Math 534
Problem Set 5

due Wednesday, October 8, 2025

1. Let $\mathscr{C}$ be a category.
   (a) Show that for any object $X \in \mathrm{Ob}(\mathscr{C})$, there is only one morphism from X to itself that has the defining property of id_X.
   (b) Suppose that $f \in \mathrm{Mor}(X, Y)$ is an isomorphism. Does this imply that f has a unique inverse in $\mathrm{Mor}(Y, X)$?
   (c) Show that the group $\mathrm{Aut}(X)$ acts on the set of isomorphisms in $\mathrm{Mor}(X, Y)$, and that this action is transitive.

2. One of the games of category theory is to take a property of objects/morphisms in a familiar category, such as ((sets)), and find a way to restate that property "category-theoretically", by describing it entirely in terms of objects and morphisms, without any reference to "elements" or "subsets". We obtain a property that makes sense for objects/morphisms in any category. We can then take another category, and see what the property amounts to in the new setting.

   For example, take the property of being *surjective*; this is a familiar property of functions between sets. If X and Y are sets, then a function $f\colon X \to Y$ is called surjective if every element of Y is the image of some element of X. We can rephrase this property in category-theoretical terms as follows: $(*)$ Whenever $g\colon Y \to Z$ and $h\colon Y \to Z$ are two morphisms such that $g \circ f = h \circ f$, then $g = h$.

   (a) Show that in ((sets)), $f\colon X \to Y$ is surjective iff f satisfies $(*)$.
   (b) Characterize the morphisms satisfying $(*)$ in ((topological spaces)).
   (c) Characterize the morphisms satisfying $(*)$ in ((groups)).
   (d) Characterize the morphisms satisfying $(*)$ in ((abelian groups)).

3. Do the following categories have a universal repelling (or attracting) object? If so, describe them; if not, explain why not.
   (a) ((sets))
   (b) ((groups))
   (c) ((fields)) (Hint: For any field F, there is a field $F(x)$ of rational functions in an indeterminate x with coefficients in F.)

(d) ((fields of characteristic 0))

4. In some categories, the objects are sets and have elements, and then we can try to describe those elements category-theoretically. For example:

   (a) Let $\mathbf{1}$ be any one-element set in ((sets)), for example $\{\varnothing\}$. Show that the *elements* of a set X are in 1-to-1 correspondence with the morphisms $\mathrm{Mor}(\mathbf{1}, X)$.

   (b) In ((groups)), show that the elements of a group G are in 1-to-1 correspondence with $\mathrm{Mor}(\mathbb{Z}, G)$.

   (c) Let F be a field, and consider the category ((F-vector spaces)) of vector spaces over the field F. Can you find a vector space V_0 in this category, such that for any vector space V over F, the elements of V are in 1-to-1 correspondence with $\mathrm{Mor}(V_0, V)$?

5. Let $\mathscr{C}$ be a category, and $A \in \mathscr{C}$ a fixed object. For every $X \in \mathrm{Ob}(\mathscr{C})$, let $F(X) = \mathrm{Mor}(A, X)$ be the set of morphisms from A to X in our category. Show that F is a functor from $\mathscr{C}$ to ((sets)).

6. Let A and B be two universal attracting (or repelling) objects in a category. Show that there is a unique isomorphism between A and B.

7. Given a topological space X, we can define a category $\mathscr{C}_X$ as follows. The objects of $\mathscr{C}_X$ are the open subsets of X, and for any two open subsets $U, V \subseteq X$, the set of morphisms in $\mathscr{C}_X$ is

$$\mathrm{Mor}(U, V) = \begin{cases} \{i_{UV}\} & \text{if } U \subseteq V, \\ \varnothing & \text{if } U \nsubseteq V. \end{cases}$$

   If $U \subseteq V \subseteq W$, we define the composition as $i_{VW} \circ i_{UV} = i_{UW}$.

   (a) Prove that $\mathscr{C}_X$ is indeed a category.

   (b) Does $\mathscr{C}_X$ have a universal attracting/repelling object?

   (c) For every open $U \subseteq X$, define $F(U)$ to be the set of continuous functions from U into $\mathbb{R}$. Show that F is a contravariant functor from $\mathscr{C}_X$ to the category (($\mathbb{R}$-vector spaces)).