Class 14. Complex harmonic theory (October 17)

The anti-holomorphic Laplacian. From now on, let (M, h) be a compact complex manifold M, of dimension n, with a Hermitian metric h. We then have the space $A^{p,q}(M)$ of smooth differential forms of type (p,q), and the $\bar\partial$-operator $\bar\partial \colon A^{p,q}(M) \to A^{p,q+1}(M)$. Recall that we defined the Dolbeault cohomology groups

$$H^{p,q}(M) = \frac{\ker\bigl(\bar\partial \colon A^{p,q}(M) \to A^{p,q+1}(M)\bigr)}{\operatorname{im}\bigl(\bar\partial \colon A^{p,q-1}(M) \to A^{p,q}(M)\bigr)}.$$

According to the discussion above, we have a Hermitian inner product on the space of (p,q)-forms, defined by

$$(\alpha, \beta)_M = \int_M \alpha \wedge *\overline{\beta},$$

and the complex-linear Hodge $*$-operator

$$* \colon A^{p,q}(M) \to A^{n-q,n-p}(M).$$

As in the real case, we define the adjoint $\bar\partial^* \colon A^{p,q}(M) \to A^{p,q-1}(M)$ by the condition that, for every $\alpha \in A^{p,q-1}(M)$ and $\beta \in A^{p,q}(M)$,

$$\bigl(\bar\partial\alpha, \beta\bigr)_M = \bigl(\alpha, \bar\partial^*\beta\bigr)_M.$$

By essentially the same calculation as in Proposition 13.4, we get an the following formula for the adjoint operator.

Proposition 14.1. *We have* $\bar\partial^* = - * \partial *$.

Proof. Fix $\alpha \in A^{p,q-1}(M)$ and $\beta \in A^{p,q}(M)$; then $\gamma = \alpha \wedge *\overline{\beta}$ is of type $(n, n-1)$, and so $d\gamma = \bar\partial\gamma$. We compute that $\bar\partial\gamma = \bar\partial\alpha \wedge *\overline{\beta} + (-1)^{p+q-1}\alpha \wedge \bar\partial(*\overline{\beta})$, and so it again follows from Stokes' theorem that

$$\bigl(\bar\partial\alpha, \beta\bigr)_M = \int_M \bar\partial\alpha \wedge *\overline{\beta} = (-1)^{p+q} \int_M \alpha \wedge \bar\partial * \overline{\beta}.$$

The adjoint is therefore given by the formula

$$\bar\partial^*\beta = (-1)^{p+q}\overline{\bigl(*^{-1}\bar\partial * \overline{\beta}\bigr)} = (-1)^{p+q} *^{-1} \partial * \beta,$$

using that $*$ is a real operator. Now $\partial * \beta \in A^{n-q+1,n-p}(M)$, and therefore $*^2 \partial * \beta = (-1)^{2n-p-q+1}\beta$; putting things together, we find that $\bar\partial^*\beta = (-1)^{2n+1} * \partial * \beta = - * \partial * \beta$, as asserted above. □

Definition 14.2. The *anti-holomorphic Laplacian* is the linear differential operator $\overline{\Box} \colon A^{p,q}(M) \to A^{p,q}(M)$, defined as $\overline{\Box} = \bar\partial \circ \bar\partial^* + \bar\partial^* \circ \bar\partial$. We say that a (p,q)-form ω is *$\bar\partial$-harmonic* if $\overline{\Box}\omega = 0$, and let $\mathcal{H}^{p,q}(M) = \ker \overline{\Box}$ denote the space of $\bar\partial$-harmonic forms.

One proves that $\overline{\Box}$ is formally self-adjoint and elliptic, and that a (p,q)-form ω is $\bar\partial$-harmonic iff $\bar\partial\omega = 0$ and $\bar\partial^*\omega = 0$. In particular, any such form defines a class in Dolbeault cohomology. By a variant of the fundamental theorem on elliptic operators, we have an orthogonal decomposition

$$A^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \operatorname{im}\bigl(\overline{\Box} \colon A^{p,q}(M) \to A^{p,q}(M)\bigr).$$

It implies the following version of the Hodge theorem for (p,q)-forms by the same argument as in the proof of Theorem 13.7.

Theorem 14.3. *Let (M,h) be a compact Hermitian manifold. Then the natural map $\mathcal{H}^{p,q}(M) \to H^{p,q}(M)$ is an isomorphism; in other words, every Dolbeault cohomology class contains a unique $\bar{\partial}$-harmonic form.*

Similarly, one can define the adjoint $\partial^* \colon A^{p,q}(M) \to A^{p-1,q}(M)$ and the holomorphic Laplacian $\Box = \partial \circ \partial^* + \partial^* \circ \partial$, and get a representation theorem for the cohomology groups of the ∂-operator.

Note. In general, there is *no* relationship between the real Laplace operator $\Delta = d \circ d^* + d^* \circ d$ and the holomorphic Laplacian $\overline{\Box} = \bar{\partial} \circ \bar{\partial}^* + \bar{\partial}^* \circ \bar{\partial}$. This means that a $\bar{\partial}$-harmonic form need not be harmonic, and in fact, not even d-closed.

One can prove that if $\omega \in A^{p,q}(M)$ is $\bar{\partial}$-harmonic, then $*\bar{\omega} \in A^{n-p,n-q}(M)$ is again $\bar{\partial}$-harmonic; this is due to the fact that $\bar{\partial}^* = - * \partial *$. This observation implies the following duality theorem.

Corollary 14.4. *The two vector spaces $H^{p,q}(M)$ and $H^{n-p,n-q}(M)$ have the same dimension.*

Kähler metrics. Let M be a compact complex manifold with a Hermitian metric h; then $g = \operatorname{Re} h$ also defines a Riemannian metric on the underlying smooth manifold. Consequently, we can represent any class in $H^k(M,\mathbb{R})$ by a harmonic form (in the kernel of Δ), and any class in $H^{p,q}(M)$ by a $\bar{\partial}$-harmonic form (in the kernel of $\overline{\Box}$). As already mentioned, there is in general no relation between those two kinds of harmonic forms. For instance, a $\bar{\partial}$-harmonic form need not be d-closed; and conversely, if we decompose a harmonic form α by type as $\alpha = \sum_{p+q=k} \alpha^{p,q}$, then none of the $\alpha^{p,q}$ need be harmonic or $\bar{\partial}$-harmonic. In a nutshell, this is due to a lack of compatibility between the metric and the complex structure.

There is, however, a large class of complex manifolds on which the two theories interact very nicely: the so-called *Kähler manifolds*. Recall that the projective space $\mathbb{P}^n$ has a very natural Hermitian metric, namely the Fubini-Study metric h_{FS}. Its associated $(1,1)$-form ω_{FS}, after pulling back via the map $q \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, is given by the formula

$$q^* \omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log\bigl(|z_0|^2 + |z_1|^2 + \dots + |z_n|^2\bigr).$$

The formula shows that $d\omega_{FS} = 0$, which means that ω_{FS} is a closed form. This simple condition turns out to be the key to the compatibility between the metric and the complex structure.

Definition 14.5. A Hermitian metric h on a complex manifold M is said to be *Kähler* if its associated $(1,1)$-form ω satisfies $d\omega = 0$. A *Kähler manifold* is a complex manifold M that admits at least one Kähler metric.

Any complex submanifold N of a Kähler manifold (M,h) is again a Kähler manifold; indeed, if we give N the induced metric, then $\omega_N = i^*\omega$, where $i \colon N \to M$ is the inclusion map, and so $d\omega_N = d(i^*\omega) = i^* d\omega = 0$. In particular, since $\mathbb{P}^n$ is Kähler, any projective manifold is automatically a Kähler manifold. To a large extent, this accounts for the usefulness of complex manifold theory in algebraic geometry.

Let's look at the Kähler condition in local holomorphic coordinates $z_1, \dots, z_n$ on M. With $h_{j,k} = h(\partial/\partial z_j, \partial/\partial z_k)$, the associated $(1,1)$-form is given by the formula

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k.$$

Note that the matrix with entries $h_{j,k}$ is Hermitian-symmetric, and therefore, $h_{k,j} = \overline{h_{j,k}}$. Now we compute that

$$d\omega = \frac{i}{2} \sum_{j,k,l} \frac{\partial h_{j,k}}{\partial z_l} dz_l \wedge dz_j \wedge d\bar{z}_k + \frac{i}{2} \sum_{j,k,l} \frac{\partial h_{j,k}}{\partial \bar{z}_l} dz_j \wedge d\bar{z}_k \wedge d\bar{z}_l,$$

and so $d\omega = 0$ iff $\partial h_{j,k}/\partial z_l = \partial h_{l,k}/\partial z_j$ and $\partial h_{j,k}/\partial \bar{z}_l = \partial h_{j,l}/\partial \bar{z}_k$. The second condition is actually equivalent to the first (by conjugating), and this proves that the metric h is Kähler iff

$$\frac{\partial h_{j,k}}{\partial z_l} = \frac{\partial h_{l,k}}{\partial z_j} \tag{14.6}$$

for every $j, k, l \in \{1, \dots, n\}$.

Note that the usual Euclidean metric on $\mathbb{C}^n$ has associated $(1,1)$-form

$$\frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$

and is therefore Kähler. The following lemma shows that, conversely, any Kähler metric agrees with the Euclidean metric to second order, in suitable local coordinates.

Lemma 14.7. *A Hermitian metric h is Kähler iff, at every point $p \in M$, there is a holomorphic coordinate system $z_1, \dots, z_n$ centered at p such that*

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j + O(|z|^2).$$

Proof. One direction is very easy: If we can find such a coordinate system centered at a point p, then $d\omega$ clearly vanishes at p; this being true for every $p \in M$, it follows that $d\omega = 0$, and so h is Kähler.

Conversely, assume that $d\omega = 0$, and fix a point $p \in M$. Let $z_1, \dots, z_n$ be arbitrary holomorphic coordinates centered at p, and set $h_{j,k} = h(\partial/\partial z_j, \partial/\partial z_k)$; since we can always make a linear change of coordinates, we may clearly assume that $h_{j,k}(0) = \mathrm{id}_{j,k}$ is the identity matrix. Using that $h_{j,k} = \overline{h_{k,j}}$, we then have

$$h_{j,k} = \mathrm{id}_{j,k} + E_{j,k} + \overline{E_{k,j}} + O(|z|^2),$$

where each $E_{j,k}$ is a linear form in $z_1, \dots, z_n$. Since h is Kähler, (14.6) shows that $\partial E_{j,k}/\partial z_l = \partial E_{l,k}/\partial z_j$; this condition means that there exist (homogeneous) quadratic polynomials $q_k(z)$ such that $E_{j,k} = \partial q_k/\partial z_j$ and $q_k(0) = 0$. Now let

$$w_k = z_k + q_k(z);$$

since the Jacobian $\partial(w_1, \dots, w_n)/\partial(z_1, \dots, z_n)$ is the identity matrix at $z = 0$, the functions $w_1, \dots, w_n$ give holomorphic coordinates in a small enough neighborhood of the point p. By construction,

$$dw_j = dz_j + \sum_{k=1}^{n} \frac{\partial q_j}{\partial z_k} dz_k = dz_j + \sum_{k=1}^{n} E_{k,j} dz_k.$$

and so we have, up to second-order terms,

$$\begin{aligned}\frac{i}{2}\sum_{j=1}^{n} dw_j \wedge d\bar{w}_j &\equiv \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j + \frac{i}{2}\sum_{j,k=1}^{n} dz_j \wedge \overline{E_{k,j}} d\bar{z}_k + \frac{i}{2}\sum_{j,k=1}^{n} E_{k,j} dz_k \wedge d\bar{z}_j \\ &= \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j + \frac{i}{2}\sum_{j,k=1}^{n} \bigl(E_{j,k} + \overline{E_{k,j}}\bigr) dz_j \wedge d\bar{z}_k \\ &\equiv \frac{i}{2}\sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k.\end{aligned}$$

which shows that $\omega = \frac{i}{2}\sum_{j,k} dw_j \wedge d\bar{w}_k + O(|w|^2)$ in the new coordinate system. □

This lemma is extremely useful for proving results about arbitrary Kähler metrics by only looking at the Euclidean metric on $\mathbb{C}^n$.

Kähler metrics and differential geometry. To show how the condition $d\omega = 0$ implies that the metric is compatible with the complex structure, we shall now look at equivalent formulations of the Kähler condition.

We begin by reviewing some Riemannian differential geometry. Let M be a smooth manifold, with real tangent bundle $T_{\mathbb{R}}M$, and let $T(M)$ denote the set of smooth vector fields on M. Recall that vector fields can be viewed as operators on smooth functions: if $\xi \in T(M)$, then $\xi \cdot f$ is a smooth function for any smooth function f. In local coordinates $x_1, \dots, x_n$, we can write $\xi = \sum a_i \partial/\partial x_i$, with smooth functions $a_1, \dots, a_n$, and then

$$\xi \cdot f = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}.$$

Given any two vector fields ξ and η, their commutator $[\xi, \eta] \in T(M)$ acts on smooth functions by the rule $[\xi, \eta] \cdot f = \xi \cdot (\eta f) - \eta \cdot (\xi f)$. In local coordinates, $\xi = \sum a_j \partial/\partial x_i$ and $\eta = \sum b_i \partial/\partial x_i$, and then

$$[\xi, \eta] = \sum_{i,j=1}^{n} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j},$$

as a short computation will show.

Since $T_{\mathbb{R}}M$ is a vector bundle, there is no intrinsic way to differentiate its sections; this requires the additional data of a *connection*. Such a connection is a mapping

$$\nabla \colon T(M) \times T(M) \to T(M), \quad (\xi, \eta) \mapsto \nabla_\xi \eta,$$

linear in the first argument, and satisfying the Leibniz rule

$$\nabla_\xi (f \cdot \eta) = (\xi f) \cdot \eta + f \cdot \nabla_\xi \eta.$$

In other words, a connection gives a way to differentiate vector fields, and $\nabla_\xi \eta$ should be viewed as the derivative of η in the direction of ξ.

Proposition 14.8. *On a Riemannian manifold (M, g), there is a unique connection that is both* compatible with the metric*, in the sense that*

$$\xi \cdot g(\eta, \zeta) = g\bigl(\nabla_\xi \eta, \zeta\bigr) + g\bigl(\eta, \nabla_\xi \zeta\bigr),$$

and torsion-free*, in the sense that*

$$\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta].$$

This connection is known as the Levi-Cività connection *associated to the metric.*

Proof. Let $x_1, \dots, x_n$ be local coordinates on M, and set $\partial_i = \partial/\partial x_i$; the Riemannian metric is represented by the matrix $g_{i,j} = g(\partial_i, \partial_j)$. To describe the connection, it is sufficient to know the coefficients $\Gamma^k_{i,j}$ in the expression

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^n \Gamma^k_{i,j} \partial_k.$$

The conditions above now mean the following: the connection is torsion-free iff $\Gamma^k_{i,j} = \Gamma^k_{j,i}$, and compatible with the metric iff

$$\frac{\partial g_{j,k}}{\partial x_i} = g\big(\nabla_{\partial_i} \partial_j, \partial_k\big) + g\big(\partial_j, \nabla_{\partial_i} \partial_k\big) = \sum_{l=1}^n \left(g_{l,k} \Gamma^l_{i,j} + g_{j,l} \Gamma^l_{i,k} \right).$$

From these two identities, we compute that

$$\frac{\partial g_{i,j}}{\partial x_k} - \frac{\partial g_{i,k}}{\partial x_j} + \frac{\partial g_{j,k}}{\partial x_i} = 2 \sum_{l=1}^n g_{j,l} \Gamma^l_{i,k},$$

and so the coefficients are given by the formula

$$\Gamma^k_{i,j} = \frac{1}{2} \sum_{l=1}^n g^{k,l} \left(\frac{\partial g_{i,l}}{\partial x_j} - \frac{\partial g_{i,j}}{\partial x_l} + \frac{\partial g_{l,j}}{\partial x_i} \right),$$

where $g^{i,j}$ are the entries of the inverse matrix. □

We come back to the case of a Hermitian manifold (M, h). At each point $p \in M$, we have an isomorphism between the real tangent space $T_{\mathbb{R},p} M$ and the real vector space underlying the holomorphic tangent space $T'_p M$. As usual, we denote by $J_p \in \operatorname{End}\big(T_{\mathbb{R},p} M\big)$ the operation of multiplying by i. We can say that the complex structure on M is encoded in the map $J \in \operatorname{End}\big(T_{\mathbb{R}} M\big)$. On the other hand, $g = \operatorname{Re} h$ defines a Riemannian metric on the underlying smooth manifold, with Levi-Cività connection ∇.

Theorem 14.9. *Let (M, h) be a Hermitian manifold. The following two conditions are equivalent:*

1. *The metric h is Kähler.*
2. *The complex structure $J \in \operatorname{End}\big(T_{\mathbb{R}} M\big)$ is flat for the Levi-Cività connection, i.e., $\nabla_\xi (J\eta) = J \nabla_\xi \eta$ for any two smooth vector fields ξ, η on M.*