

Math 534
Problem Set 9

due Wednesday, November 12, 2025

In problems #1–5, R is always a commutative ring with 1, while M , N , etc. are R -modules. To abbreviate, “module” will always mean R -module, “morphism” will mean morphism of R -modules, and so on.

1. A module M is said to be *simple* if M is not the zero module and the only submodules of M are M and 0 . Show that M is a simple module if and only if $M \cong R/I$, where I is a maximal ideal of R . (Here \cong means isomorphic as R -modules.)
2. Suppose that M and N are submodules of a module P . Show that $(M + N)/M \cap N \cong (M/M \cap N) \oplus (N/M \cap N)$.
3. Suppose that R is an integral domain. An element $m \in M$ is said to be a *torsion element* if $rm = 0$ for some $r \neq 0$ in R . (In that case, one also says that r *annihilates* the element m .)
 - (a) Let M_{tor} be the set of torsion elements in M . Show that M_{tor} is a submodule of M .
 - (b) M is said to be *torsion-free* if $M_{\text{tor}} = 0$. Show that M/M_{tor} is always torsion-free.
 - (c) Suppose that $R = \mathbb{Z}$ and $M = \mathbb{R}/\mathbb{Z}$. What is M_{tor} in this case?
4. Let M be a module and let $m \in M$. The *annihilator* of m in R , usually denoted $\text{Ann}(m)$, is the set $\{r \in R \mid rm = 0\}$.
 - (a) Show that $\text{Ann}(m)$ is an ideal of R .
 - (b) A module M is said to be *cyclic* if it is generated by a single element $m_0 \in M$, in the sense that every element $m \in M$ can be written as $m = rm_0$ for some $r \in R$. Show that

$$M \text{ is cyclic} \iff M \cong R/I \text{ for some ideal } I \text{ of } R$$

5. Suppose that R is an integral domain, and let I be a nonzero ideal of R . Show that $I \cong R$ if and only if I is a principal ideal of R .
6. Let F be a field, let $F[x]$ be the ring of polynomials over F , and let V be an $F[x]$ -module that is finite-dimensional as a vector space over F .

- (a) Show that V is a torsion module. (Hint: For a given $v \in V$, consider the sequence of elements v, xv, x^2v, \dots)
 - (b) Let $T: V \rightarrow V$ be the map defined by $T(v) = xv$ for $v \in V$. Show that T is a linear transformation of the F -vector space V .
7. Let F be a field, G a finite group, and $F[G]$ the group ring. Let V be a finitely-generated $F[G]$ -module.
- (a) Show that V is a finite-dimensional vector space over F .
 - (b) Let $g \in G$, and let $T_g: V \rightarrow V$ be the map $T_g(v) = gv$. Show that T_g is a linear transformation of the F -vector space V .
 - (c) Let v_1, \dots, v_n be a basis for V over F , and for each $g \in G$, let M_g be the matrix of T_g with respect to the basis v_1, \dots, v_n . Show that $M_g \in \text{GL}_n(F)$, and that the map

$$\rho: G \rightarrow \text{GL}_n(F), \quad g \mapsto M_g$$

is a group homomorphism.

8. Let R be a noetherian ring.
- (a) Let M be a submodule of R^n for some $n \geq 1$. Let $\pi_1: R^n \rightarrow R$ be the projection to the first coordinate, so $\pi_1(r_1, \dots, r_n) = r_1$. Show that $\pi_1(M)$ is an ideal in R .
 - (b) Let $M_0 = \{m \in M \mid \pi_1(m) = 0\}$. Show that there is a finitely-generated submodule $M' \subseteq M$ such that $M = M' + M_0$.
 - (c) Use induction on $n \geq 1$ to prove that every submodule of R^n is finitely-generated.
 - (d) Deduce that every submodule of a finitely-generated R -module is again finitely-generated.