Math 534
Problem Set 4

due Wednesday, October 1, 2025

1. A *characteristic subgroup* of a group is a subgroup $H \subseteq G$ such that $\alpha(H) = H$ for every automorphism $\alpha \in \mathrm{Aut}(G)$. Show that the center Z and the commutator subgroup G' are characteristic subgroups of G.

2. Let $\phi\colon G \to \mathrm{Aut}(G)$ be the map that associates to any $g \in G$ the corresponding inner automorphism: $\phi(g) = \alpha_g$, where $\alpha_g(x) = gxg^{-1}$.

   (a) Show that ϕ is a homomorphism.

   (b) Show that the kernel of ϕ is the center $Z \subseteq G$.

   (c) Let $\mathrm{Inn}(G)$ be the image of ϕ. Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

3. Let $n \geq 2$. Show that A_n is the only subgroup of S_n of index 2.

4. Let $n \geq 3$, and let $N \subseteq A_n$ be the subgroup generated by all 3-cycles.

   (a) Show that N is a normal subgroup of A_n.

   (b) Show that $N = A_n$.

5. Let $n \geq 4$. Show that there exists an injective group homomorphism $S_n \to \mathrm{Aut}(A_n)$. What happens for $n = 3$?

6. Suppose that G is a simple group of order 60.

   (a) Show that G has exactly six Sylow 5-subgroups.

   (b) Show that the action of G on its Sylow 5-subgroups (by conjugation) gives an injective homomorphism $\phi\colon G \to S_6$, and that the image of ϕ is a subgroup of A_6 of index 6.

   (c) Show that $G \cong A_5$.

7. In this problem, we construct an exotic automorphism of S_6.

   (a) Show that S_5 has exactly 6 Sylow 5-subgroups.

   (b) Show that the action of S_5 on its Sylow 5-subgroups gives an injective homomorphism $\phi\colon S_5 \to S_6$, whose image H is a subgroup of S_6 of index 6.

(c) Let $H_k \subseteq S_6$ be the stabilizer of $k \in \{1, 2, \ldots, 6\}$. Show that H is *not* one of the subgroups $H_1, \ldots, H_6$.

(d) Show that the action of S_6 on the cosets of H gives an automorphism $\alpha\colon S_6 \to S_6$.

(e) Show that α is *not* an inner automorphism.

8. Let G be a solvable finite group. Show that there is a chain

$$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_r = \{1\},$$

such that $N_{i+1} \triangleleft N_i$, and N_i/N_{i+1} is cyclic, for every $0 \le i \le r-1$.

9. Let G be a solvable group.

   (a) Show that every subgroup of G is solvable.

   (b) Let $\phi\colon G \to H$ be a homomorphism. Show that $\phi(G)$ is solvable.

10. Let N be a normal subgroup of a group G. Show that G is solvable if and only if both N and G/N are solvable.