Class 15. The Kähler identities (October 22)

Let's start by proving the theorem from last time. Recall that M is a complex manifold with a hermitian metric h; this gives us a Riemannian metric g on the underlying smooth manifold, and we have the Levi-Civitá connection ∇. We record the complex structure on M in the operator $J \in \operatorname{End}\bigl(T_{\mathbb{R}} M\bigr)$.

Theorem 15.1. *Let (M,h) be a Hermitian manifold. The following two conditions are equivalent:*

(1) The metric h is Kähler.
(2) The complex structure $J \in \operatorname{End}\bigl(T_{\mathbb{R}} M\bigr)$ is flat for the Levi-Cività connection, i.e., $\nabla_\xi(J\eta) = J\nabla_\xi \eta$ for any two smooth vector fields ξ, η on M.

Proof. It suffices to prove the identity $\nabla_\xi(J\eta) = J\nabla_\xi \eta$ at every point $p \in M$. Since the metric is Kähler, Lemma 14.7 allows us to choose local coordinates centered at p in which the h agrees with the Euclidean metric to second order. Now the identity only involves first-order derivatives of h, as is clear from the proof of Proposition 14.8; on the other hand, it is clearly true for the Euclidean metric on $\mathbb{C}^n$. It follows that the identity remains true for h at the point p. In this way, (1) implies (2).

To show that (2) implies (1), recall that the associated $(1,1)$-form $\omega = -\operatorname{Im} h$ is related to the Riemannian metric $g = \operatorname{Re} h$ by the formula $\omega(\xi,\eta) = g(J\xi,\eta)$. Since the metric is compatible with the connection, we thus have

$$\xi \cdot \omega(\eta,\zeta) = g\bigl(\nabla_\xi(J\eta),\zeta\bigr) + g\bigl(J\eta, \nabla_\xi \zeta\bigr) = \omega\bigl(\nabla_\xi \eta, \zeta\bigr) + \omega\bigl(\eta, \nabla_\xi \zeta\bigr). \tag{15.2}$$

Expressed in a coordinate-free manner, the exterior derivative $d\omega$ is given by the formula

$$(d\omega)(\xi,\eta,\zeta) = \xi\cdot\omega(\eta,\zeta) - \eta\cdot\omega(\xi,\zeta) + \zeta\cdot\omega(\xi,\eta) + \omega\bigl(\xi,[\eta,\zeta]\bigr) - \omega\bigl(\eta,[\xi,\zeta]\bigr) + \omega\bigl(\zeta,[\xi,\eta]\bigr).$$

After substituting (15.2) and using the identity $\nabla_\xi \eta - \nabla_\eta \xi = [\xi,\eta]$, we find that $d\omega = 0$, proving that the metric is indeed Kähler. □

The Kähler identities. Our next goal is to prove that, on a Kähler manifold, the usual Laplace operator $\Delta = d\circ d^* + d^*\circ d$ and the antiholomorphic Laplace operator $\overline{\Box} = \bar\partial\circ\bar\partial^* + \bar\partial^*\circ\bar\partial$ are related by the formula

$$\Delta = 2\overline{\Box}.$$

This will imply that the two notions of harmonic form (harmonic and $\bar\partial$-harmonic) are the same; it will also imply that the Laplace operator Δ preserves the type of a form. Along the way, we shall establish several other relations between the different operators that have been introduced; these relations are collectively known as the *Kähler identities*.

Let (M,h) be a Kähler manifold; we usually refer to the associated $(1,1)$-form $\omega \in A^{1,1}(M)$ as the *Kähler form*. Since ω is real and satisfies $d\omega = 0$, it defines a class in $H^2(M,\mathbb{R})$; in the proof of Wirtinger's lemma, we have already seen that on a compact manifold, this class is nonzero because the formula

$$\operatorname{vol}(M) = \frac{1}{n!}\int_M \omega^{\wedge n}$$

shows that ω can never be exact.

Taking the wedge product with ω defines the so-called *Lefschetz operator*

$$L\colon A^k(M) \to A^{k+2}(M), \quad \alpha \mapsto \omega \wedge \alpha.$$

Since ω has type $(1,1)$, it is clear that L maps $A^{p,q}(M)$ into $A^{p+1,q+1}(M)$. Using the induced metric on the space of forms, we also define the adjoint

$$\Lambda\colon A^k(M) \to A^{k-2}(M)$$

by the (pointwise) condition that $g(L\alpha, \beta) = g(\alpha, \Lambda\beta)$. As usual, we obtain a formula for Λ involving the $*$-operator by noting that

$$g(L\alpha, \beta) \cdot \mathrm{vol}(g) = \omega \wedge \alpha \wedge *\beta = \alpha \wedge (\omega \wedge *\beta) = \alpha \wedge (L * \beta);$$

consequently, $\Lambda\beta = *^{-1} L * \beta = (-1)^k * L * \beta$ because $*^2 = (-1)^k \,\mathrm{id}$. From this formula, we see that Λ maps the space $A^{p,q}(M)$ into the space $A^{p-1,q-1}(M)$. When M is compact, we can further integrate the identity over M to obtain

$$(L\alpha, \beta)_M = \int_M L\alpha \wedge *\beta = \int_M \alpha \wedge *\Lambda\beta = (\alpha, \Lambda\beta)_M,$$

and so Λ is also the adjoint with respect to the inner products on the spaces $A^k(M)$ and $A^{k+2}(M)$. Whereas the Lefschetz operator is easy to understand, it is less clear what the adjoint Λ is doing.

We also have the exterior derivative $d = \partial + \bar{\partial}$, and its adjoint $d^* = \partial^* + \bar{\partial}^*$. When M is compact, we used integration by parts to prove that

$$d^* = - * d *, \quad \bar{\partial}^* = - * \partial *, \quad \partial^* = - * \bar{\partial} * .$$

When M is not compact, we can use these formulas as the definition for the operators d^*, $\bar{\partial}^*$, and ∂^*. The Kähler identies relate the operators ∂ and $\bar{\partial}$ and their adjoints with the help of the operator Λ.

Theorem 15.3. *On a Kähler manifold (M, h), the following identities are true:*

$$[\Lambda, \bar{\partial}] = -i\partial^* \quad \text{and} \quad [\Lambda, \partial] = i\bar{\partial}^*$$

Here and in what follows, the commutator $[A, B]$ of two operators is always defined as $[A, B] = A \circ B - B \circ A$. With the help of the $*$-operator and the above formulas for the adjoints, we can rewrite the Kähler identities in the following equivalent way:

$$[\Lambda, d] = [\Lambda, \partial + \bar{\partial}] = i\bar{\partial}^* - i\partial^* = * \, i(\bar{\partial} - \partial) * \tag{15.4}$$

This is the version that we are actually going to prove; note that it makes perfect sense even when M is not compact. The standard proof for the Kähler identities is to reduce everything to the standard metric on $\mathbb{C}^n$ (using Lemma 14.7), and then to prove the identities directly by a long and not very enlightening calculation. I am going to present a nonstandard proof in class, which is partly inspired by the note "A new proof of the Weil identity" by Ben Anthes (but I will also keep the standard proof in the notes). The goal is to do as much as possible by "pure thought", and to avoid explicit calculations. In order to make this work, we are going to use some very basic representation theory.

Representation theory. To get started, we introduce yet another operator

$$H\colon A^k(M) \to A^k(M), \quad H(\alpha) = (k-n)\alpha.$$

This basically records the degree of a form, but we subtract $n = \dim X$ in order to make H show the inherent symmetry in the degrees. For example, we know that the $*$-operator gives an isomorphism between $A^{n-k}(M)$ and $A^{n+k}(M)$; on the first space, H acts as multiplication by $-k$, and on the second as multiplication by k. Because the Lefschetz operator L increases the degree of a form by 2, we get

$$[H, L]\alpha = H(L\alpha) - L(H\alpha) = H(\omega \wedge \alpha) - (k-n)\omega \wedge \alpha = 2\omega \wedge \alpha = 2L\alpha.$$

In other words, $[H, L] = 2L$; similarly, we have $[H, \Lambda] = -2\Lambda$. In fact, the three operators H, L, and Λ satisfy one additional relation.

Proposition 15.5. *We have* $[L, \Lambda] = H$.

This is a pointwise statement, without any derivatives in it, and so it is enough to check it separately at each point $p \in M$. Fix such a point, and set $V = T^*_{\mathbb{R},p}M$, so that $\dim_{\mathbb{R}} V = 2n$. This has an inner product $g = g_p$ and a natural orientation; if we let $e_1, \dots, e_{2n} \in V$ be a positively oriented orthonormal basis, then $V \cong \mathbb{R}^{2n}$ with the standard inner product. Recall from Class 13 the simple formulas for the Hodge $*$-operator $*\colon \bigwedge^k V \to \bigwedge^{2n-k} V$ in terms of the orthonormal basis. Finally, the Kähler form ω gives us an element $\omega = \omega_p \in \bigwedge^2 V$; in our orthonormal basis, it takes the form

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2n-1} \wedge e_{2n}.$$

This gives us the two operators $L\alpha = \omega \wedge \alpha$ and $\Lambda\alpha = (-1)^{\deg \alpha} * L * \alpha$. We are now ready to prove the identity $[L, \Lambda] = H$.

Proof. First, we treat the case $n = 1$. Here $V = \mathbb{R}^2$, with basis e_1, e_2; the other basis elements are $1 \in \bigwedge^0 V$ and $e_1 \wedge e_2 \in \bigwedge^2 V$. Now we simply compute everything. We already know that

$$*(1) = e_1 \wedge e_2, \quad *(e_1) = e_2, \quad *(e_2) = -e_1, \quad *(e_1 \wedge e_2) = 1,$$

due to the fact that $e_1 \wedge *e_1 = e_1 \wedge e_2$. From $\Lambda = (-1)^{\deg} * L *$, we get

$$\Lambda(1) = 0, \quad \Lambda(e_1) = 0, \quad \Lambda(e_2) = 0, \quad \Lambda(e_1 \wedge e_2) = 1.$$

For degree reasons, $[L, \Lambda](e_1) = [L, \Lambda](e_2) = 0$. In the other degrees, we have

$$[L, \Lambda](1) = -\Lambda L(1) = -\Lambda(e_1 \wedge e_2) = -1$$
$$[L, \Lambda](e_1 \wedge e_2) = L\Lambda(e_1 \wedge e_2) = L(1) = e_1 \wedge e_2.$$

This shows that $[L, \Lambda] = H$ in all four cases.

In order to do the general case, we use induction on the dimension. If $n \geq 2$, we can certainly decompose $V = V_1 \oplus V_2$, for example by letting V_1 be the span of $e_1, \dots, e_{2n-2}$, and letting V_2 be the span of e_{2n-1} and e_{2n}. By induction, we already know the result on V_1 and V_2. So it is enough to prove that our identity $[L, \Lambda] = H$ is compatible with direct sums.

So let us consider more generally the case where $V = V_1 \oplus V_2$. Suppose that each $\dim V_j = 2n_j$ is even, and that we have a positively oriented orthonormal basis in each V_j; let us denote by $*_j$ the $*$-operator on V_j, and by $\omega_j \in \bigwedge^2 V_j$ the "Kähler form". We use the notation L_j, Λ_j, and H_j for the three operators on V_j; by induction on the dimension, we can assume that we already know the identity $[L_j, \Lambda_j] = H_j$ on V_j. The union of the two orthonormal bases, with the same ordering, gives us a positively oriented orthonormal basis on $V = V_1 \oplus V_2$. We keep the symbols $*$, L, Λ, and H for the resulting operators on V. Using this basis, it is easy to see that we have an isomorphism

$$\bigwedge^k V \cong \bigoplus_{i+j=k} \bigwedge^i V_1 \otimes \bigwedge^j V_2.$$

We can write this more concisely as $\bigwedge V \cong \bigwedge V_1 \otimes \bigwedge V_2$. If $\alpha \in \bigwedge^i V_1$ and $\beta \in \bigwedge^j V_2$, we will denote the corresponding element of $\bigwedge^{i+j} V$ by the symbol $\alpha \wedge \beta$. From the formula for the $*$-operator in Class 13, it is easy to see that

$$*(\alpha \wedge \beta) = (-1)^{\deg \alpha \cdot \deg \beta} (*_1 \alpha) \wedge (*_2 \beta).$$

We also have $\omega = \omega_1 \wedge 1 + 1 \wedge \omega_2$, and therefore

$$L(\alpha \wedge \beta) = (L_1 \alpha) \wedge \beta + \alpha \wedge (L_2 \beta).$$

Now let us prove a similar formula for the operator $\Lambda = (-1)^{\deg} * L *$. Take two elements $\alpha \in \bigwedge^i V_1$ and $\beta \in \bigwedge^j V_2$. Then $\alpha \wedge \beta \in \bigwedge^{i+j} V$, and so

$$\Lambda(\alpha \wedge \beta) = (-1)^{i+j} * L * (\alpha \wedge \beta) = (-1)^{i+j+ij} * L(*_1 \alpha \wedge *_2 \beta).$$

Using the above identities for L and $*$, we can rewrite this as

$$\begin{aligned} &(-1)^{i+j+ij} * \big((L_1 *_1 \alpha) \wedge (*_2 \beta) + (*_1 \alpha) \wedge (L_2 *_2 \beta) \big) \\ &\quad = (-1)^{i+j} \big((*_1 L_1 *_1 \alpha) \wedge (*_2^2 \beta) + (*_1^2 \alpha) \wedge (*_2 L_2 *_2 \beta) \big) = (\Lambda_1 \alpha) \wedge \beta + \alpha \wedge (\Lambda_2 \beta), \end{aligned}$$

due to the fact that $*_1^2 \alpha = (-1)^i$ and $*_2^2 \beta = (-1)^j$.

We can now compute the commutator $[L, \Lambda]$. The result is that

$$\begin{aligned} [L, \Lambda](\alpha \wedge \beta) &= L\big(\Lambda_1 \alpha \wedge \beta + \alpha \wedge \Lambda_2 \beta\big) - \Lambda\big(L_1 \alpha \wedge \beta + \alpha \wedge L_2 \beta\big) \\ &= L_1 \Lambda_1 \alpha \wedge \beta + \Lambda_1 \alpha \wedge L_2 \beta + L_1 \alpha \wedge \Lambda_2 \beta + \alpha \wedge L_2 \Lambda_2 \beta \\ &- \Lambda_1 L_1 \alpha \wedge \beta - L_1 \alpha \wedge \Lambda_2 \beta - \Lambda_1 \alpha \wedge L_2 \beta - \alpha \wedge \Lambda_2 L_2 \beta \\ &= [L_1, \Lambda_1] \alpha \wedge \beta + \alpha \wedge [L_2, \Lambda_2] \beta \\ &= H_1 \alpha \wedge \beta + \alpha \wedge H_2 \beta. \end{aligned}$$

Because $H_1 \alpha = (i - n_1)\alpha$ and $H_2 \beta = (j - n_2)\beta$, this proves that $[L, \Lambda]$ is multiplication by $i + j - (n_1 + n_2) = i + j - n$, as required. □

As a consequence, we now have the following three identities:

$$[L, \Lambda] = H, \quad [H, L] = 2L, \quad [H, \Lambda] = -2\Lambda$$

Jean-Pierre Serre observed that these are exactly the relations among the three standard generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and so the graded vector space

$$A^{n+*}(M) = \bigoplus_{k=-n}^{n} A^{n+k}(M)$$

becomes becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$. (The representation theory of Lie algebras is a topic in its own right, but we only need some very basic results that we can easily prove by hand.)

Recall that $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of complex 2×2-matrices with trace zero. It is 3-dimensional, with basis the three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F,$$

which are exactly the same as the relations among H, L, and Λ. This means that we have a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, defined by

$$\rho \colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End}\big(A(M)\big), \quad \rho(H) = H, \quad \rho(E) = L, \quad \rho(F) = \Lambda.$$

We are going to use this representation to understand the Kähler identities (and also the action by the Hodge $*$-operator).