H)$ into $E_{\lambda-2}(H)$. Because $F^n v = 0$ for $n \gg 0$, we can look at the minimal value of n such that $F^n v \neq 0$ but $F^{n+1}v = 0$. After replacing v by $F^n v$, we may therefore assume in addition that $Fv = 0$. So from now on, $Hv = \lambda v$ and $Fv = 0$.

Now consider the sequence of vectors

$$v, Ev, E^2 v, \dots, E^n v, \dots$$

The action of E takes each vector to the next one; the following lemma describes the action of F. We write the formulas in terms of $-\lambda$ because it will turn out soon that $\lambda \leq 0$.

Lemma 16.1. *If $Hv = \lambda v$ and $Fv = 0$, then one has*

$$FE^n v = n(-\lambda - n + 1)E^{n-1}v,$$
$$F^n E^n v = (n!)^2 \binom{-\lambda}{n} v,$$

for every $n \geq 0$.

Proof. We prove the first identity by induction on $n \geq 0$, with the case $n = 0$ being trivial. For $n \geq 0$, the relation $EF - FE = H$ gives

$$FE^{n+1}v = -HE^n v + EFE^n v = -(\lambda + 2n)E^n v + n(-\lambda - n + 1)E^n v$$
$$= (n+1)(-\lambda - n)E^n v,$$

as required. From this, we get

$$F^{n+1}E^{n+1}v = (n+1)(-\lambda - n)F^n E^n v,$$

and so the second identity again follows by induction on $n \geq 0$. □

Now let $n \geq 0$ be the unique integer such that $E^n v \neq 0$ and $E^{n+1}v = 0$; this exists for the same reason as above. The lemma shows that the subspace generated by $v, Ev, \dots, E^n v$ is stable under the action of H, E, and F; as V is an irreducible representation, it follows that

$$V = \mathbb{C}\langle v, Ev, \dots, E^n v\rangle,$$

and therefore $\dim V = n+1$. Now we can easily show that λ must be a nonpositive integer. Again from the lemma,

$$0 = FE^{n+1}v = (n+1)(-\lambda - n)E^n v,$$

and because $E^n v \neq 0$, we get $\lambda = -n$. The following proposition summarizes what we have found about the structure of irreducible representations.

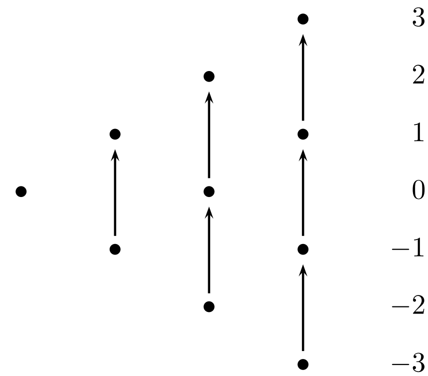
Proposition 16.2. *If V is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with $\dim V = n+1$, then there is a nonzero vector $v \in V$ with $Hv = -nv$ and $Fv = 0$, such that*

$$V = \mathbb{C}\langle v, Ev, \dots, E^n v\rangle.$$

In this basis, the operator H is diagonal, with eigenvalues $-n, -n+2, \dots, n$; and the operators E and F are given by the formulas

$$E(E^k v) = E^{k+1}v, \quad F(E^k v) = k(n-k+1)E^{k-1}v.$$

In particular, $\mathfrak{sl}_2(\mathbb{C})$ has, up to isomorphism, a unique irreducible representation of every finite dimension. The simplest example is the *standard representation* on $\mathbb{C}^2$, where H, E, and F act as 2×2-matrices; the eigenvalues of H are of course -1 and 1. The following schematic picture shows a typical representation:



Each dot stands for a one-dimensional subspace, the vertical arrows indicate the action of E, and the numbers on the right are the eigenvalues of H, which are usually called the *weights* of the representation. Note that the weights are symmetric around 0.

The Lefschetz decomposition. Let us now prove that every representation is a direct sum of irreducible representations. Changing the notation slightly, suppose

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$

is a graded vector space that is also a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We assume that each $V_k = E_k(H)$ is exactly the k-eigenspace of H; it follows that $E(V_k) \subseteq V_{k+2}$ and $F(V_k) \subseteq V_{k-2}$. We are going to impose the same boundedness condition as before: for every $v \in V$, one has $E^n v = 0$ and $F^n v = 0$ for $n \gg 0$.

Definition 16.3. A nonzero vector $v \in V_k$ is called *primitive* if $Fv = 0$. We saw above that primitive vectors only exist for $k \leq 0$.

We saw above that each irreducible representation is spanned by a primitive vector and its images under powers of E. The following result generalizes this to arbitrary representations (under the usual boundedness condition).

Proposition 16.4. *Every vector $v \in V_k$ can be uniquely written as a finite sum*

$$v = \sum_{j \geq \max(k,0)} \frac{E^j}{j!} v_j$$

with $v_j \in V_{k-2j} \cap \ker F$, called the Lefschetz decomposition*.*

The proof that I presented in class was incomplete; here is the correct argument. (The point is that we need to use the boundedness condition for both E and F, because the statement is wrong for unbounded representations.) Let me first explain where the range of the summation comes from. We have seen above that if $v \in V$ is a nonzero vector with $Hv = -nv$ and $Fv = 0$, then one has $E^n v \neq 0$ and

$E^{n+1}v = 0$. Applied to $v_j \in V_{k-2j}$, this says that $E^{2j-k+1}v_j = 0$, and so $E^j v_j = 0$ whenever $j \geq 2j - k + 1$. This is why only terms with $j \geq k$ appear in the sum.

Now for the proof. The uniqueness of the decomposition is easy. By linearity, we only need to show that if we have a finite sum

$$0 = \sum_{j \geq \max(k,0)} \frac{E^j}{j!} v_j$$

for certain vectors $v_j \in V_{k-2j} \cap \ker F$, then every $v_j = 0$. If we apply F and use Lemma 16.1, we get

$$0 = \sum_{j \geq \max(k,1)} (j + 1 - k) \frac{E^{j-1}}{(j-1)!} v_j,$$

because $F v_0 = 0$. This sum has fewer terms, and so by induction, we get $v_j = 0$ for $j \geq \max(k, 1)$. But then also $v_0 = 0$. This gives uniqueness.

To show that every vector $v \in V_k$ has a Lefschetz decomposition, we have to work a bit harder. First, we prove the analogue of Lemma 16.1 by the same method.

Lemma 16.5. *If $Hv = \lambda v$ and $Ev = 0$, then $\lambda \in \mathbb{N}$, and one has*

$$EF^n v = n(\lambda - n + 1)F^{n-1}v,$$
$$E^n F^n v = (n!)^2 \binom{\lambda}{n} v,$$

for every $n \geq 0$.

For $v \in V_k$ with $k \geq 1$, the proposition is claiming in particular that v lies in the image of $E^k \colon V_{-k} \to V_k$. The next step is to prove this assertion.

Lemma 16.6. *The mapping $E^k \colon V_{-k} \to V_k$ is surjective for $k \geq 1$.*

Proof. Suppose that $v \in V_k$ with $k \geq 1$. By our boundedness assumption, there is a unique $n \in \mathbb{N}$ with $E^n v \neq 0$ and $E^{n+1}v = 0$. The proof is by induction on n. If $n = 0$, then $Ev = 0$. The formulas in Lemma 16.5 (with $\lambda = k$) give

$$E^k F^k v = (k!)^2 v$$

and so v lies in the image of E^k. In the general case, we have $E^{n+1}v = 0$, and so $E^n v \in V_{k+2n} \cap \ker E$. We use Lemma 16.5 again, this time with $\lambda = k + 2n$, to get

$$E^{k+2n} F^{k+2n} E^n v = ((k + 2n)!)^2 E^n v.$$

This shows that $v - \frac{1}{((k+2n)!)^2} E^{k+n} F^{k+2n} E^n v \in \ker E^n$; by induction, this vector lies in the image of E^k, and so the same is true for v itself. □

With this lemma in hand, it is now enough to consider the case where $v \in V_k$ with $k \leq 0$. Here the proposition is claiming that there is a finite sum

$$v = \sum_{j \geq 0} \frac{E^j}{j!} v_j \tag{16.7}$$

with $v_j \in V_{k-2j} \cap \ker F$. As before, there is a unique $n \in \mathbb{N}$ such that $F^n v \neq 0$ and $F^{n+1}v = 0$, and we prove the existence of the decomposition by induction on n. In the case $n = 0$, we have $Fv = 0$, and so v itself primitive (and is therefore its own Lefschetz decomposition). In general, consider $Fv \in V_{k-2}$. By induction,

$$Fv = \sum_{j \geq 0} \frac{E^j}{j!} v_j'$$

with $v_j' \in V_{k-2-2j} \cap \ker F$. If a decomposition as in (16.7) exists, then after applying F to it and using Lemma 16.1, we would get

$$Fv = \sum_{j \geq 1} (j+1-k) \frac{E^{j-1}}{(j-1)!} v_j.$$

By comparing the two formulas, we see that we need to take $v_j = \frac{1}{j+1-k} v_{j-1}'$ for $j \geq 1$; the denominator is nonzero because $k \leq 0$. With this choice, the vector

$$v_0 = v - \sum_{j \geq 1} \frac{E^j}{j!} v_j$$

belongs to $\ker F$, and so we obtain the desired decomposition for v. This completes the proof of Proposition 16.4.

The Lefschetz decomposition reduces most questions about arbitrary vectors in an $\mathfrak{sl}_2(\mathbb{C})$-representation to the special case of primitive vectors. One example is the symmetry of the weight spaces V_k around $k = 0$.

Corollary 16.8. *For every $k \geq 1$, the map $E^k \colon V_{-k} \to V_k$ is an isomorphism.*

Proof. We showed in Lemma 16.6 that E^k is surjective. Injectivity follows from the uniqueness of the Lefschetz decomposition. □

The Weyl element. In a bounded representation of $\mathfrak{sl}_2(\mathbb{C})$, the weight spaces V_k are symmetric around $k = 0$. There is another useful way to see this symmetry, involving the Lie group $\mathrm{SL}_2(\mathbb{C})$. Consider the so-called *Weyl element*

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

A brief matrix computation shows that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and therefore $w = \exp(E) \exp(-F) \exp(E)$. In principle, the exponential

$$\exp(E) = \mathrm{id} + E + \frac{1}{2!} E^2 + \frac{1}{3!} E^3 + \cdots$$

is given by an infinite series, but because our representation is bounded,

$$w(v) = \exp(E) \exp(-F) \exp(E)\, v$$

only has finitely many nonzero terms for every given vector $v \in V$.

The reason for introducing w is that it explains the symmetry between the weight spaces V_k and V_{-k}. Namely, the Lie group $\mathrm{SL}_2(\mathbb{C})$ acts on its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by conjugation, and under this action, one has

$$wXw^{-1} = -Y, \quad wYw^{-1} = -X, \quad wHw^{-1} = -H.$$

Again, this is easily checked by direct computation: for example,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The identity $wHw^{-1} = -H$ means that the action of w interchanges the two weight spaces V_k and V_{-k}; because $w^4 = \mathrm{id}$, it follows that

$$w \colon V_{-k} \to V_k$$

is an isomorphism for every $k \in \mathbb{Z}$. The point here is that the isomorphism given by the Weyl element behaves better than the isomorphism given by E^k.

Note. The element

$$w^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

acts on the weight space V_k as $(-1)^k$, and *not* as multiplication by -1. One can see this for example from the fact that $w^2 = (-1)^H$, where the right-hand side is an abbreviation for the power series $\exp(i\pi H)$.

If $v \in V_{-n}$ is a primitive vector, we can compute $w(v) \in V_n$ with the help of the formula $w = \exp(E)\exp(-F)\exp(E)$. Indeed,

$$w \exp(-E)v = \exp(E)\exp(-F)v = \exp(E)v,$$

using $Fv = 0$ in the second step. Expanding both sides into power series gives

$$w \sum_{j=0}^{\infty} (-1)^j \frac{E^j}{j!} v = \sum_{j=0}^{\infty} \frac{E^j}{j!} v,$$

and after projecting to the weight space V_n, we find that

$$w(v) = \frac{E^n}{n!} v. \tag{16.9}$$

This shows that, at least as far as the action of w is concerned, the most natural basis for the irreducible representation generated by v is really

$$v, Ev, \frac{E^2}{2!} v, \dots, \frac{E^n}{n!} v.$$

If we instead project to the weight space V_{n-2j}, we get the useful identity

$$w \frac{E^j}{j!} v = (-1)^j \frac{E^{n-j}}{(n-j)!} v, \tag{16.10}$$

valid for any $v \in V_{-n} \cap \ker F$. (Convince yourself that, contrary to appearance, this identity is actually symmetric in j and $n-j$.)

We can use the Lefschetz decomposition to get a relatively simple formula for the action of the Weyl element w on arbitrary vectors.

Proposition 16.11. *Write the Lefschetz decomposition of $v \in V_k$ as*

$$v = \sum_{j \geq \max(k,0)} \frac{E^j}{j!} v_j$$

with $v_j \in V_{k-2j} \cap \ker F$ primitive. Then one has

$$w(v) = \sum_{j \geq \max(k,0)} (-1)^j \frac{E^{j-k}}{(j-k)!} v_j.$$

Proof. Since $v_j \in V_{k-2j}$ is primitive, this follows from (16.10) with $n = 2j - k$. $\square$

Operations on representations. We also need to know a little bit about direct sums and tensor products of representations. Suppose that V and V' are two representations of $\mathfrak{sl}_2(\mathbb{C})$. The direct sum $V \oplus V'$ is also a representation in the obvious way. More interesting is the case of the tensor product $V \otimes V'$. Here E, F, and H act in the following way:

$$\begin{aligned} E(v \otimes v') &= Ev \otimes v' + v \otimes Ev' \\ F(v \otimes v') &= Fv \otimes v' + v \otimes Fv' \\ H(v \otimes v') &= Hv \otimes v' + v \otimes Hv' \end{aligned}$$

This makes sense if we think of the action by a Lie algebra as the derivative of the action by the corresponding Lie group. Correspondingly, the Weyl elements acts as

$$w(v \otimes v') = w(v) \otimes w(v').$$

In a similar manner, the space of $\mathbb{C}$-linear maps $\operatorname{Hom}(V, V')$ becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$ by setting

$$\begin{aligned} (E \cdot f)(v) &= Ef(v) - f(Ev), \\ (F \cdot f)(v) &= Ff(v) - f(Fv), \\ (H \cdot f)(v) &= Hf(v) - f(Hv). \end{aligned}$$

The Weyl element acts as $(w \cdot f)(v) = wf(w^{-1}v)$. With this definition, a linear map $f \colon V \to V'$ is a morphism of $\mathfrak{sl}_2(\mathbb{C})$-representations exactly when the element $f \in \operatorname{Hom}(V, V')$ belongs to $\ker H \cap \ker E \cap \ker F$.

In the special case where $V = V'$, it follows that E, F, and H by commutators: for $f \in \operatorname{End}(V)$, the formula $(E \cdot f)(v) = Ef(v) - f(Ev)$ is saying that

$$E \cdot f = E \circ f - f \circ E = [E, f].$$

If we use the notation $\operatorname{ad} E = [E, -]$ for the operation of taking the commutator with E, then the $\mathfrak{sl}_2(\mathbb{C})$-representation on $\operatorname{End}(V)$ is given by $\operatorname{ad} E$, $\operatorname{ad} F$, and $\operatorname{ad} H$. Similarly, the Weyl element w acts by conjugation:

$$w \cdot f = w \circ f \circ w^{-1} = (\operatorname{Ad} w)(f)$$

The following simple lemma is crucial for understanding the Kähler identities.

Lemma 16.12. *If $v \in V$ is such that $Hv = v$ and $Ev = 0$, then $Fv = -w(v)$.*

Proof. From $v \in V_1$, we get $w(v) \in V_{-1}$. Lemma 16.5 gives $F^2 v = 0$, and so

$$w(v) = \exp(E)\exp(-F)\exp(E)v = \exp(E)\exp(-F)v = \exp(E)(v - Fv).$$

But on the right-hand side, the only term that belongs to V_{-1} is $-Fv$. $\square$

The Kähler identities. Let's now go back to the case where M is an n-dimensional Kähler manifold with Kähler form $\omega \in A^{1,1}(M)$, and

$$L(\alpha) = \omega \wedge \alpha, \quad \Lambda(\alpha) = (-1)^{\deg \alpha} * L * \alpha, \quad H(\alpha) = \big(\deg \alpha - n\big)\alpha.$$

They determine a (bounded) representation of $\mathfrak{sl}_2(\mathbb{C})$ on the vector space

$$A^{n+*}(M) = \bigoplus_{k=-n}^{n} A^{n+k}(M).$$

We would like to prove the Kähler identity

$$[\Lambda, d] = *i(\bar{\partial} - \partial)* . \tag{16.13}$$

Consider the exterior derivative $d\colon A^k(M) \to A^{k+1}(M)$ as a $\mathbb{C}$-linear endomorphism of $A^{n+*}(M)$. Because d increases the degree of a form by 1, we have

$$(\operatorname{ad} H)(d) = [H, d] = d.$$

The Kähler condition $d\omega = 0$ gives us $(\operatorname{ad} L)(d) = [L, d] = 0$, because

$$L(d\alpha) - d(L\alpha) = \omega \wedge d\alpha - d(\omega \wedge \alpha) = 0.$$

So with respect to the $\mathfrak{sl}_2(\mathbb{C})$-representation on the space of endomorphisms of $A^{n+*}(X)$, the element d has weight 1 and is in the kernel of $E = \operatorname{ad} L$. We can therefore compute its image under $F = \operatorname{ad} \Lambda$ using Lemma 16.12. The result is that

$$[\Lambda, d] = (\operatorname{ad} \Lambda)(d) = -(\operatorname{Ad} w)(d) = -w \circ d \circ w^{-1}.$$

This is basically (16.13), once we know the following formula for w, usually called *Weil's identity.* (Mind that "Weil" and "Weyl" are two different people!)

Proposition 16.14. *One has $w(\alpha) = \varepsilon(\deg \alpha) \cdot J(*\alpha)$ for every $\alpha \in A^k(X)$.*

Here $\varepsilon(k) = (-1)^{k(k-1)/2}$ is a sign factor, and the operator $J\colon A^k(X) \to A^k(X)$ is defined by the rule

$$J\alpha = J \sum_{p+q=k} \alpha^{p,q} = \sum_{p+q=k} i^{p-q} \alpha^{p,q},$$

using the decomposition of forms by type. We will see next time that J is actually a real operator.