**Math 534**
**Problem Set 7**

due Wednesday, October 29, 2025

1. Let $F$ be a field. Prove that the polynomial ring $F[x]$ is a PID.

2. Let $R$ be a PID.

   (a) Show that any two elements $a, b \in R$ have a least common multiple $\mathrm{lcm}(a, b)$, which is unique up to multiplication by units.

   (b) Show that $\gcd(a, b)\,\mathrm{lcm}(a, b)$ equals $ab$, up to a unit.

3. Let $R$ be an integral domain in which every *prime* ideal is principal. The goal of this exercise is to show that $R$ must be a PID.

   (a) Show that if $R$ is not a PID, then there is an ideal $I$ that is not principal, and is maximal with respect to this property.

   (b) Since $I$ cannot be prime, there are elements $a, b \in R$ with $a, b \notin I$ and $ab \in I$. Show that $(a) + I = (c)$ for some $c \in R$.

   (c) Show that the ideal $J = \{\, r \in R \mid rc \in I \,\}$ is principal.

   (d) Conclude that $I$ itself must be principal, which is a contradiction.

4. Exhibit all the ideals in the ring $F[x]/(p(x))$, where $F$ is a field and $p(x)$ is a polynomial in $F[x]$. (Describe them in terms of the factorization of $p(x)$.)

5. Let $p \in \mathbb{Z}$ be a prime with $p \equiv 3 \mod 4$. Prove that the quotient ring $\mathbb{Z}[i]/(p)$ is a field with $p^2$ elements.

6. Show that $(x, y)$ is not a principal ideal in $\mathbb{Q}[x, y]$.

7. Suppose that $f(x)$ and $g(x)$ are two polynomials with rational coefficients, whose product $f(x)g(x)$ has integer coefficients. Show that the product of any coefficient of $f(x)$ with any coefficient of $g(x)$ is an integer.

8. Let $F$ be a field. A subring $R \subseteq F$ is called a *valuation ring* if, for every nonzero $x \in F$, at least one of $x$ and $x^{-1}$ belongs to $R$.

   (a) Show that $R$ has a unique maximal ideal.

   (b) Show that the ideals of $R$ are totally ordered under inclusion.