Einstein Constants

and

Differential Topology

Claude LeBrun
Stony Brook University

Riemannian Geometry and Related Topics,
Dipartimento di Matematica "Giuseppe Peano,"
Università di Torino, lunedì 15 dicembre 2025.

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— A. Einstein, to G. Gamow

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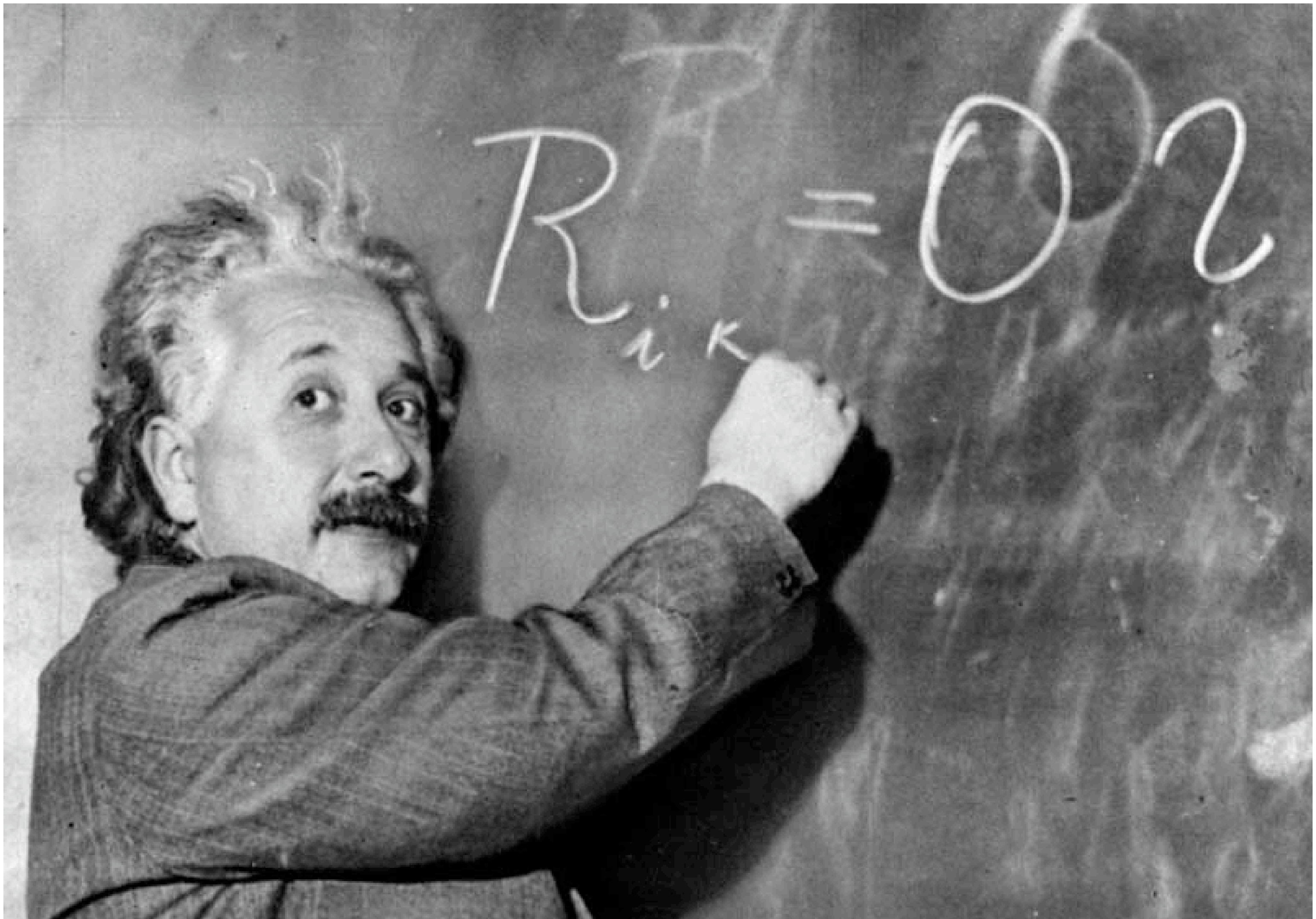

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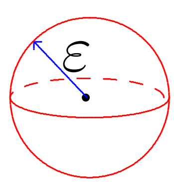
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Ergebnisse der Mathematik und ihrer Grenzgebiete
3. Folge · Band 10
A Series of Modern Surveys in Mathematics

Arthur L. Besse

Einstein Manifolds

Springer-Verlag Berlin Heidelberg GmbH

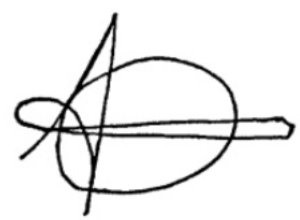
Acknowledgements

Pour rassembler les éléments un peu disparates qui constituent ce livre, j'ai dû faire appel à de nombreux amis, heureusement bien plus savants que moi. Ce sont, entre autres, Geneviève Averous, Lionel Bérard-Bergery, Marcel Berger, Jean-Pierre Bourguignon, Andrei Derdzinski, Dennis M. DeTurck, Paul Gauduchon, Nigel J. Hitchin, Josette Houillot, Hermann Karcher, Jerry L. Kazdan, Norihito Koiso, Jacques Lafontaine, Pierre Pansu, Albert Polombo, John A. Thorpe, Liane Valère.

Les institutions suivantes m'ont prêté leur concours matériel, et je les en remercie: l'UER de mathématiques de Paris 7, le Centre de Mathématiques de l'Ecole Polytechnique, Unités Associées du CNRS, l'UER de mathématiques de Chambéry et le Conseil Général de Savoie.

Enfin, qu'il me soit permis de saluer ici mon prédecesseur et homonyme Jean Besse, de Zürich, qui s'est illustré dans la théorie des fonctions d'une variable complexe (voir par exemple [Bse]).

Vôtre,



Arthur Besse

Le Faux, le 15 septembre 1986

Besse en Chandesse

L'Auvergne

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But it turns out that the answer is actually Yes!

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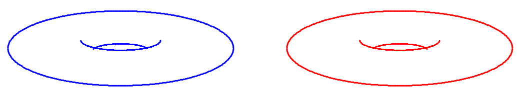

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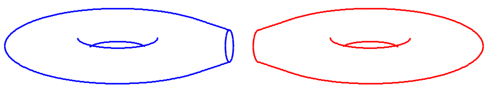

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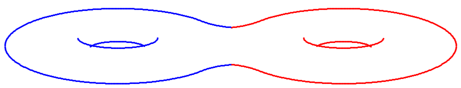

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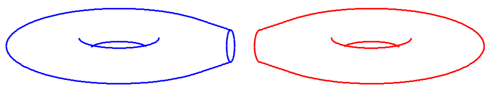

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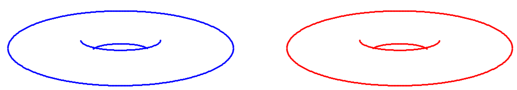

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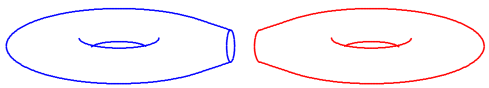

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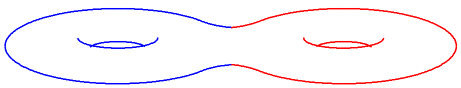

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$\Longleftrightarrow$ they have the same Einstein constant λ.

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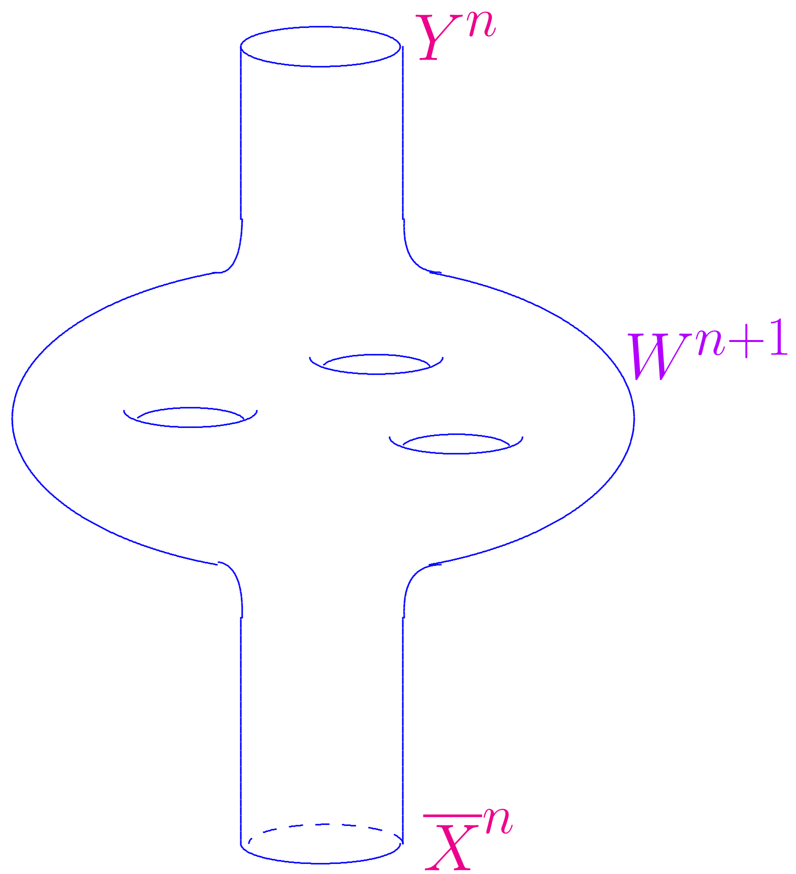
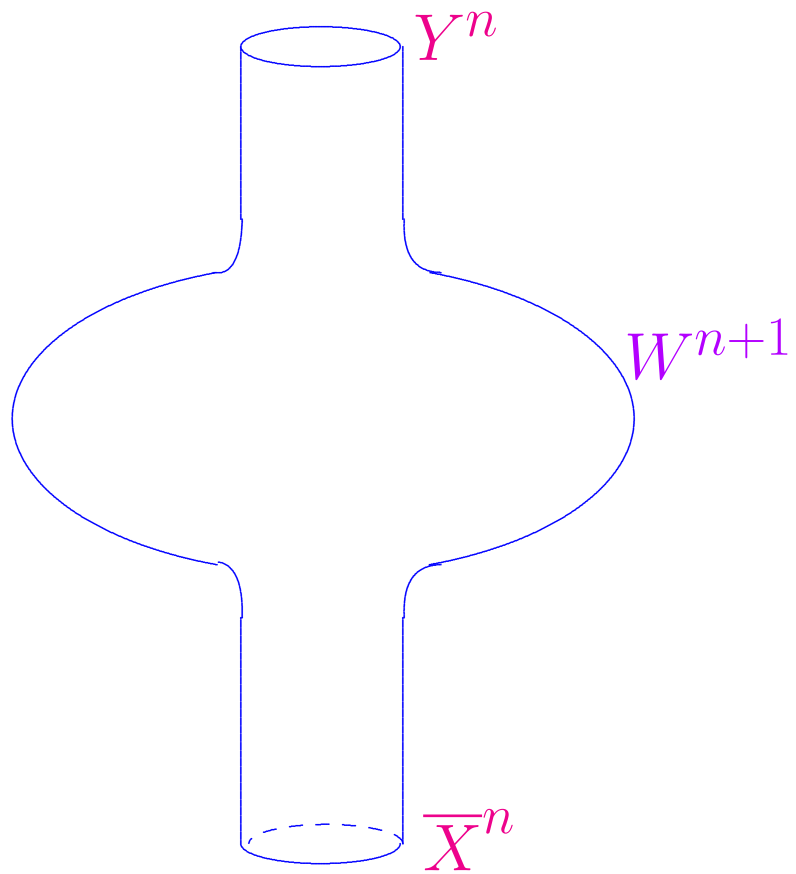
- Smale's h-cobordism theorem.

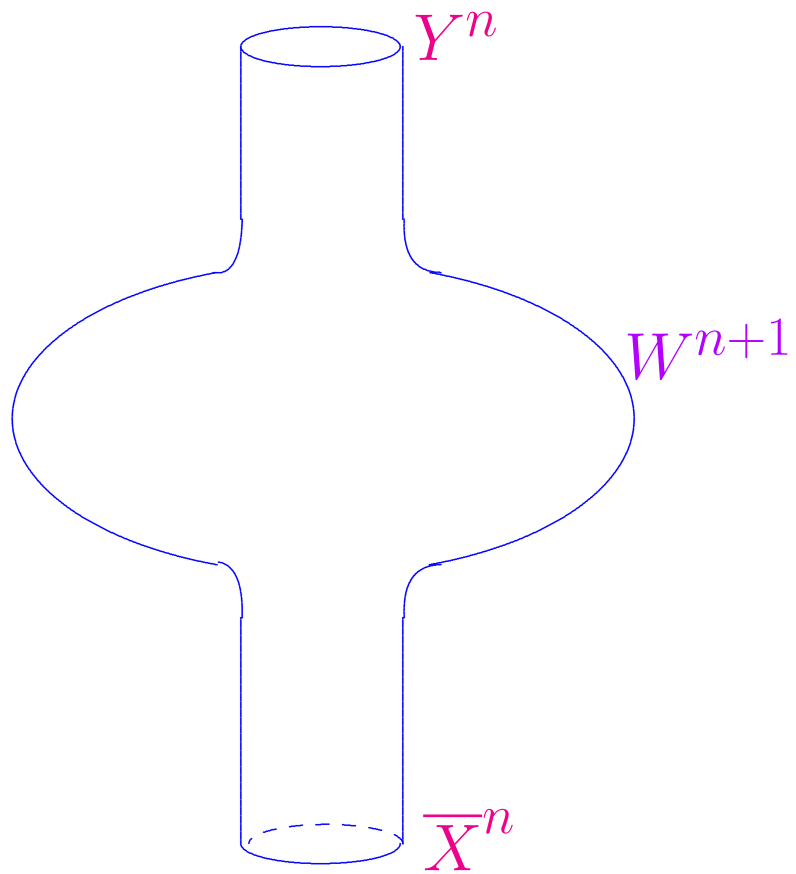


Cobordism

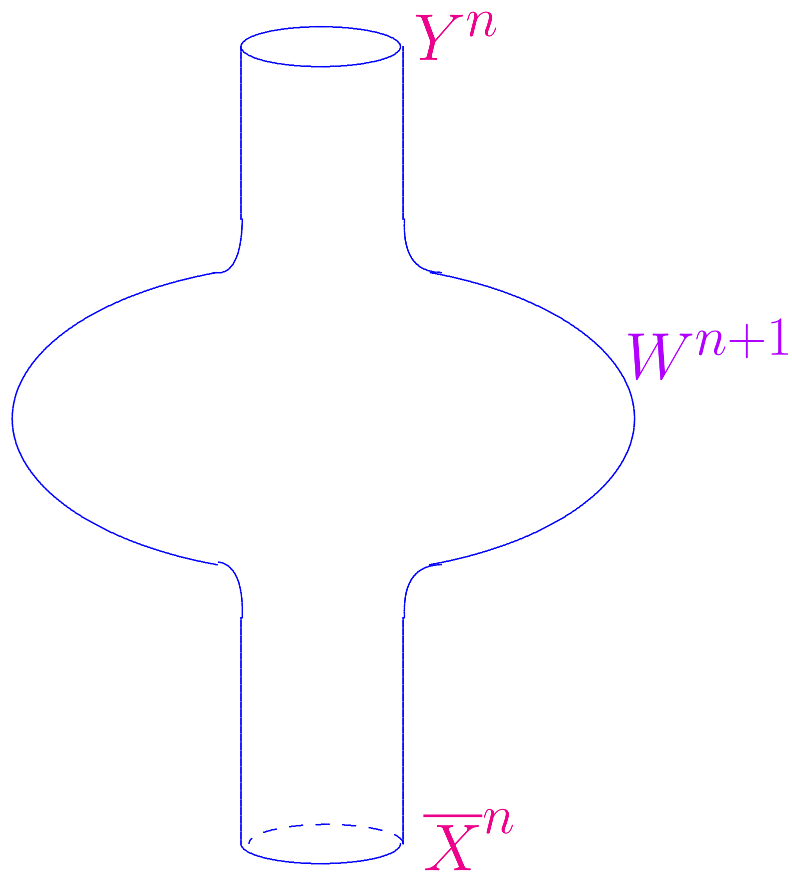


h-Cobordism

if $\quad X \hookrightarrow W, \quad Y \hookrightarrow W \quad$ both homotopy equivalences



Smale: Suppose that X^n is h-cobordant to Y^n. If $\pi_1 = 0$ and $n > 4$, then X is diffeomorphic to Y.



Wall: Suppose that X^4 homotopy equivalent to Y^4.
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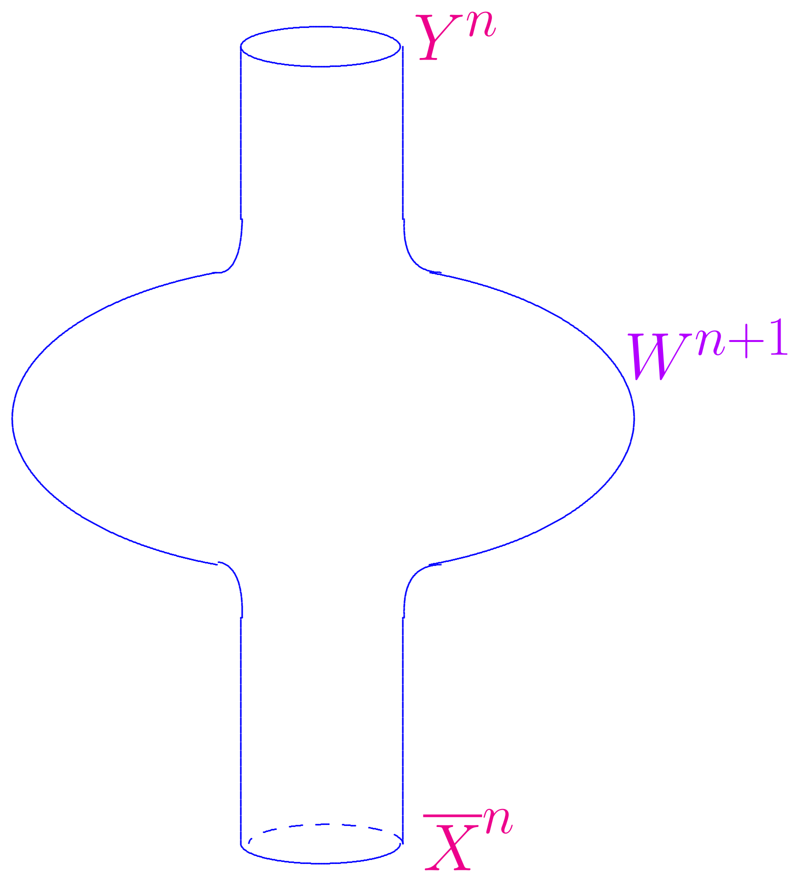

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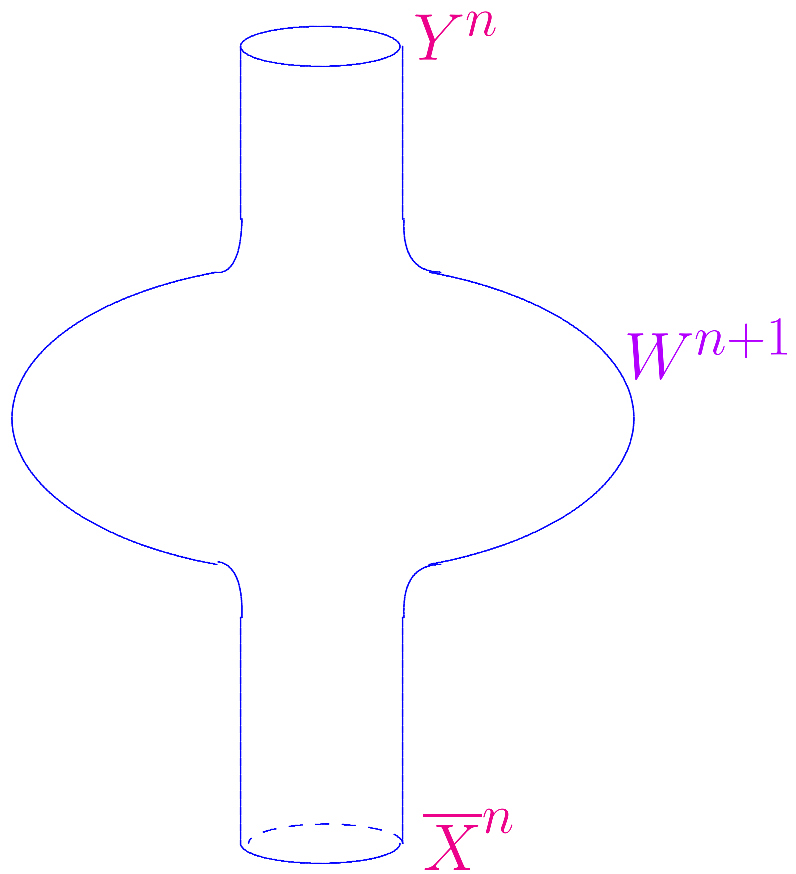
But Smale doesn't apply when $n = 4$!



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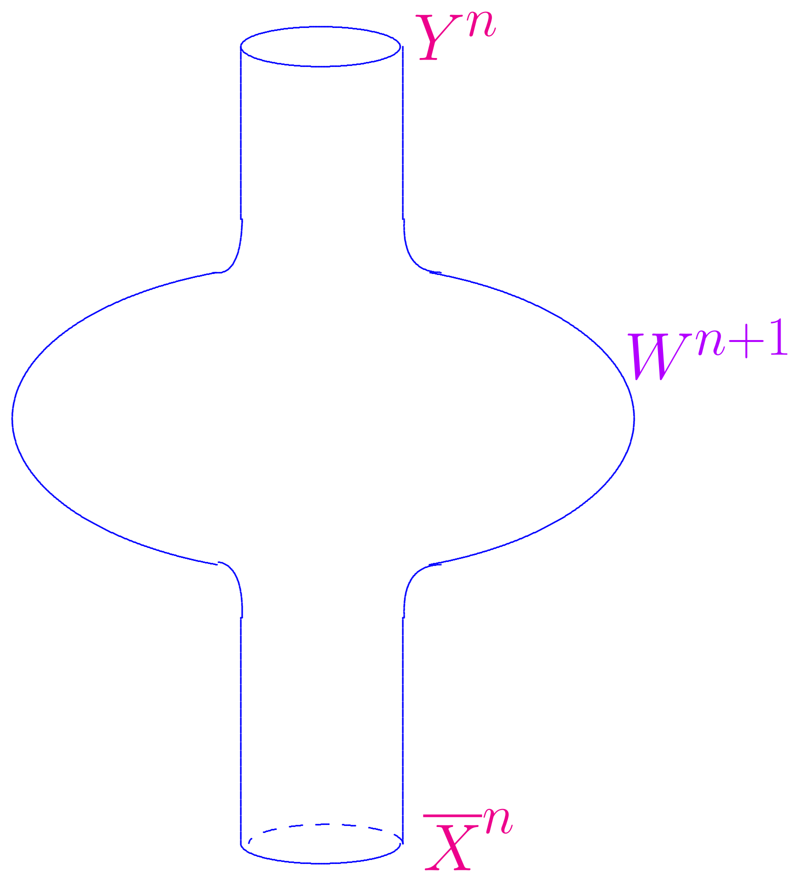

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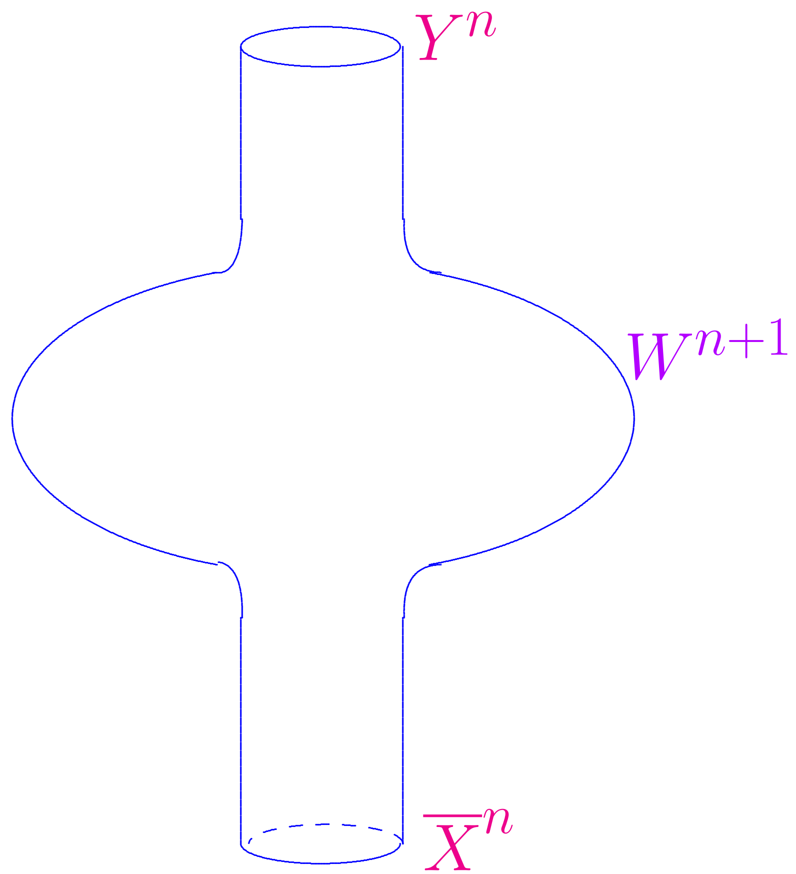
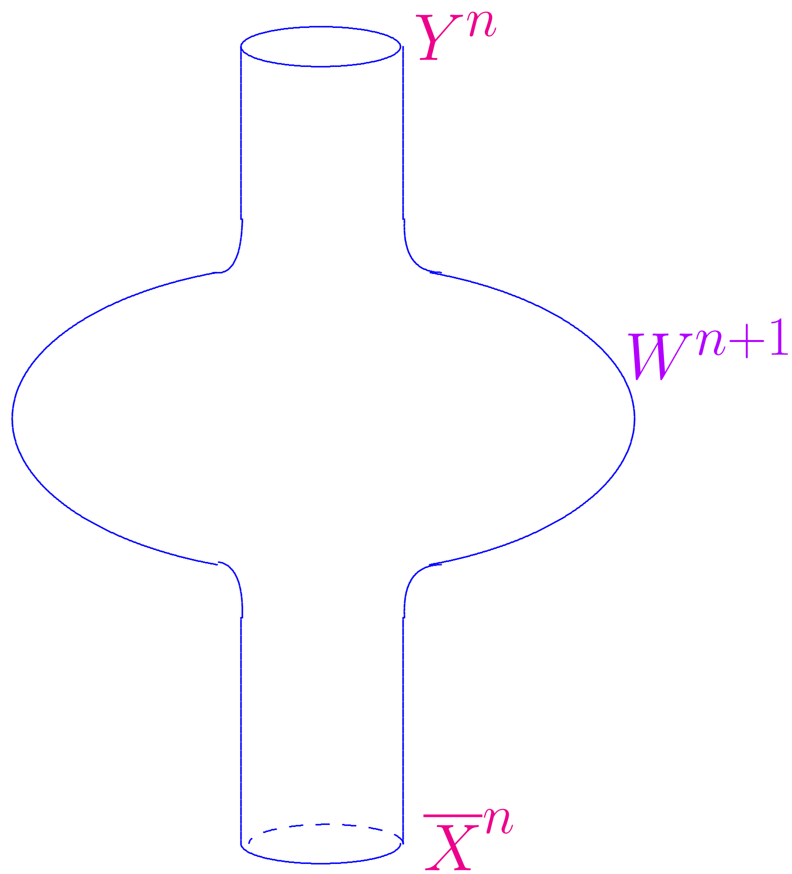
Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then $X \times X$ is actually diffeomorphic to $Y \times Y$.*



Indeed, if W is h-cobordism $X \sim Y$, then

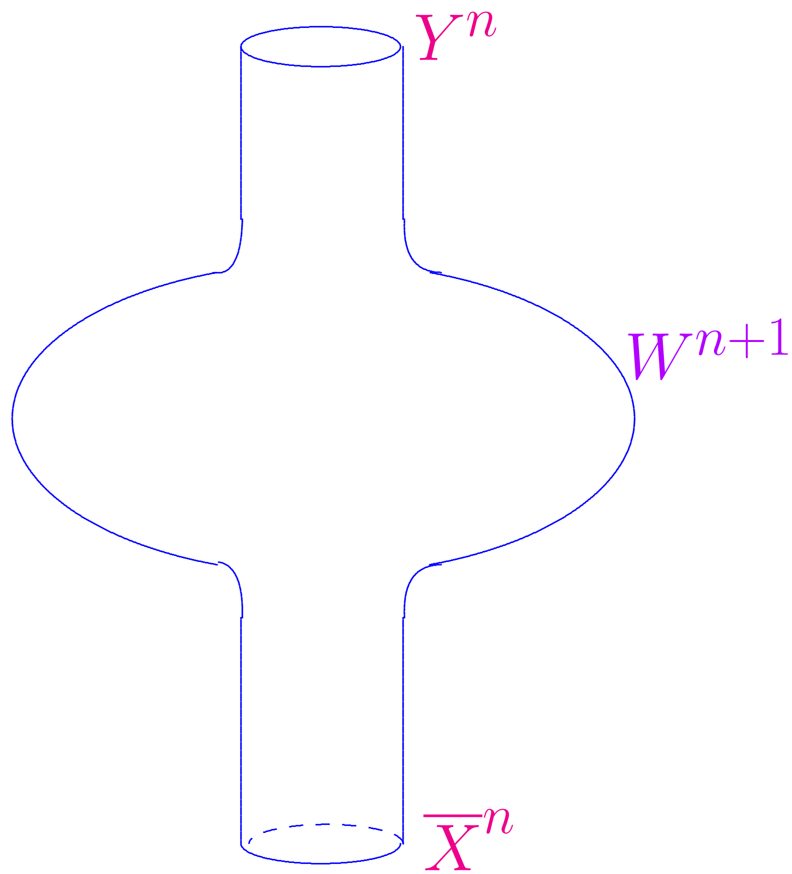
$$(X \times W) \cup_{X \times Y} (W \times Y)$$

is an h-cobordism $(X \times X) \sim (Y \times Y)$.



Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then*

$$\underbrace{X \times \cdots \times X}_{k} \approx_{\mathsf{diff}} \underbrace{Y \times \cdots \times Y}_{k}$$



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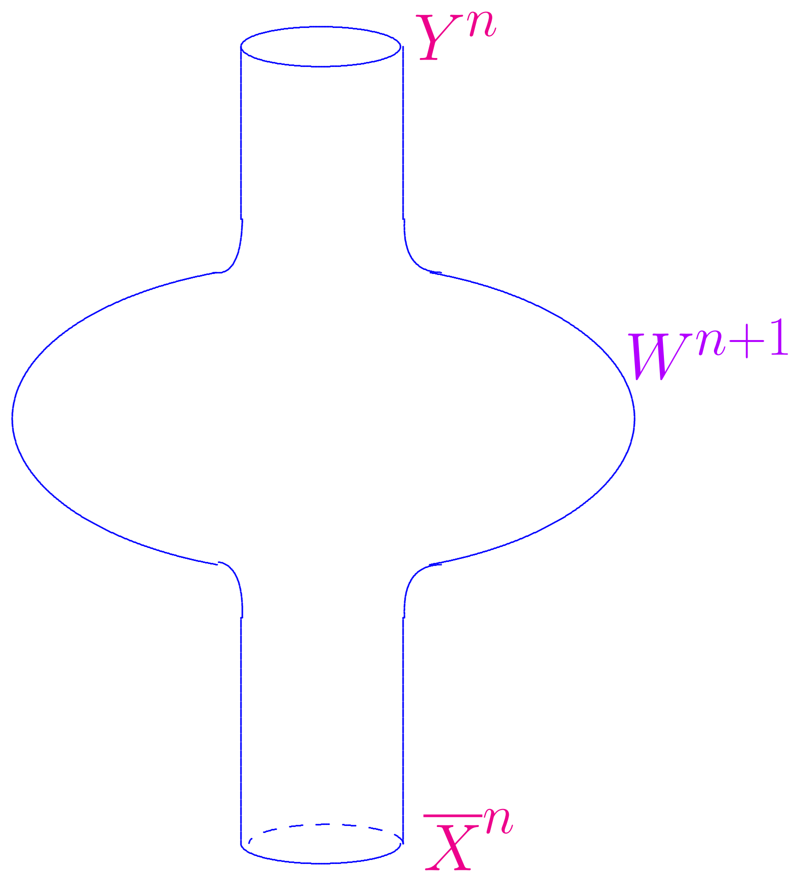

Lemma. *If* X^4 *and* Y^4 *are simply connected, non-spin, with* $\chi(X) = \chi(Y)$, $\tau(X) = \tau(Y)$, *then*

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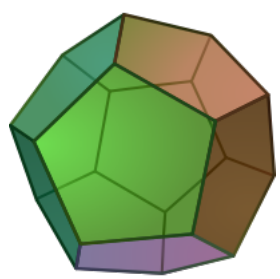

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Barlow surface X is minimal resolution of Q.

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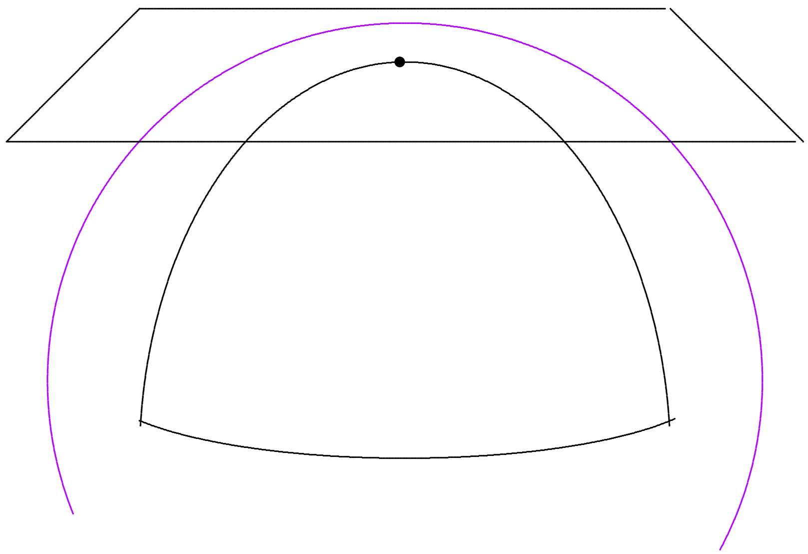

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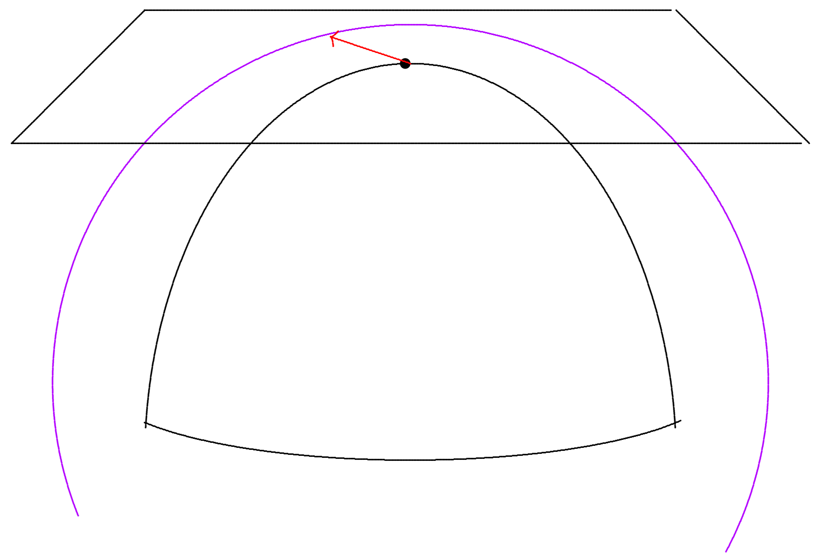

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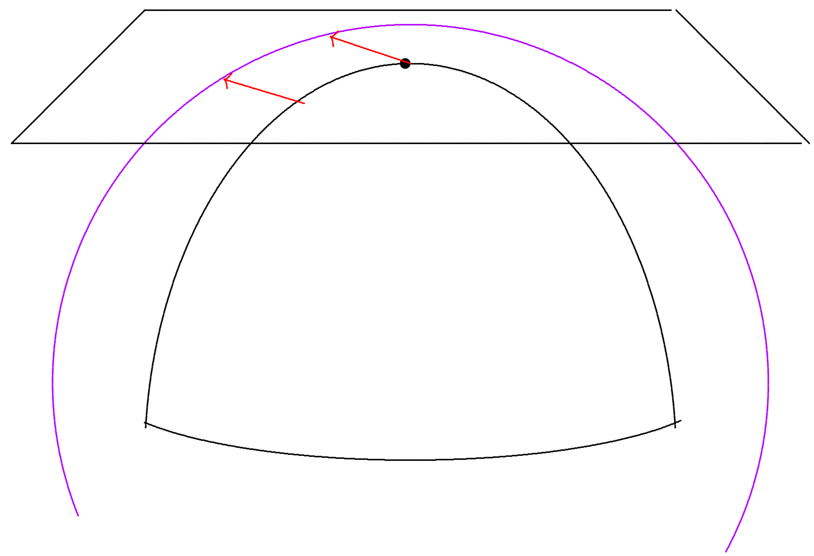

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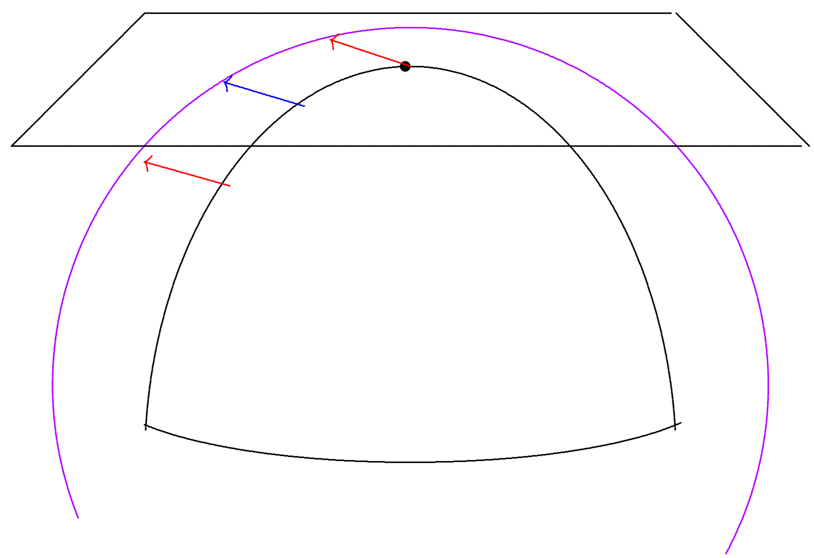

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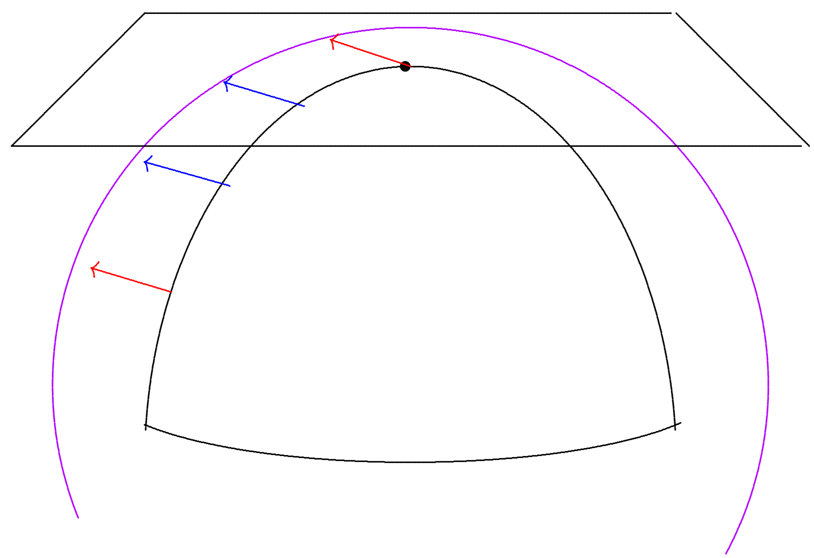

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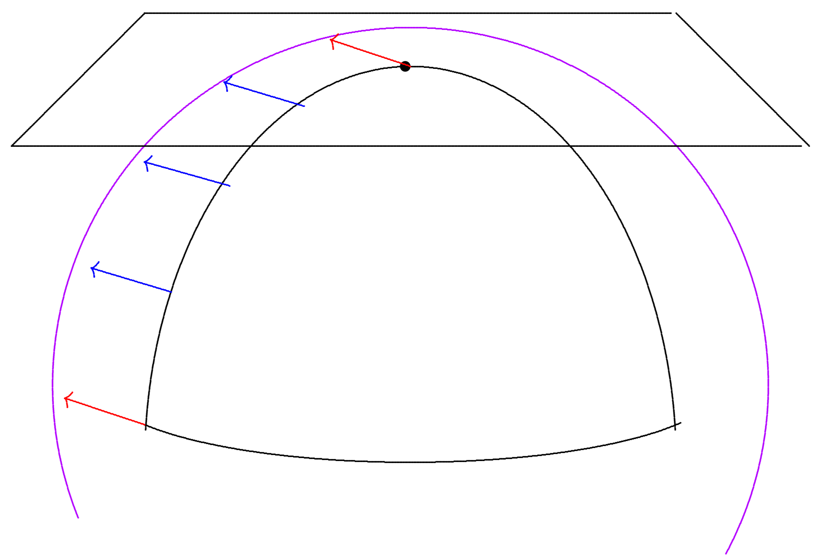

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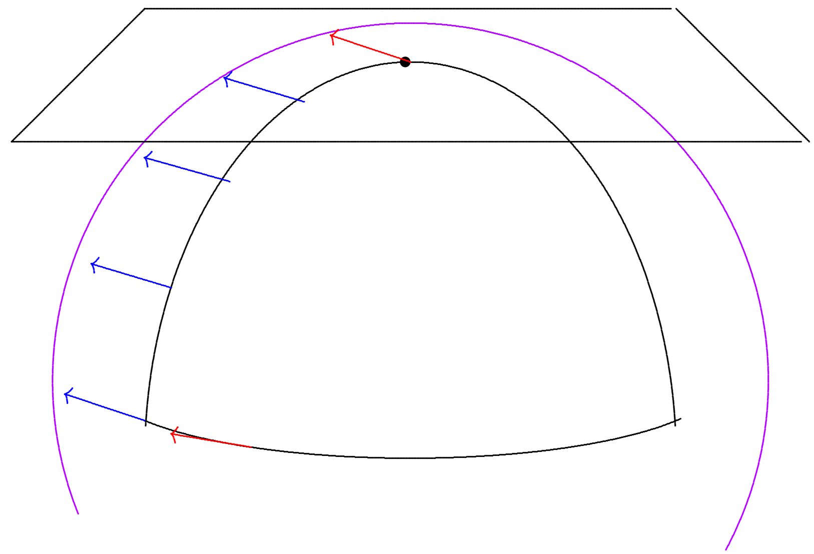

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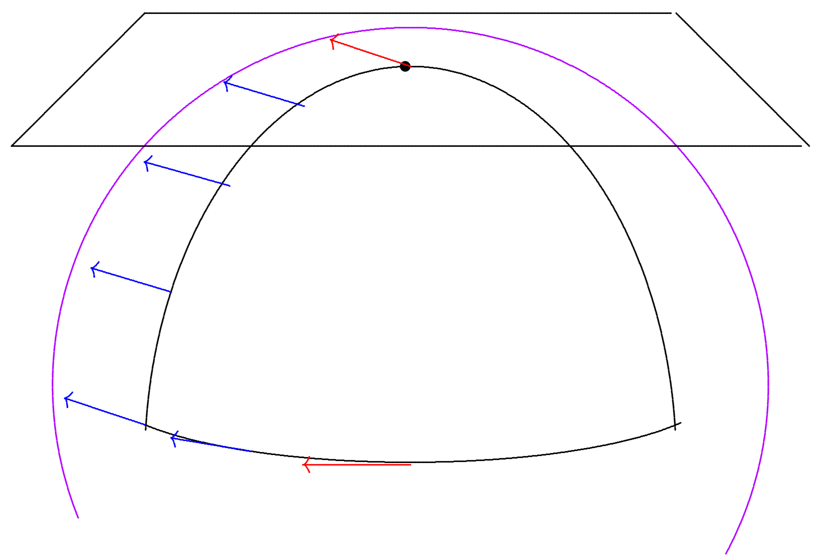

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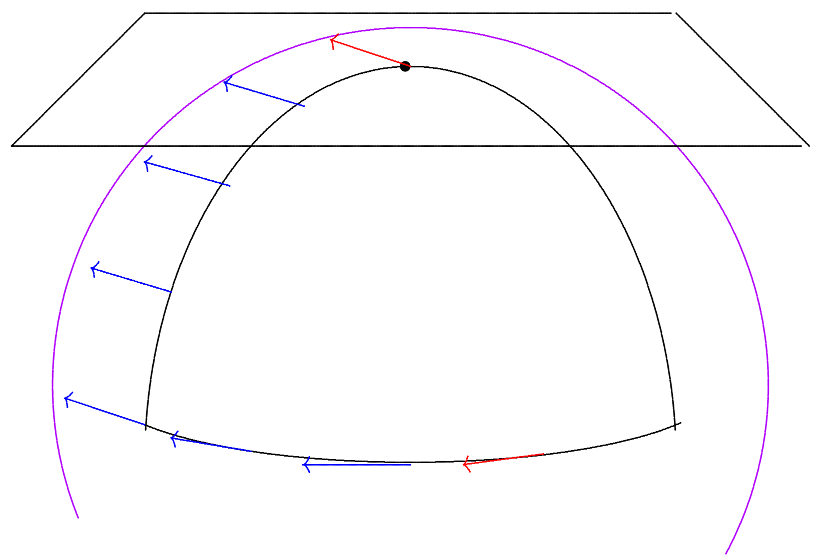

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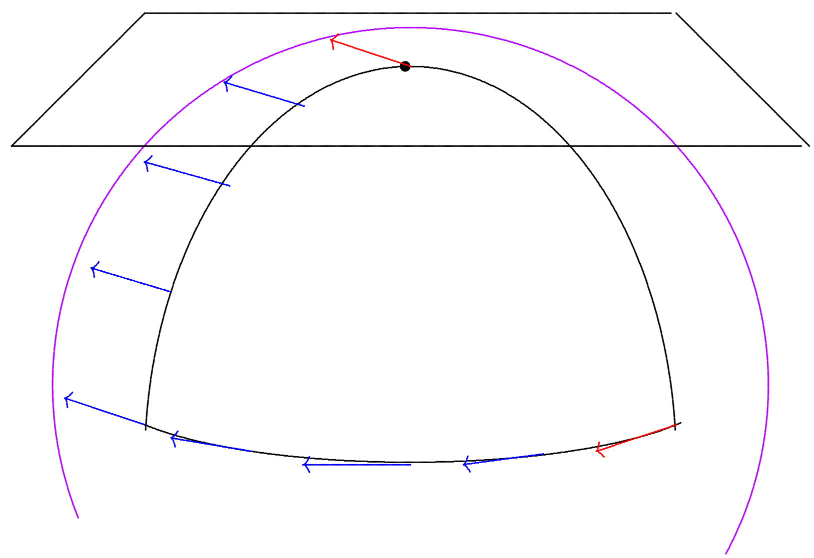

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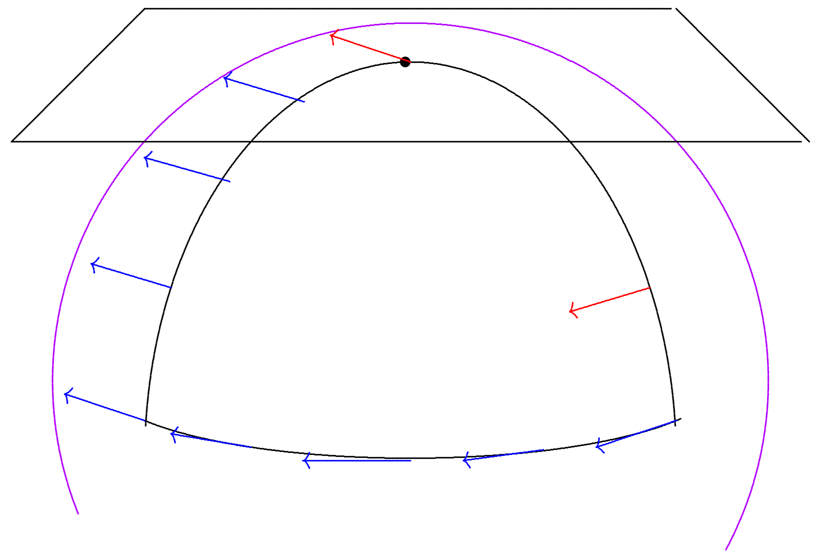

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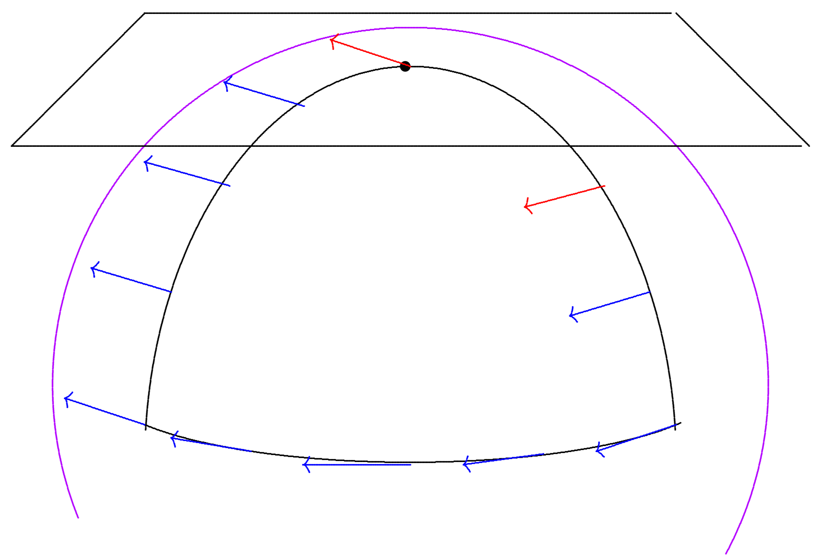

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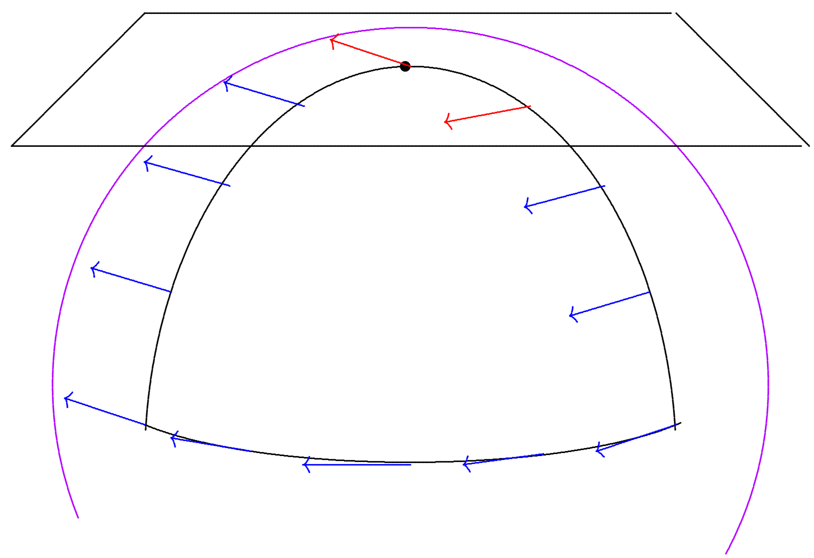

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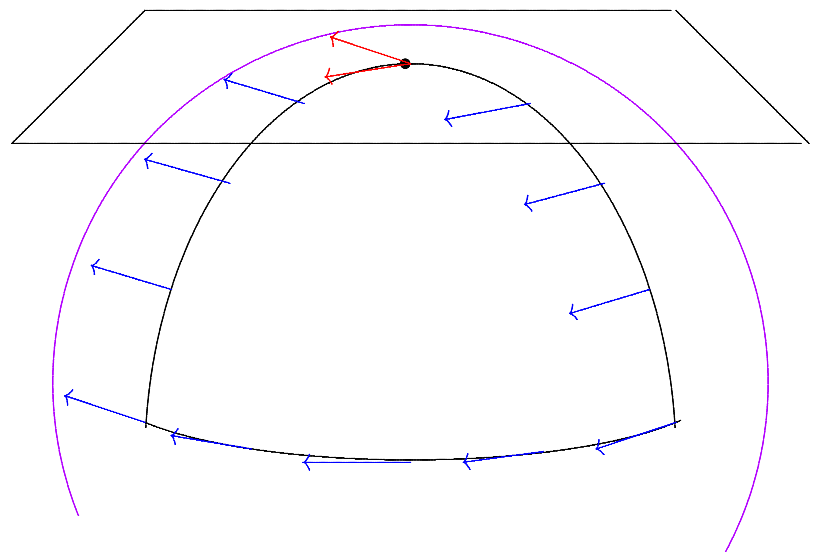

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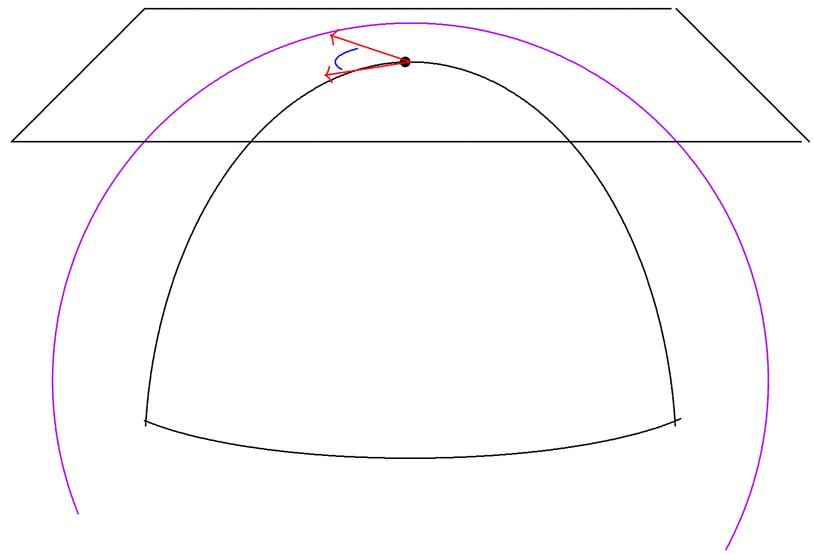
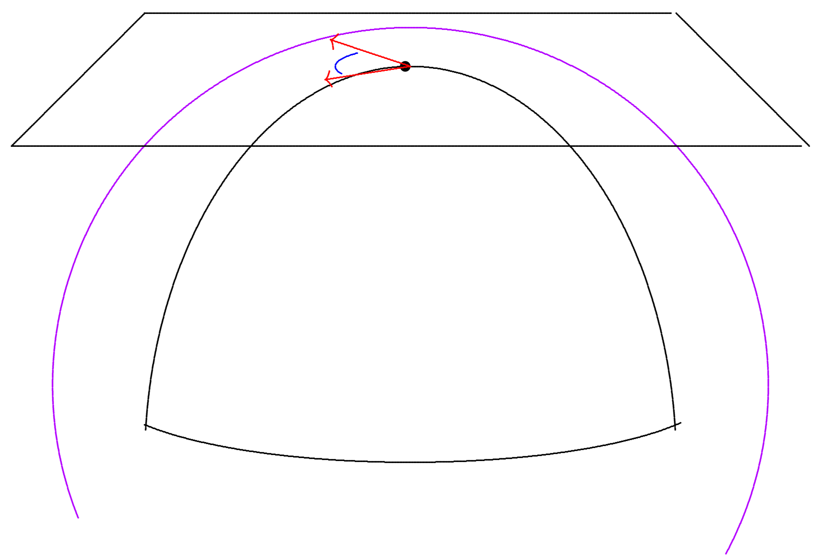


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

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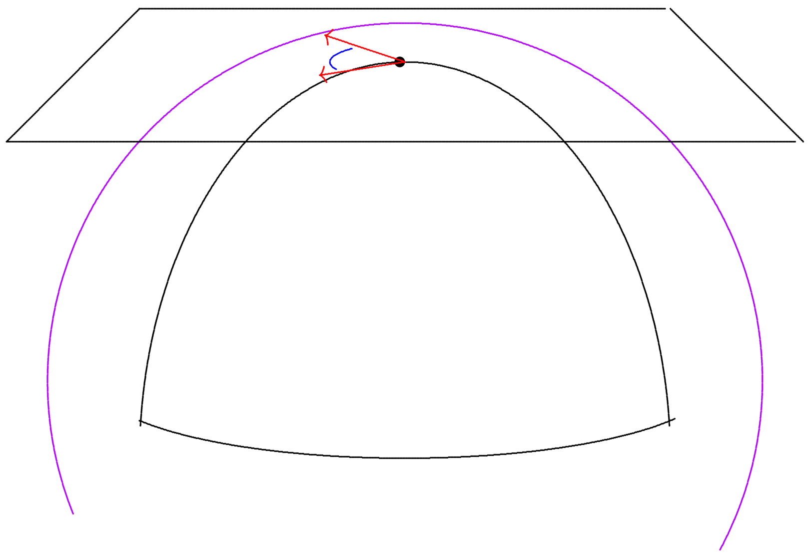

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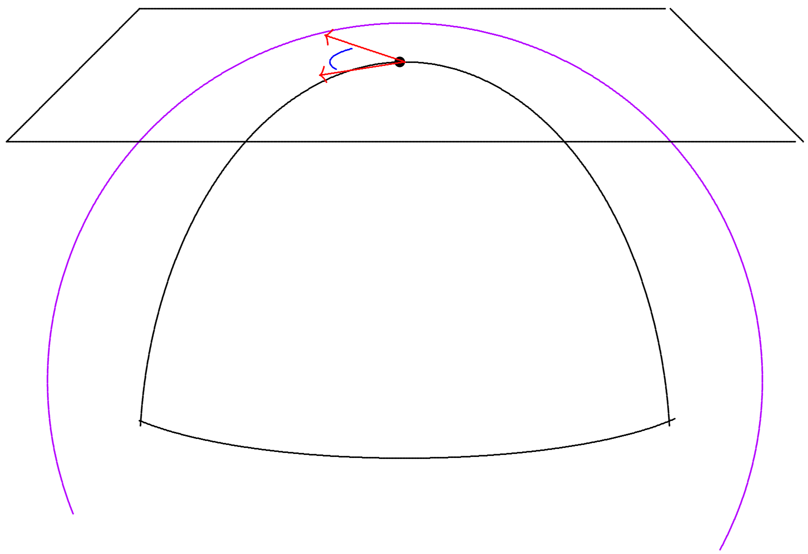

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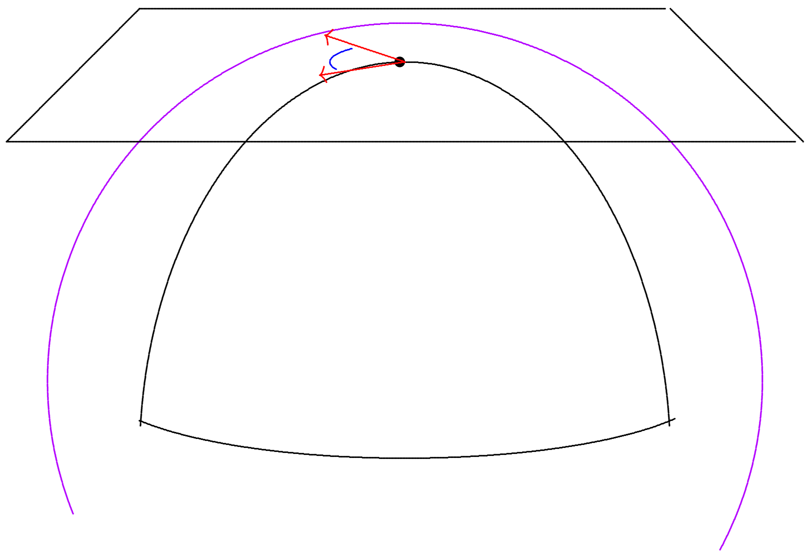

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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Kähler, but for wildly different complex structures.

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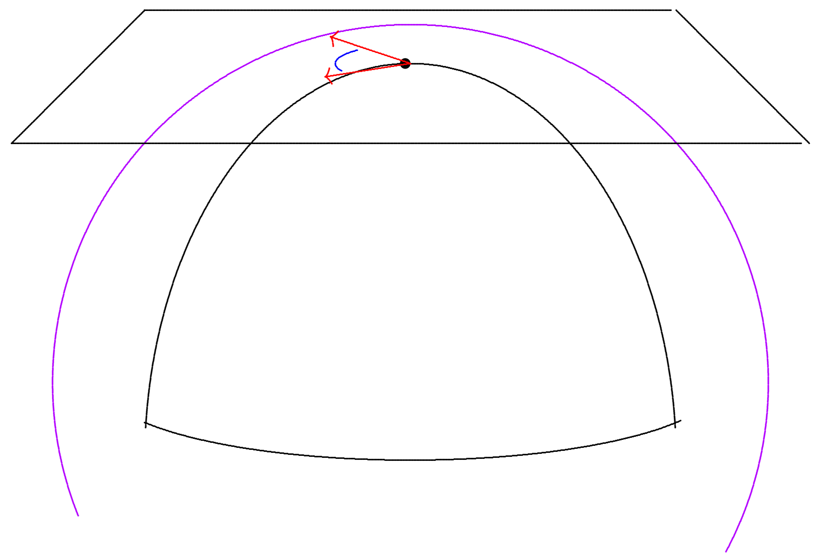

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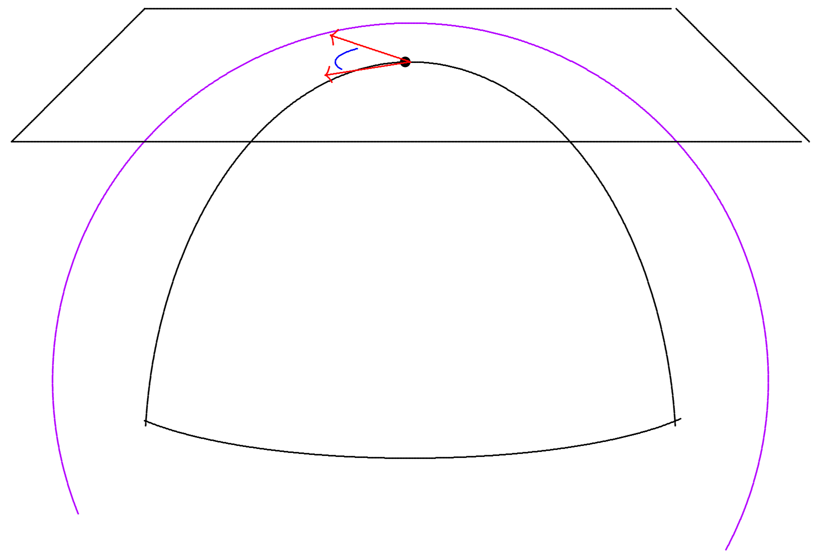

What are the possible holonomy groups?

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Alekseevsky, Calabi, Hitchin, Salamon, Bryant, Joyce. . .

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dimension n	$\mathrm{Hol}(M^n, g)$	geometry	Einstein?
n	$\mathbf{SO}(n)$	generic	?
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$2m$	$\mathbf{SU}(m)$	Calabi-Yau	$\checkmark$
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n	$\mathbf{SO}(n)$	generic	?
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The latter seems to be especially delicate!

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Can such metrics coexist?

Alas, no!

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Theorem (L '25)**.**

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Theorem. *No smooth compact* M^7

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To make this plausible,

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To make this plausible, will first illustrate assertion for prototypical examples due to S. Kobayashi '63.

Prototypical examples:

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M unit circle bundle in $K^* = \Lambda^3 T^{1,0} X$.

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Sub-complex of deRham complex:

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Replace Hard Lefschetz on $H^*(X, \mathbb{R})$ with transverse version on $H^*_B(M, \mathfrak{F})$ due to El Kacimi-Alaoui.

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Theorem. *No smooth compact M^7 can admit both a Sasaki-Einstein metric g_1 and a metric g_2 with holonomy $\subset \mathbf{G}_2$.*

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CTC Wall: Set of invariants that determines when a given 6-manifold is diffeomorphic to one of these.

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But no one has yet to discover a Calabi-Yau partner for any of these Fano manifolds!

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Buon compleanno, Simon!

E cento di questi giorni!

Happy Birthday, Simon!

And Many Happy Returns!