Einstein Metrics, Four-Manifolds, and Differential Topology

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A Riemannian metric g on a smooth manifold M is said to be *Einstein* if it has constant Ricci curvature, or in other words if

$$r = \lambda g, \tag{1}$$

where r is the Ricci tensor of g and λ is some real constant [**7**]. We still do not know if there are any obstructions to the existence of Einstein metrics on high-dimensional manifolds, but it has now been known for three decades that not every 4-manifold admits such metrics [**20, 37**]. Only recently, however, has it emerged that there are also obstructions to the existence of Einstein metrics which depend on the differentiable structure rather than just on the homotopy type of a 4-manifold [**27, 23, 28, 22**]. This article will attempt to give a concise explanation of this state of affairs.

Our current understanding of the problem rests largely on certain curvature estimates which are deduced from the Seiberg-Witten equations. The strongest of these [**28**] was originally proved indirectly, by invoking the solution to a generalized version of the Yamabe problem. One main purpose of the present article is to give (§4) a new and simpler proof of this estimate, using conformal rescaling only in order to introduce a generalized form of the Seiberg-Witten equations. But this article will also attempt to clarify the nature of the resulting obstructions, by systematically reformulating them in terms of a new diffeomorphism invariant, called $\alpha(M)$, which is introduced in §3.

As this article will make abundantly clear, Blaine Lawson's work on spin geometry and scalar curvature has had a deep and lasting impact on my own research. On a more personal level, Blaine has also been tremendous source of inspiration and encouragement throughout my many years at Stony Brook. I am lucky indeed to be able to call him a friend and colleague, and it is a very great pleasure for me to be able to contribute an article to this volume.

1. Differential Geometry on 4-Manifolds

The curvature and topology of 4-manifolds are interrelated in a number of ways that have no adequate analogs in other dimensions. Many of these phenomena

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are intimately related to the fact that the bundle Λ^2 of 2-forms over an oriented Riemannian 4-manifold (M, g) invariantly decomposes as the direct sum

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \tag{2}$$

of the eigenspaces of the Hodge star operator

$$\star : \Lambda^2 \to \Lambda^2.$$

The sections of Λ^+ are characterized by $\star\varphi = \varphi$, and so are called *self-dual 2-forms*, whereas the sections of Λ^- satisfy $\star\varphi = -\varphi$, and are called *anti-self-dual 2-forms*. Writing an arbitrary 2-from uniquely as

$$\varphi = \varphi^+ + \varphi^-,$$

where $\varphi^\pm \in \Lambda^\pm$, we then have

$$\varphi \wedge \varphi = \Big(|\varphi^+|^2 - |\varphi^-|^2\Big) d\mu_g,$$

where $d\mu_g$ denotes the metric volume form associated with the fixed orientation. The real importance of all this stems from the fact that the curvature of any connection on a vector bundle over M is a bundle-valued 2-form, and (2) therefore always gives rise to a decomposition of any curvature tensor into simpler pieces.

The Riemann curvature tensor of g is a good case in point. By raising an index, we can turn the Riemannian curvature into a self-adjoint linear map

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

called the *curvature operator*. Decomposing the 2-forms as in (2), this linear endomorphism of Λ^2 can then be decomposed into irreducible pieces

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W^- + \frac{s}{12} \end{array} \right). \tag{3}$$

Here s denotes the scalar curvature, whereas $\overset{\circ}{r} = r - \frac{s}{4}g$ denotes the trace-free part of the Ricci curvature. The tensors $W_\pm$ are the trace-free pieces of the appropriate blocks, and are respectively called the *self-dual* and *anti-self-dual Weyl curvature* tensors. The corresponding pieces of the Riemann tensor enjoy the remarkable property of being *conformally invariant*, in the sense that they remain unaltered when the metric is multiplied by an arbitrary smooth positive function. Notice that Einstein condition (1) is satisfies iff $\mathcal{R}$ commutes with the star operator; in more elementary terms, this amounts to the statement that a Riemannian 4-manifold is Einstein iff the sectional curvatures coincide for every orthogonal pair of 2-planes $P, P^\perp \subset T_xM$, $x \in M$.

Now let us suppose that (M, g) is a *compact* oriented Riemannian 4-manifold. The Hodge theorem then tells us that every de Rham class on M has a unique harmonic representative, so that we have a canonical identification

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d \star \varphi = 0\}.$$

However, the Hodge star operator $\star$ defines an involution of the right-hand side. We therefore have a direct-sum decomposition

$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g, \tag{4}$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

are the spaces of self-dual and anti-self-dual harmonic forms.

The intersection form

$$\begin{aligned} Q : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) &\mapsto \int_M \varphi \wedge \psi \end{aligned}$$

becomes positive-definite when restricted to $\mathcal{H}^+_g$, and negative-definite when restricted to $\mathcal{H}^-_g$. Moreover, these two subspaces are mutually orthogonal with respect to Q. Indeed, combining an L^2-orthonormal basis for $\mathcal{H}^+_g$ with an L^2-orthonormal basis for $\mathcal{H}^-_g$ gives one a basis for $H^2(\mathbb{R})$ in which the intersection form is represented by the diagonal matrix

$$\left[\begin{array}{cc} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{b_+(M)} & \\ & b_-(M)\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{array}\right].$$

The numbers $b_\pm(M) = \dim \mathcal{H}^\pm_g$ are thus oriented homotopy invariants of M; namely, b_+ (respectively, b_-) is the maximal dimension possible for a linear subspace of $H^2(M,\mathbb{R})$ on which the restriction of Q is positive (respectively, negative) definite.

The assignment $g \mapsto \mathcal{H}^+_g$ gives one an important canonical map

$$\{\text{Riemannian metrics on } M\} \longrightarrow Gr^+_{b_+}[H^2(M,\mathbb{R})]$$

from the infinite-dimensional space of all metrics to the finite-dimensional Grassmannian of $b_+(M)$-dimensional subspaces of $H^2(M,\mathbb{R})$ on which the intersection form is positive definite. This map is called the *period map* of M. It is obviously invariant under the action of the identity component $\mathit{Diff}_0(M)$ of the diffeomorphism group, and can also be shown to be invariant under the action of the smooth functions $M \to \mathbb{R}^+$ by conformal rescaling. A more subtle fact is that the period map is smooth, and has no critical points [**13**].

The difference $\tau(M) = b_+(M) - b_-(M)$ is called the *signature* of M. Like the Euler characteristic $\chi(M) = 2 - 2b_1(M) + b_2(M)$, it is the index of a geometric elliptic operator, and so may be expressed as a curvature integral. Indeed, to be explicit, one has

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left[\frac{s^2}{24} + |W^+|^2 + |W^-|^2 - \frac{|\mathring{r}|^2}{2}\right] d\mu$$

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left[|W^+|^2 - |W^-|^2\right] d\mu$$

for absolutely any Riemannian metric g on M. In particular, we may combine these to obtain the Gauss-Bonnet-like formulæ

$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left[\frac{s^2}{24} + 2|W_\pm|^2 - \frac{|\mathring{r}|^2}{2}\right] d\mu. \tag{5}$$

Notice, however, that the integrand in this last expression is non-negative for any Einstein metric. This gives us an important constraint, discovered independently by Thorpe [**37**] and Hitchin [**20**], on the topology of 4-dimensional Einstein manifolds.

Theorem 1 (Hitchin-Thorpe Inequality). *If the smooth compact oriented 4-manifold M admits an Einstein metric g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality iff (M, g) is finitely covered by a flat 4-torus T^4 or by $K3$ with a hyper-Kähler metric. Moreover, (M, g) must also satisfy

$$(2\chi - 3\tau)(M) \geq 0,$$

and this inequality is strict unless (M, g) is finitely covered by a flat 4-torus T^4 or by the orientation-reversed version of $K3$ with a hyper-Kähler metric.

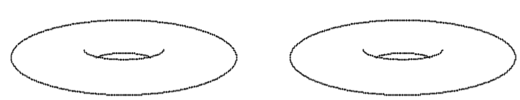
Here a smooth Riemannian metric g on an oriented 4-manifold M is called *hyper-Kähler* if the induced connection on Λ^+ is flat and trivial. Up to diffeomorphism, there is exactly one simply connected compact 4-manifold admits such a metric, called $K3$, in honor of Kummer, Kähler, and Kodaira. This 4-manifold is spin (meaning that its tangent bundle has $w_2 = 0$), and has $b_+ = 3$, $b_- = 19$. One model of $K3$ is the quartic hypersurface

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$$

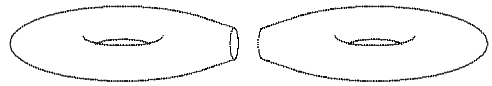
in $\mathbb{CP}_3$. The important point is that $K3$ is the underlying smooth oriented 4-manifold of a simply connected compact complex surface with $c_1 = 0$, because a truly remarkable theorem of Kodaira asserts that all the complex surfaces satisfying these conditions are deformation equivalent, and hence mutually diffeomorphic [**4**].

In order to put the Hitchin-Thorpe inequality to work, let us consider the following important way of constructing new manifolds.

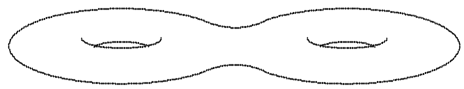
Definition 1. *Let M_1 and M_2 be smooth connected compact oriented n-manifolds.*



Their connected sum $M_1 \# M_2$ is then the smooth connected oriented n-manifold obtained by deleting a small ball from each manifold



and identifying the resulting S^{n-1} boundaries



via a reflection.

If M_1 and M_2 are simply connected 4-manifolds, then $M = M_1 \# M_2$ is also simply connected, and has $b_\pm(M) = b_\pm(M_1) + b_\pm(M_2)$. Now let us use $\mathbb{CP}_2$ denote the complex projective plane with its *standard* orientation, and $\overline{\mathbb{CP}}_2$ denote the same smooth 4-manifold with the *opposite* orientation. Then the iterated connected sum

$$k\mathbb{CP}_2 \# \ell\overline{\mathbb{CP}}_2 = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{k} \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{\ell}$$

is a simply connected 4-manifold with $b_+ = k$ and $b_- = \ell$. In particular,

$$(2\chi + 3\tau)(k\mathbb{CP}_2 \# \ell\overline{\mathbb{CP}}_2) = 4 + 5k - \ell,$$

so this construction gives us lots of simply connected 4-manifolds which do not admit Einstein metrics, by taking ℓ to be sufficiently large with respect to k.

We have now seen that Euler characteristic χ and signature τ have a vital rôle to play in the theory of 4-dimensional Einstein manifolds. However, knowing the Euler characteristic and signature of a *simply connected* 4-manifold is equivalent to knowing the invariants $b_\pm$, or in other words knowing the intersection form Q up to isomorphism as a quadratic form over $\mathbb{R}$. Now a celebrated result of Michael Freedman [**14**] asserts that simply connected smooth 4-manifolds are determined up to homeomorphism by their intersection forms over $\mathbb{Z}$. However, the integer coefficient version of the intersection form certainly contains more information than just $b_\pm$; indeed, the *parity* of the form (even or odd) determines whether or not the 4-manifold is spin. Indefinite quadratic forms over $\mathbb{Z}$ turn out to classified [**16**] by parity and $b_\pm$. On the other hand, a gauge-theoretic argument of Donaldson [**12**] shows that only the simplest definite forms can arise as intersection forms of 4-manifolds. One thus obtains the following remarkable classification result:

Theorem 2 (Freedman). *Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if*

- *they have the same Euler characteristic χ;*
- *they have the same signature τ; and*
- *both are spin, or both are non-spin.*

As a consequence, any smooth compact simply connected non-spin 4-manifold is homeomorphic to a connected sum $k\mathbb{CP}_2 \# \ell\overline{\mathbb{CP}}_2$. For spin manifolds, the situation is a bit more unsettled, but the connected sums $mK3\#n(S^2 \times S^2)$ and their orientation-reversed versions, together with S^4, at least exhaust all the simply connected homeotypes for which the additional constraint $\chi \geq \frac{11}{8}|\tau| + 2$ is satisfied.

The so-called $11/8$ conjecture asserts that this last inequality is in fact automatically satisfied, or in other words that the above list of spin homeotypes is complete. A strong partial result in this direction has been proved by Furuta [**15**].

For the purposes of Riemannian geometry, however, this beautiful classification is somewhat aside from the point, since it is concerned with classification up to *homeomorphism*, not *diffeomorphism*. In order to do differentiable geometry, we need a differentiable structure. However, many of the above topological manifolds turn out to have *infinitely many* inequivalent differentiable structures, and the difference between these smooth structures is often detectable by asking questions about the curvature of Riemannian metrics. This article will attempt to explain some of the ramifications that this interplay between Riemannian geometry and differential topology is now known to have for the existence and uniqueness of Einstein metrics on 4-manifolds.

2. Examples of Einstein Manifolds

Most of the currently available examples of compact 4-dimensional Einstein manifolds are Kähler. Recall that a Riemannian manifold (M, g) is called *Kähler* if it admits an almost-complex structure $J : TM \to TM$, $J^2 = -1$, which is invariant under parallel transport with respect to g. Such an almost-complex structure is automatically integrable, and (M, J) may therefore be viewed as a complex manifold.

One may ask when a given compact complex manifold admits a compatible Kähler metric which is also Einstein. For Einstein metrics of negative Ricci curvature, the definitive solution to this problem was found independently by Aubin [**3**] and Yau [**40**]:

THEOREM 3 (Aubin/Yau). *A compact complex manifold (M^{2m}, J) admits a compatible Kähler-Einstein metric with $s < 0$ iff its canonical line bundle $K = \Lambda^{m,0}$ is ample. When such a metric exists, it is unique, up to an overall multiplicative constant.*

Here a holomorphic line bundle $L \to M$ is called *ample* if it it has a positive power $L^{\otimes k}$ with enough holomorphic sections to yield an embedding $M \hookrightarrow \mathbb{CP}_n$. As it turns out, a compact complex manifold (M, J) of real dimension 4 has ample canonical line bundle K iff it is a minimal complex surface of general type without (-2)-curves [**4**]. These exist in great profusion.

Yau [**40**] also gave a definitive solution to the analogous problem for Ricci-flat metrics:

THEOREM 4 (Yau). *A compact complex manifold (M, J) admits a compatible Kähler-Einstein metric with $s = 0$ iff (M, J) admits a Kähler metric and $K^{\otimes \ell}$ is trivial for some positive integer ℓ. When this happens, there is exactly one such metric in each Kähler class.*

In real dimension 4, there are exactly two diffeotypes of compact Kähler manifolds for which K is trivial, namely $K3$ and T^4. The Ricci-flat Kähler metrics on these manifolds are exactly the hyper-Kähler metrics alluded to in our previous discussion. Any other compact Kähler surface with K^ℓ trivial for some positive ℓ is the quotient of $K3$ or T^4 by the free action of a finite group of isometries of one of these hyper-Kähler metrics.

The existence of Kähler-Einstein metrics is much more delicate in the case of positive Ricci curvature. In real dimension 4, however, a complete solution to the problem was given by Tian [**38**]:

THEOREM 5 (Tian). *A compact complex surface (M^4, J) admits a compatible Kähler-Einstein metric with $s > 0$ iff its anti-canonical line bundle K^{-1} is ample and its Lie algebra of holomorphic vector fields is reductive.*

The 4-manifolds which carry Einstein metrics by virtue of this last result are $\mathbb{CP}_2$, $S^2 \times S^2$, and $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $3 \leq k \leq 8$.

Until quite recently, only sporadic examples of non-Kähler compact 4-dimensional Einstein manifolds were known. An interesting case in point is the Page metric [**7, 31**] on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$, which is beautiful, but has yet to lead to the construction of other compact examples. Of course, we have long known [**9**] that there is an infinite class of compact hyperbolic manifolds $\mathcal{H}^4/\Gamma$, but it is easy to dismiss these constant-sectional-curvature spaces as a bit boring in the small. A recent construction of Michael Anderson [**2**], however, puts these hyperbolic manifolds in a new, non-trivial context. Anderson's construction begins with a complete, non-compact hyperbolic manifold of finite volume, replaces the cusps with Schwarzschild-anti-deSitter metrics, and then, under mild additional technical hypotheses, perturbs the resulting metric so as to make it Einstein. Of course, these new Einstein manifolds still bear a family resemblance to their hyperbolic cousins, as they always have infinite fundamental group and vanishing signature, and always contain large regions where the sectional curvature is nearly constant. Nevertheless, Anderson's construction does seem to represent the first systematic method of constructing such a large class of compact non-locally-symmetric Einstein manifolds of general holonomy.

3. The Seiberg-Witten Equations

If M is a smooth oriented 4-manifold, $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ is always in the image of the natural homomorphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$ induced by $\mathbb{Z} \to \mathbb{Z}_2$. Consequently, we can always find Hermitian line bundles $L \to M$ such that $c_1(L) \equiv w_2(TM) \bmod 2$. For any such L, and for any Riemannian metric g on M, one can then find rank-2 Hermitian vector bundles $\mathbb{V}_\pm$ which formally satisfy

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2},$$

where $\mathbb{S}_\pm$ are the locally defined left- and right-handed spinor bundles of (M, g). Such a choice of $\mathbb{V}_\pm$, up to isomorphism, is called a spin^c structure $\mathfrak{c}$ on M. Moreover, $\mathfrak{c}$ is completely determined by the first Chern class $c_1(L) = c_1(\mathbb{V}_\pm) \in H^2(M, \mathbb{Z})$ if we assume that $H_1(M, \mathbb{Z})$ does not contain any elements of order 2.

Every unitary connection A on L induces a connection

$$\nabla_A : \Gamma(\mathbb{V}_+) \to \Gamma(\Lambda^1 \otimes \mathbb{V}_+),$$

and composition of this with the natural *Clifford multiplication* homomorphism

$$\Lambda^1 \otimes \mathbb{V}_+ \to \mathbb{V}_-$$

gives one a spin^c version

$$D_A : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

of the Dirac operator [**19, 25**]. This is an elliptic first-order differential operator, and in many respects closely resembles the usual Dirac operator of spin geometry. In particular, one has the Weitzenböck formula

$$\langle \Phi, D_A^* D_A \Phi \rangle = \frac{1}{2}\Delta|\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4}|\Phi|^2 + 2\langle -iF_A^+, \sigma(\Phi)\rangle \tag{6}$$

for any $\Phi \in \Gamma(\mathbb{V}_+)$, where F_A^+ is the self-dual part of the curvature of A, and where $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map satisfying

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

This, of course, generalizes the Weitzenböck formula used by Lichnerowicz [**29**] to prove that metrics with $s > 0$ cannot exist when M is spin and $\tau(M) = -8\hat{A}(M) \neq 0$. However, one cannot hope to derive interesting geometric information about the Riemannian metric g by just using (6) for an arbitrary connection A, since one would have no control at all over the F_A^+ term. Witten [**39**], however, had the brilliant insight that one could remedy this by considering both Φ and A as unknowns, subject to the *Seiberg-Witten equations*

$$D_A\Phi = 0 \tag{7}$$
$$F_A^+ = i\sigma(\Phi). \tag{8}$$

These equations are non-linear, but they become an elliptic first-order system once one imposes the 'gauge-fixing' condition

$$d^*(A - A_0) = 0$$

to eliminate the natural action of the 'gauge group' of automorphisms of the Hermitian line bundle $L \to M$.

Because the Seiberg-Witten equations are non-linear, one cannot use something like an index formula to predict that they must have solutions. Nonetheless, there exist spin^c structures on many 4-manifolds for which there is at least one solution for every metric g. This situation is conveniently described by the following terminology [**24**]:

Definition 2. *Let M be a smooth compact oriented 4-manifold with $b_+ \geq 2$. An element $a \in H^2(M, \mathbb{Z})/\text{torsion}$ will be called a* **monopole class** *of M iff there exists a spin^c structure $\mathfrak{c}$ on M with first Chern class*

$$c_1(L) \equiv a \mod \text{torsion}$$

which has the property that the corresponding Seiberg-Witten equations (7–8) have a solution for every Riemannian metric g on M.

For example, if (M, ω) is a symplectic 4-manifold with $b_+ \geq 2$, and if J is any almost-complex structure which is compatible with ω, then $\pm c_1(M, J)$ are both monopole classes [**35**]. Usually, one detects the presence of a monopole class by thinking of the moduli space of the Seiberg-Witten equations (that is, solutions modulo gauge equivalence) as a cycle which represents an element of the homology of a certain configuration space [**39, 36, 30**]. The resulting homology class is then metric-independent, and one may, for example, then check that it is non-zero by analyzing the moduli space for some particular metric. A sophisticated recent refinement of this idea, however, instead detects the presence of a monopole class by means of an element of a stable cohomotopy group [**5, 6, 21**].

Because the Seiberg-Witten equations imply the Weitzenböck formula

$$0 = 2\Delta|\Phi|^2 + 4|\nabla\Phi|^2 + s|\Phi|^2 + |\Phi|^4, \tag{9}$$

where $\nabla = \nabla_A$, one immediately sees that they admit no solution with $\Phi \not\equiv 0$ relative to a metric g with $s > 0$. This allows one to prove, in particular, that there are lots of simply connected non-spin 4-manifolds which do not admit positive-scalar-curvature metrics, in complete contrast to the situation in higher dimensions [**18**]. But indeed, this Weitzenböck formula makes an even more remarkable prediction concerning the behavior of the scalar curvature [**39, 26**]:

PROPOSITION 6. *Let (M, g) be a smooth compact oriented Riemannian manifold, let $\mathfrak{c}$ be a spin^c structure on M, and let c_1^+ denote the self-dual part of the harmonic 2-form representing the first Chern class $c_1(L)$ of $\mathfrak{c}$. If there is a solution of the Seiberg-Witten equations (7–8) on M for g and $\mathfrak{c}$, then the scalar curvature s_g of g satisfies*

$$\int_M s_g^2 d\mu_g \geq 32\pi^2[c_1^+]^2.$$

When $[c_1^+] \neq 0$, moreover, equality can only occur if g is a Kähler metric of constant scalar curvature.

PROOF. Integrating (9), we have

$$0 = \int [4|\nabla\Phi|^2 + s|\Phi|^2 + |\Phi|^4] d\mu,$$

and it follows that

$$\int (-s)|\Phi|^2 d\mu \geq \int |\Phi|^4 d\mu.$$

Applying the Cauchy-Schwarz inequality to the left-hand side therefore gives us

$$\left(\int s^2 d\mu\right)^{1/2} \left(\int |\Phi|^4 d\mu\right)^{1/2} \geq \int |\Phi|^4 d\mu,$$

so that

$$\int s^2 d\mu \geq \int |\Phi|^4 d\mu = 8 \int |F_A^+|^2 d\mu,$$

and the inequality is strict unless $\nabla\Phi \equiv 0$ and s is constant. However, $F_A^+ - 2\pi c_1^+$ is an exact form plus a co-exact form, and so is L^2-orthogonal to the harmonic forms. This gives us the inequality

$$\int |F_A^+|^2 d\mu \geq 4\pi^2 \int |c_1^+|^2 d\mu = 4\pi^2 \int c_1^+ \wedge c_1^+,$$

and the last expression may be re-interpreted as the intersection pairing $[c_1^+]^2$ of the de Rham class of c_1^+ with itself. This gives us the desired inequality

$$\int s^2 d\mu \geq 32\pi^2[c_1^+]^2,$$

and, when the right-hand side is non-zero, equality can only happen if g has special holonomy and constant scalar curvature. □

One important application of this estimate is the following fundamental fact:

PROPOSITION 7. *Let M be any smooth compact oriented 4-manifold with $b_+(M) \geq 2$. Then $\mathcal{C} = \{$monopole classes of $M\}$ is a finite set.*

Proof. Let $g_1 = g$ be any Riemannian metric on M, and let $\mathbf{e}_1 = [\omega_1]$ be the cohomology class of a harmonic self-dual form with respect to g, normalized so that $\mathbf{e}_1^2 := Q(\mathbf{e}_1, \mathbf{e}_1) = 1$. Because every metric is a regular point of the period map [**13**, Prop. 4.3.14], it follows that $[\omega_1]$ has an open neighborhood in $H^2(M,\mathbb{R})$ in which every element can be represented by a self-dual harmonic form relative to some perturbation of g. However, any open set in a finite-dimensional vector space spans the entire space. Hence we can find a basis $\{\mathbf{e}_j\}$ for $H^2(M,\mathbb{R})$, together with a collection of Riemannian metrics $\{g_j \mid j = 1, \ldots, b_2\}$, such that the g_j-harmonic 2-form ω_j representing the de Rham class $\mathbf{e}_j$ is self-dual with respect to g_j.

For convenience, let us now normalize these basis elements so that $\mathbf{e}_j^2 = 1$ for each j, and let $L_j : H^2(M,\mathbb{R}) \to \mathbb{R}$ be the linear functionals $L_j(x) = \mathbf{e}_j \cdot x := Q(\mathbf{e}_j, x)$. Then Proposition 6, together with the Cauchy-Schwarz inequality, implies that any monopole class $a \in H^2(M,\mathbb{Z})/\text{torsion}$ satisfies

$$|L_j(a)| = |\mathbf{e}_j \cdot a^+_{g_j}| \leq \sqrt{(a^+_{g_j})^2} \leq \left(\frac{1}{32\pi^2}\int_M s_{g_j}^2 d\mu_{g_j}\right)^{1/2} = \kappa_j ,$$

where the constant κ_j is independent of a. This shows that $\mathcal{C} \subset H^2(M,\mathbb{R})$ is contained in the $b_2(M)$-dimensional parallelepiped

$$\left\{ x \in H^2(M,\mathbb{R}) \;\Big|\; |L_j(x)| \leq \kappa_j \;\forall j = 1, \ldots, b_2(M) \right\},$$

which is a compact set. Since $\mathcal{C} \subset H^2(M,\mathbb{Z})/\text{torsion}$ is also discrete, it follows that $\mathcal{C}$ is finite. $\square$

We now want to extract a numerical invariant from the set of monopole classes which captures those features of Seiberg-Witten theory which are of the greatest relevance to problems in Riemannian geometry. To this end, let us once again consider the open Grassmannian $\mathbb{G} = Gr^+_{b_+}[H^2(M,\mathbb{R})]$ of all maximal linear subspaces $\mathcal{H}$ of the second cohomology for which the restriction $Q|_{\mathcal{H}}$ of the intersection pairing is positive definite. Each element $\mathcal{H} \in \mathbb{G}$ then determines an orthogonal decomposition

$$H^2(M,\mathbb{R}) = \mathcal{H} \oplus \mathcal{H}^\perp$$

with respect to Q. Let

$$\mathfrak{C} := \mathcal{C} - \{0\} \subset H^2(M,\mathbb{Z})/\text{torsion} \subset H^2(M,\mathbb{R})$$

denote the set of all the non-zero monopole classes $a \neq 0$ of M. Given a monopole class $a \in \mathfrak{C}$ and a positive subspace $\mathcal{H} \in \mathbb{G}$, we may then define a^+ to be the orthogonal projection of a into $\mathcal{H}$ with respect to Q. Using this, we now define an oriented-diffeomorphism invariant $\alpha(M) \in [0, \infty)$.

Definition 3. *Let M be a smooth compact oriented manifold with $b_+ \geq 2$, and let $\mathfrak{C} \subset H^2(M,\mathbb{Z})/$torsion be the set of the non-zero monopole classes of M. If $\mathfrak{C} = \emptyset$, we declare that $\alpha(M) = 0$. Otherwise, we set*

$$\alpha(M) = \inf_{\mathcal{H}\in\mathbb{G}} \left[\max_{a\in\mathfrak{C}} \sqrt{(a^+)^2}\right] .$$

As a matter of notational convenience, let us also set $\alpha^2(M) := [\alpha(M)]^2$.

Needless to say, this is not the simplest invariant that one can cook up using the Seiberg-Witten equations. However, it precisely captures those aspects of Seiberg-Witten theory which are of the greatest relevance to many problems in Riemannian geometry. In particular, Proposition 6 has the following important consequence:

PROPOSITION 8. *Let M be any smooth compact oriented 4-manifold with $b_2 \geq 2$. Then every Riemannian metric g on M satisfies*

$$\int_M s_g^2 d\mu_g \geq 32\pi^2\alpha^2(M). \tag{10}$$

Moreover, if $\alpha(M) \neq 0$, then equality can hold only if g is a Kähler metric of constant negative scalar curvature.

In particular, any metric, or sequence of metrics, on M gives one an explicit upper bound for $\alpha(M)$. On the other hand, one can obtain a lower bound for $\alpha(M)$ by replacing $\mathfrak{C}$ with any known set of non-zero monopole classes. Remarkably, the upper and lower bounds obtained in this manner actually coincide for many 4-manifolds, and in these circumstances one can therefore determine the invariant α even without necessarily knowing all the monopole classes on M. For example, the results of [**27, 21**] allow one to read off the following:

PROPOSITION 9. *Let M be the underlying smooth oriented 4-manifold of any compact complex surface (M, J) with $b_+ > 1$. Let X be the minimal model of (M, J). Then*

$$\alpha^2(M) = c_1^2(X).$$

PROPOSITION 10. *Let X_1, X_2, X_3 be minimal, simply connected complex surfaces with $b_+ \equiv 3 \bmod 4$, and let $M = X_1 \# X_2 \# X_3$. Then*

$$\alpha^2(M) = c_1^2(X_1) + c_1^2(X_2) + c_1^2(X_3).$$

Recall that a complex surface is said to be *minimal* if it is not obtained from another complex surface by blowing up. Any complex surface M can be obtained from some minimal surface X, called its minimal model, by blowing up a finite number of times [**4**], and this minimal model X is unique when $b_+ > 1$. As smooth manifolds, one then has

$$M \approx X \# k\overline{\mathbb{CP}}_2$$

for some integer $k \geq 0$, and the statement that X is minimal amounts to saying that this value of k is maximal.

To see that all this is relevant to the study of Einstein metrics, notice that (10) can be rewritten as

$$\frac{1}{4\pi^2}\int_M \frac{s^2}{24} d\mu \geq \frac{1}{3}\alpha^2(M).$$

Comparing the left-hand side with our Gauss-Bonnet formula (5) for $(2\chi - 3\tau)(M)$, we thus immediately obtain the following improvement of one of the Hitchin-Thorpe inequalities:

THEOREM 11. *If a smooth compact oriented 4-manifold M with $b_+ > 1$ admits an Einstein metric g, then*

$$(2\chi - 3\tau)(M) \geq \frac{1}{3}\alpha^2(M).$$

Moreover, if $\alpha(M) \neq 0$, equality occurs if and only if (M, g) is a compact quotient $\mathbb{C}\mathcal{H}_2/\Gamma$ of the complex hyperbolic plane, equipped with a constant multiple of its standard Kähler-Einstein metric.

In particular, this tells us that the Einstein metric on a complex-hyperbolic manifold $\mathbb{C}\mathcal{H}_2/\Gamma$ is *unique*, modulo diffeomorphisms and rescaling [**27**].

Now one can obviously imitate this argument by using $2\chi+3\tau$ instead of $2\chi-3\tau$. However, one would expect for the results obtained in this way [**27**] to be quite far from optimal, since the Kähler metrics which saturate (10) when $\alpha \neq 0$ always have $W_+ \neq 0$. In the next section, we will remedy this, by replacing Proposition 6 with an estimate which involves the self-dual Weyl curvature W_+ as well as the scalar curvature s.

4. The Main Curvature Estimate

The Dirac operator is conformally invariant, provided that the relevant spinors are viewed as having appropriate conformal weights [**25, 32**]. Thus, if $\hat{\Phi}$ solves the spin^c Dirac equation $\hat{D}_A\hat{\Phi} = 0$ with respect to the conformally rescaled metric $f^{-2}g$, there is a corresponding solution Φ of the equation $D_A\Phi = 0$ with respect to g, with $|\Phi|_g = f^{-3/2}|\hat{\Phi}|_{\hat{g}}$ and $\sigma(\Phi) = f\hat{\sigma}(\hat{\Phi})$. If $(\hat{\Phi}, A)$ is a solution of the Seiberg-Witten equations with respect to $f^{-2}g$, it thus follows that (Φ, A) solves the *rescaled Seiberg-Witten equations*

$$D_A\Phi = 0 \tag{11}$$
$$F_A^+ = if\sigma(\Phi). \tag{12}$$

with respect to g. We will now show that directly studying equations (11–12) for an appropriate family of choices of f will yield an efficient avenue for deducing curvature estimates for the fixed metric g.

Indeed, plugging (12) into the Weitzenböck formula for (11) gives us

$$\begin{aligned} 0 &= 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s|\Phi|^2 + 8\langle -iF_A^+, \sigma(\Phi)\rangle, \\ &= 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s|\Phi|^2 + f|\Phi|^4, \end{aligned}$$

so that multiplying by $|\Phi|^2$ gives us

$$0 = 2|\Phi|^2\Delta|\Phi|^2 + 4|\Phi|^2|\nabla_A\Phi|^2 + s|\Phi|^4 + f|\Phi|^6.$$

Integration therefore yields the inequality

$$0 \geq \int_M \left[4|\Phi|^2|\nabla_A\Phi|^2 + s|\Phi|^4 + f|\Phi|^6\right] d\mu. \tag{13}$$

We will now use this to obtain an estimate involving f and curvature of g.

Lemma 12. *Let (M, g) be a compact oriented Riemannian 4-manifold, and let $f > 0$ be a smooth positive function on M. Suppose, for some spin^c structure with first Chern class $c_1(L)$, that there is a a solution of the Seiberg-Witten equations with respect to the conformally related metric $f^{-2}g$. Let c_1^+ denote the self-dual part of the harmonic 2-form representing $c_1(L)$, relative to the metric g. Then*

$$\left(\int_M f^4 d\mu_g\right)^{1/3}\left(\int_M |s_g + 3w_g|^3 f^{-2}d\mu_g\right)^{2/3} \geq 72\pi^2[c_1^+]^2, \tag{14}$$

where $s_g : M \to \mathbb{R}$ denotes the scalar curvature of g and $w_g : M \to \mathbb{R}$ is the lowest eigenvalue of $W_+ : \Lambda^+ \to \Lambda^+$ at each $x \in M$.

PROOF. Any self-dual 2-form ψ on any oriented 4-manifold satisfies the Weitzenböck formula [**10**]

$$(d+d^*)^2\psi = \nabla^*\nabla\psi - 2W_+(\psi,\cdot) + \frac{s}{3}\psi, \tag{15}$$

where W_+ is the self-dual Weyl tensor. It follows that

$$\int_M \left(|\nabla\psi|^2 - 2W_+(\psi,\psi) + \frac{s}{3}|\psi|^2\right) d\mu \geq 0,$$

and hence that

$$\int_M |\nabla\psi|^2 d\mu \geq \int_M \left(2w - \frac{s}{3}\right)|\psi|^2 d\mu.$$

However, the particular self-dual 2-form $\sigma(\Phi)$ satisfies

$$\begin{aligned} |\sigma(\Phi)|^2 &= \frac{1}{8}|\Phi|^4, \\ |\nabla\sigma(\Phi)|^2 &\leq \frac{1}{2}|\Phi|^2|\nabla\Phi|^2. \end{aligned}$$

Setting $\psi = \sigma(\Phi)$, we thus have

$$\int_M 4|\Phi|^2|\nabla\Phi|^2\, d\mu \geq \int_M \left(2w - \frac{s}{3}\right)|\Phi|^4 d\mu.$$

But (13) tells us that

$$0 \geq \int_M \left[4|\Phi|^2|\nabla_A\Phi|^2 + s|\Phi|^4 + f|\Phi|^6\right] d\mu,$$

so we obtain

$$0 \geq \int_M \left[\left(\frac{2}{3}s + 2w\right)|\Phi|^4 + f|\Phi|^6\right] d\mu,$$

which we may rewrite as

$$\int_M \left[-\frac{2}{3}(s+3w)\, f^{-2/3}\right]\left(f^{2/3}|\Phi|^4\right) d\mu \geq \int_M f|\Phi|^6 d\mu.$$

Applying the Hölder inequality to the left-hand side now gives

$$\left[\int_M \left|\frac{2}{3}(s+3w)\right|^3 f^{-2} d\mu\right]^{1/3}\left[\int_M f|\Phi|^6 d\mu\right]^{2/3} d\mu \geq \int_M f|\Phi|^6 d\mu,$$

and we therefore deduce that

$$\int_M \left|\frac{2}{3}(s+3w)\right|^3 f^{-2} d\mu \;\geq\; \int_M f|\Phi|^6 d\mu.$$

But the Hölder inequality also tells us that

$$\left(\int_M f^4 d\mu\right)^{1/3}\left(\int_M f|\Phi|^6 d\mu\right)^{2/3} \geq \int_M f^{4/3}(f^{2/3}|\Phi|^4) d\mu\ ,$$

so it follows that

$$\left(\int_M f^4 d\mu\right)^{1/3}\left(\int_M \left|\frac{2}{3}(s+3w)\right|^3 f^{-2} d\mu\right)^{2/3} \geq \int_M f^2|\Phi|^4 d\mu\ .$$

However, since $-iF_A^+ = f\sigma(\Phi)$, we also have

$$\int_M f^2|\Phi|^4 d\mu = 8\int_M |F_A^+|^2 d\mu \geq 8\int_M |2\pi c_1^+|^2 d\mu = 32\pi^2[c_1^+]^2$$

because $iF_A^+ = 2\pi c_1^+ + d\theta - d^*(\star\theta)$ for some 1-form θ. Thus

$$\left(\int_M f^4 d\mu\right)^{1/3}\left(\int_M \left|\frac{2}{3}(s+3w)\right|^3 f^{-2}d\mu\right)^{2/3} \geq 32\pi^2[c_1^+]^2.$$

Multiplying both sides by $9/4$ now yields the promised inequality. □

LEMMA 13. *Let M be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$, and let $a \in H^2(M,\mathbb{Z})/torsion$ be a monopole class. Let g be any Riemannian metric on M, and let $u > 0$ be any smooth positive function on M for which*

$$u \geq |s+3w|$$

at each point $x \in M$. Then

$$\int u^2 d\mu_g \geq 72\pi^2(a^+)^2.$$

PROOF. For any such function u, set $f = u^{1/2}$. and notice that we have

$$\left(\int u^2 d\mu\right)^{1/3} = \left(\int f^4 d\mu\right)^{1/3}$$

and

$$\left(\int u^2 d\mu\right)^{2/3} = \left(\int u^3 f^{-2} d\mu\right)^{2/3} \geq \left(\int |s+3w|^3 f^{-2} d\mu\right)^{2/3}.$$

Setting $c_1(L) = a$ and invoking Lemma 12, we therefore have

$$\int_M u^2 d\mu_g \geq \left(\int_M f^4 d\mu_g\right)^{1/3}\left(\int_M |s+3w|_g^3 f^{-2} d\mu_g\right)^{2/3} \geq 72\pi^2(a^+)^2,$$

as claimed. □

LEMMA 14. *Let M be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$, and let $a \in H^2(M,\mathbb{Z})/torsion$ be a monopole class. Then every Riemannian metric g on M satisfies*

$$\int (s+3w)^2\, d\mu_g \geq 72\pi^2(a^+)^2.$$

PROOF. Let u_j be a sequence of smooth positive functions satisfying $u_j > |s+3w|$ and with $u_j \to |s+3w|$ in the C^0 topology. (Since the smooth functions are dense in C^0, such a sequence may of course be constructed by setting $u_j = \frac{1}{j} + v_j$, where v_j is a smooth function whose sup-norm distance from the continuous function $|s+3w|$ is less than $1/j$.) Then Lemma 13 tells us that

$$\int_M (s+3w)^2 d\mu = \inf_j \int_M u_j^2 d\mu \geq 72\pi^2(a^+)^2,$$

as claimed. □

Note that Lemma 14 essentially reproves [**28**, Theorem 2.3], but does so in a much more elementary manner. What has been lost here, however, is a precise understanding of what happens when the inequality is saturated. For the purpose of finding obstructions to the existence of Einstein metrics, however, this shortcoming will in practice turn out to be irrelevant.

Proposition 15. *Let M be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. Then every Riemannian metric g on M satisfies*

$$||s|| + \sqrt{6}||W_+|| \geq 6\sqrt{2}\pi\alpha(M),$$

where $||\cdot||$ denotes the L^2 norm with respect to g. If equality occurs, moreover, then the two largest eigenvalues of $W_+ : \Lambda^+ \to \Lambda^+$ are equal at each $x \in M$, and $|W_+|$ is a constant multiple of the scalar curvature s.

Proof. By Lemma 14, we have

$$||s + 3w|| \geq 6\pi\sqrt{2(a^+)^2},$$

so the triangle inequality tells us that

$$||s|| + 3||w|| \geq ||s + 3w|| \geq 6\pi\sqrt{2(a^+)^2},$$

and that equality can only hold if w and s are proportional, as vectors in L^2.

Now if $\alpha(M) = 0$, the claim is of course trivial. Otherwise, the definition of $\alpha(M)$ guarantees that for every metric g there is a monopole class a with $(a^+)^2 \geq \alpha^2(M)$. Now invoking our calculations for this choice of a, we then have

$$||s|| + 3||w|| \geq 6\pi\sqrt{2}\alpha(M).$$

Moreover, since a 4-manifold M with $\alpha(M) \neq 0$ cannot admit metrics with $s \equiv 0$, equality can only occur if w is a constant times s.

On the other hand, because W_+ is a trace-free endomorphism of Λ^+, we have the point-wise inequality

$$\sqrt{\frac{2}{3}}|W_+| \geq |w|,$$

with equality only when the two largest eigenvalues of W_+ are equal. Hence

$$||s|| + \sqrt{6}||W_+|| \geq ||s|| + 3||w|| \geq 6\pi\sqrt{2}\alpha(M).$$

The resulting inequality is now strict unless the largest eigenvalue of W_+ has multiplicity ≥ 2 everywhere, and unless the functions $|W_+|$ is a constant multiple of the scalar curvature s. □

Lemma 16. *Let M be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. Then every Riemannian metric g on M satisfies*

$$\frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu \geq \frac{2}{3}\alpha^2(M).$$

Moreover, equality could only possibly hold for metrics for which $s \equiv -8\sqrt{6}|W_+|$ and for which the largest eigenvalue of W_+ has multiplicity two at each point of M.

Proof. By Proposition 15, we have

$$(1, \frac{1}{\sqrt{8}}) \cdot \left(||s||, \sqrt{48}||W_+||\right) \geq 6\sqrt{2}\pi\alpha(M),$$

where the left-hand side is to be read as a dot product in $\mathbb{R}^2$. Applying the Cauchy-Schwarz equality to this dot product now gives us

$$\left(1+\frac{1}{8}\right)^{1/2}\left(\|s\|^2+48\|W_+\|^2\right)^{1/2}\geq (72\pi^2\alpha^2(M))^{1/2},$$

so that

$$\frac{1}{4\pi^2}\int_M\left(\frac{s^2}{24}+2|W_+|^2\right)d\mu\geq\frac{2}{3}\alpha^2(M), \tag{16}$$

as claimed. If equality holds, we must have $\left(\|s\|,\sqrt{48}\|W_+\|\right)\propto(1,\frac{1}{\sqrt{8}})$ in addition to the previously deduced conditions, and the value of the ratio of $|W_+|$ and s then follows from this, together with the fact that there cannot be any metrics with $s\geq 0$ when $\alpha(M)\neq 0$. □

Lemma 17. *Let M be a smooth compact oriented 4-manifold with $b_+(M)\geq 2$, and suppose that g is a Riemannian metric on M with constant scalar curvature and harmonic self-dual Weyl curvature:*

$$(\delta W_+)_{bcd}=-\nabla^a(W_+)_{abcd}=0.$$

Then either g satisfies the strict inequality

$$\frac{1}{4\pi^2}\int_M\left(\frac{s^2}{24}+2|W_+|^2\right)d\mu>\frac{2}{3}\alpha^2(M),$$

or else $\alpha(M)=0$, g is a hyper-Kähler metric, and M is diffeomorphic to either $K3$ or T^4.

Proof. Suppose that g is a metric with constant s which saturates inequality (16). Then $|W_+|\equiv -s/8\sqrt{6}$ is constant, and the largest two eigenvalues of W_+ coincide at each point of M. We now want to ask what this tells us if we also assume that $\delta W_+=0$.

One way this may certainly happen is for s and W_+ to both vanish identically. In this case, however, the Weitzenböck formula (15) then implies that every self-dual harmonic form on M is parallel, and, since $b_+(M)\geq 2$ by assumption, it then follows that the orientable rank-3 vector bundle Λ^+ is trivialized by parallel sections, so that g is hyper-Kähler. The classification of complex surfaces then tells us that M must be diffeomorphic to $K3$ or T^4.

On the other hand, the $s\neq 0$ case is ruled out by an observation due to Derdziński [**11**]. Indeed, in this case we would know that W_+ was a non-zero constant times $\omega\otimes\omega-\frac{1}{3}\mathbf{1}_{\Lambda^+}$, where ω is a (sign-ambiguous) self-dual 2-form of norm $\sqrt{2}$ spanning the negative eigenspace of W_+. The harmonicity of W_+ thus implies that

$$\omega_{ab}\nabla^a\omega_{cd}+\omega_{cd}\nabla^a\omega_{ab}=0.$$

But since ω has constant length, contraction with ω^{cd} tells us that $\delta\omega=0$, and plugging this back into the original equation then tells us that $\nabla\omega=0$. This shows that (M,g) is locally Kähler. However, any Kähler manifold of real dimension 4 satisfies $|s|\equiv 2\sqrt{6}|W_+|$, whereas in our case we can already take it as given that $|s|\equiv 8\sqrt{6}|W_+|$. This contradiction eliminates the $s\neq 0$ case, and we are done. □

This now allows us to recover [**28**, Theorem 3.3]:

THEOREM 18. *Let M be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. If M admits an Einstein metric g, then*

$$(2\chi + 3\tau)(M) \geq \frac{2}{3}\alpha^2(M),$$

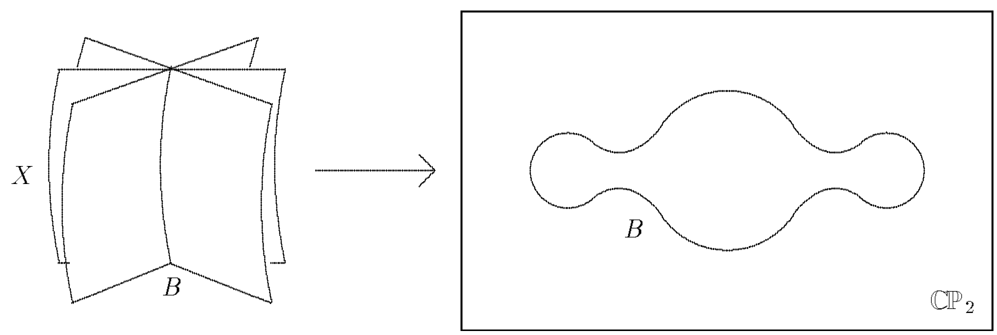
with equality only if both sides vanish, in which case M must be diffeomorphic to $K3$ or T^4, and g must be a hyper-Kähler metric.

PROOF. If (M, g) is a compact oriented 4-dimensional Einstein manifold, the generalized Gauss-Bonnet formula (5) tells us that the left-hand side of (16) is given by $(2\chi + 3\tau)(M)$, and the desired inequality therefore follows. However, the Bianchi identities also tell us that a 4-dimensional Einstein manifold has constant s and harmonic W_+, so Lemma 17 tells us that this inequality can only be saturated when g is hyper-Kähler. □

5. An Illustration

Theorem 18 gives us an obstruction to the existence of Einstein metrics involving the differential topology, rather than just the homeomorphism type, of the smooth manifold in question. The following example should help to clarify this point.

Let X be a triple cyclic cover of of $\mathbb{CP}_2$, ramified at a non-singular complex curve B of degree 6.



To be more explicit, we could, for example, take X to consist of those elements η of the $\mathcal{O}(2)$ line bundle over $\mathbb{CP}_2 = \{[x : y : z]\}$ which satisfy the equation

$$\eta^3 = x^6 + y^6 + z^6,$$

where the right-hand side is of course interpreted as a section of $\mathcal{O}(6)$. The canonical line bundle of the compact complex surface X is then exactly the pull-back of the $\mathcal{O}(1)$ line bundle on $\mathbb{CP}_2$, and from this we read off that $c_1^2(X) = 3 \cdot 1^2 = 3$, and $h^{2,0}(X) = h^0(\mathbb{CP}_2, \mathcal{O}(1)) = 3$. Moreover, an easy application of the Lefschetz hyperplane-section theorem tells us that X is simply connected.

Now let M be obtained from X by blowing up a point, so that

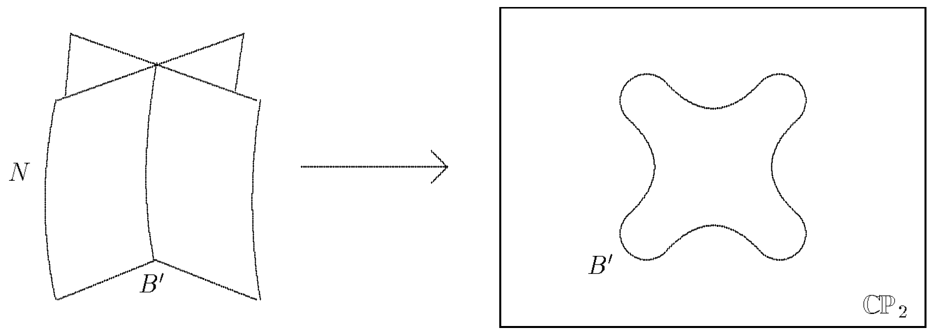
$$M \approx X \# \overline{\mathbb{CP}}_2.$$

Then $\alpha^2(M) = c_1^2(X) = 3$, whereas $(2\chi + 3\tau)(M) = c_1^2(M) = c_1^2(X) - 1 = 2$. Since $(2\chi + 3\tau)(M) = \frac{2}{3}\alpha^2(M) \neq 0$, Theorem 18 therefore tells us that the smooth compact 4-manifold M cannot admit an Einstein metric.

Next, for the sake of comparison, let N be the double branched cover of $\mathbb{CP}_2$, ramified at a non-singular complex curve B' of degree 8. To be more explicit, one could take N to consist of those elements ζ of the $\mathcal{O}(4)$ line bundle over $\mathbb{CP}_2$ which satisfy the equation

$$\zeta^2 = x^8 + y^8 + z^8.$$

Once again, this branched cover of $\mathbb{CP}_2$ is simply connected. Careful inspection also reveals that, in this example, the canonical line bundle is once again exactly the pull-back of the $\mathcal{O}(1)$ line bundle on $\mathbb{CP}_2$; thus we read off that $c_1^2(N) = 2 \cdot 1^2 = 2$, and $h^{2,0}(N) = h^0(\mathbb{CP}_2, \mathcal{O}(1)) = 3$, which is to say that these two numerical invariants are exactly the same for M and N.



But in another respect, N is wildly unlike from M. Indeed, because it is a compact complex surface with $c_1 < 0$. Theorem 3 tells us that N, unlike M, admits an Einstein metric.

However, each of these compact complex surfaces is simply connected and non-spin. Moreover, both of them have $c_1^2 = 2$ and $h^{2,0} = 3$, and from this it can be deduced that both have $b_+ = 7$ and $b_- = 37$. Thus Theorem 2 tells us that both M and N are homeomorphic to

$$7\mathbb{CP}_2 \# 37\overline{\mathbb{CP}}_2.$$

This topological 4-manifold therefore admits one smooth structure for which there is an Einstein metric, and yet another smooth structure for which no such metric can exist.

6. The Big Picture

The problem of understanding Einstein metrics on compact 4-manifolds has been an extremely active area of research in recent years, and the Seiberg-Witten techniques which have been highlighted in this article do not by any means represent the only strand of thought which has led to significant progress. It is therefore appropriate that this article conclude with a rough indication of some of these other developments, without pretending to offer a definitive survey of the subject.

One such important strand of thought grew out of the work of Gromov [**17**], who, for example, first introduced minimal volume invariants of a smooth manifold. If M is a smooth compact n-manifold, one such invariant is

$$\mathrm{Vol}_r(M) = \inf\{\mathrm{Vol}(M, g) \mid r_g \geq -(n-1)g\},$$

and is sometimes called the *minimal volume with respect to Ricci curvature*. Gromov showed that $\mathrm{Vol}_r(M)$ is positive for certain compact manifolds with infinite fundamental group by giving a lower bound for it in terms of a homotopy invariant which he called the *simplicial volume*. He then went on to observe that this implies that there are obstructions to the existence of Einstein metrics in dimension 4 which are not detected by the Hitchin-Thorpe inequality.

While Gromov's bounds are actually quite weak in practice, they nonetheless represented a breakthrough in the subject, and eventually led to a hunt for sharper estimates of the same flavor. This culminated in the work of Besson, Courtois, and Gallot [**8**], who related Gromov's work to the entropy of the geodesics flow. One of their most remarkable results goes as follows:

THEOREM 19 (Besson-Courtois-Gallot). *Let (X, g_0) be a compact hyperbolic n-manifold, $n > 2$, and let M be a compact manifold of the same dimension. Let $f : M \to X$ be any smooth map. Then*

$$\mathrm{Vol}_r(M) \geq \deg(f)\ \mathrm{Vol}(X, g_0), \tag{17}$$

where $\deg(f)$ denotes the degree of f. Moreover, if equality holds, and if the infimum (17) is achieved by some metric g, then (M, g) is an isometric Riemannian covering of (X, g_0), with covering map $M \to X$ homotopic to f.

When $n = 4$ and f is the identity map $X \to X$, this implies that g_0 is the only Einstein metric on X, up to rescalings and diffeomorphisms. Moreover, this same result also gives rise to new obstructions to the existence of Einstein metrics. Indeed, Sambusetti [**34**] used this method to prove the following:

THEOREM 20 (Sambusetti). *Any integer pair (χ, τ) with $\chi \equiv \tau \bmod 2$ can be realized as the Euler characteristic and signature of a smooth compact oriented 4-manifold M which does not admit any Einstein metrics.*

This nicely highlights how much there is to be said about the subject beyond the Hitchin-Thorpe inequality. Note, however, that Sambusetti's examples all have huge fundamental group, and are never even *homotopy equivalent* to an Einstein manifold.

Anderson's work [**1**] on Gromov-Hausdorff limits with Ricci-curvature bounds represents another major area of progress in understanding 4-dimensional Einstein manifolds. Anderson shows that any sequence of unit-volume Einstein manifolds of bounded Euler characteristic and *bounded diameter* must necessarily have a subsequence which converges in the Gromov-Hausdorff sense to a compact Einstein orbifold. In particular, this implies that a topological 4-manifold can only admit finitely many smooth structures for which there exists an Einstein metric of unit volume and diameter $< D$.

It is interesting to reconsider the Kähler-Einstein case in this light. Mumford's finiteness theorem [**4**] implies that a 4-manifold can only admit finitely many smooth structures for which there exist *Kähler*-Einstein metrics, although this finite number can be arbitrarily large [**23, 33**]. Notice that this conclusion holds without the imposition of an extraneous diameter bound. On the other hand, inspection of the Kähler-Einstein case also reveals that the diameter can tend to infinity even for sequences of unit-volume Einstein metrics on a fixed smooth 4-manifold, so a Mumford-type finiteness statement certainly cannot be deduced from a compactness result like Anderson's.

While we do not know at present whether there can only be finitely many smooth structures admitting Einstein metrics on a fixed topological 4-manifold, we *do* at least know that there are often infinitely many smooth structures for which Einstein metrics *don't* exist, even for simply connected manifolds for which the Hitchin-Thorpe inequality is strict [**22**]; cf. [**23**]. In the non-spin case, such an assertion can actually be made for a large sector of choices for (χ, τ). The spin case is much more delicate, however, and the range of homeotypes for which one can make such an assertion is, at present, much more tightly constrained.

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