Math 534
Problem Set 3

due Wednesday, September 24, 2025

1. Let G be a finite group, P a Sylow p-subgroup of G. If N is a normal subgroup of G, show that $P \cap N$ is a Sylow p-subgroup of N, and PN/N is a Sylow p-subgroup of G/N.

2. Let G be a finite group, p a prime, and H a p-subgroup of G. Show that there is a Sylow p-subgroup of G containing H.

3. Let P and Q be Sylow p-subgroups of a finite group G. Suppose the center $Z(P)$ of P is contained in Q and is normal in Q. Prove that $Z(P) = Z(Q)$.

4. Let G be a group of order p^2q, where p and q are distinct primes. Prove that G has a normal Sylow subgroup.

5. Let G be a finite group, P a Sylow p-subgroup of G. Suppose that H is a subgroup of G containing $N(P)$. Show that $(G\colon H) \equiv 1 \mod p$.

6. Let K be a normal subgroup of a finite group G. If P is a Sylow p-subgroup of K, prove that $G = K \cdot N_G(P)$, where $N_G(P)$ is the normalizer of P in G.

7. Let G be a finite group of order 80. Show that G is not simple.

8. Suppose that the number n_p of Sylow p-subgroups of a finite group G satisfies $n_p \not\equiv 1 \mod p^2$. Prove that there are two distinct Sylow p-subgroups P and Q such that $|P \cap Q| = |P|/p$.

9. Let $G = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_r = \{1\}$ be a *composition series* for a finite group G, meaning that each N_{i+1} is a normal subgroup of N_i, and that the factor groups N_i/N_{i+1} are all simple. Suppose that one of the N_i is a Sylow p-subgroup of G (for some prime p). Prove that N_i must be normal in G.

10. Suppose that G is a nontrivial finite p-group.
    (a) Show that the center of G is nontrivial. (Hint: Use Problem Set 2, #4.)
    (b) Show that G is simple if and only if $|G| = p$.

(c) Let $|G| = p^n$, and let $G = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_r = \{1\}$ be a composition series for G. Show that $r = n$ and $|N_i| = p^{n-i}$ for $i = 0, 1, \ldots, n$.

(d) Show that we can find a composition series for G such that each N_i is normal in G.