Desingularizations of

Conformally Kähler,

Einstein Orbifolds

Claude LeBrun
Stony Brook University

"Conformal Geometry, Einstein Metrics,
and Relativity" Workshop
Simons Center for Geometry and Physics
Stony Brook, January 26, 2026

Most recent results:

Joint work with

Joint work with

Tristan Ozuch

Joint work with

Tristan Ozuch
Massachusetts Institute of Technology

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e-print:
arXiv:2601.19215 [math.DG]

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— A. Einstein, to G. Gamow

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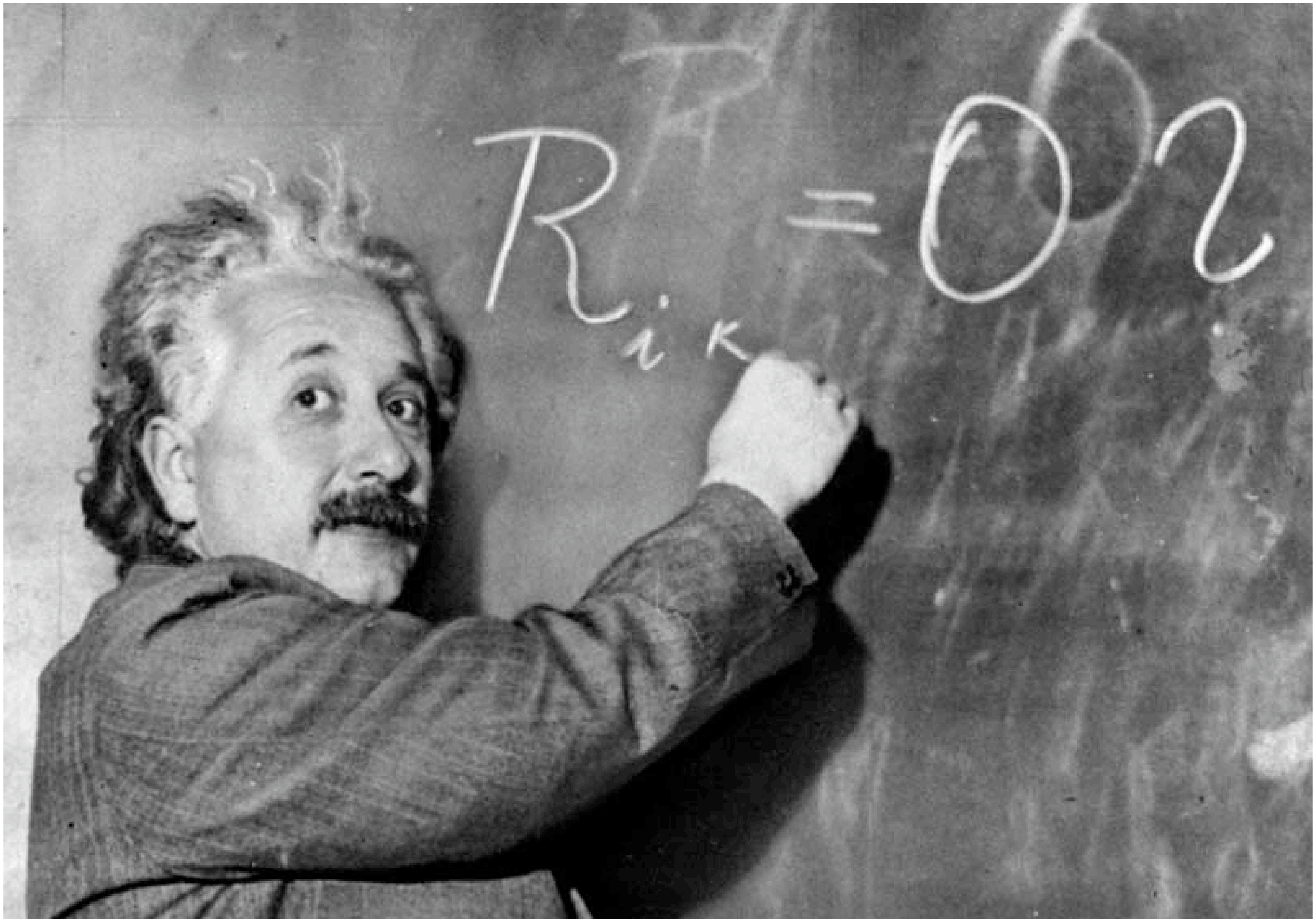

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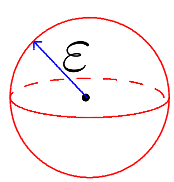

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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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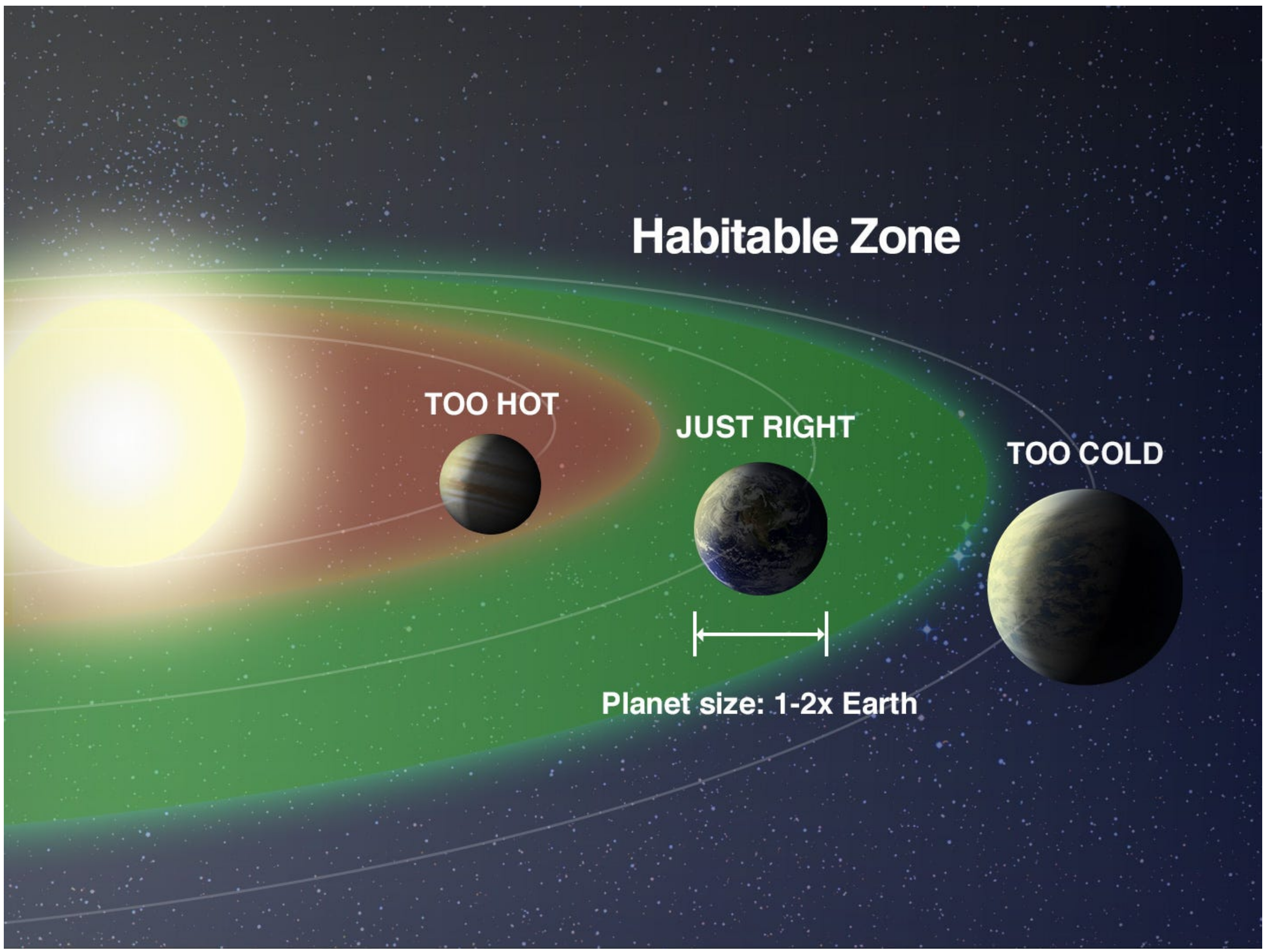

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Λ^+ self-dual 2-forms.
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W_- = anti-self-dual Weyl curvature ″

Which 4-manifolds admit Einstein metrics?

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$$d\omega = 0, \qquad \lrcorner\omega : TM \stackrel{\cong}{\to} T^*M.$$

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$$\omega = dx \wedge dy + dz \wedge dt$$

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)? What if we also require $\lambda > 0$?*

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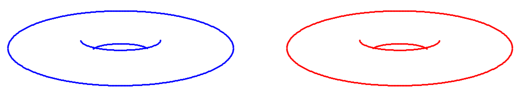

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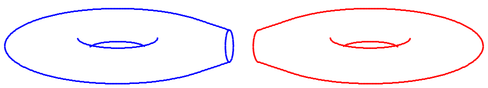

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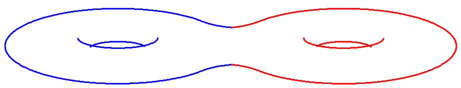

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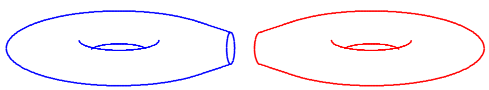

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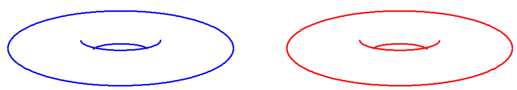

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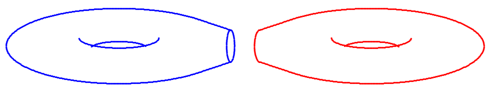

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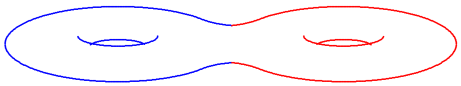

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Diffeotypes: exactly the Del Pezzo surfaces.

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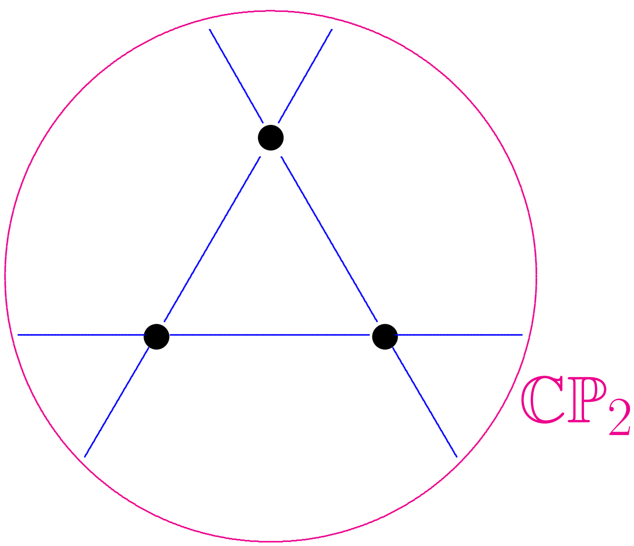

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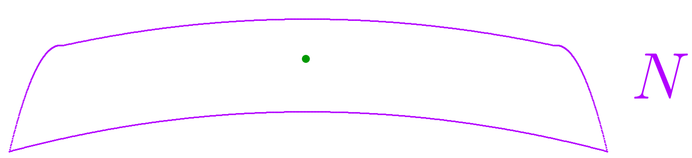

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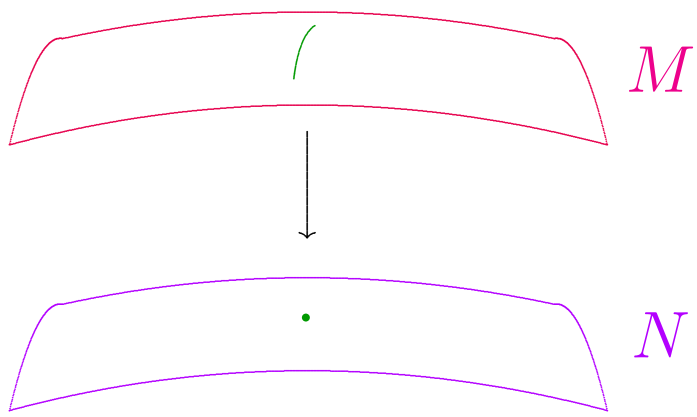

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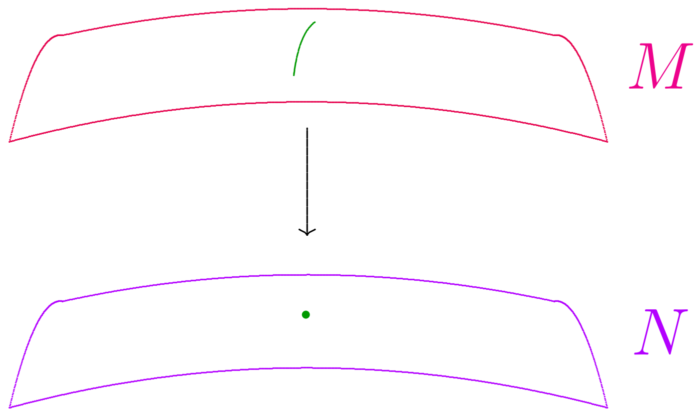

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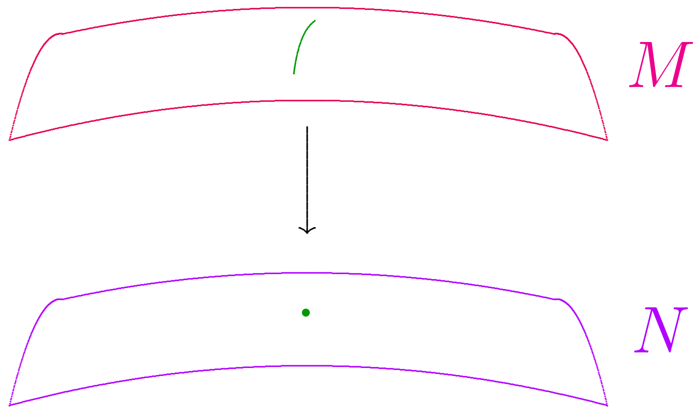

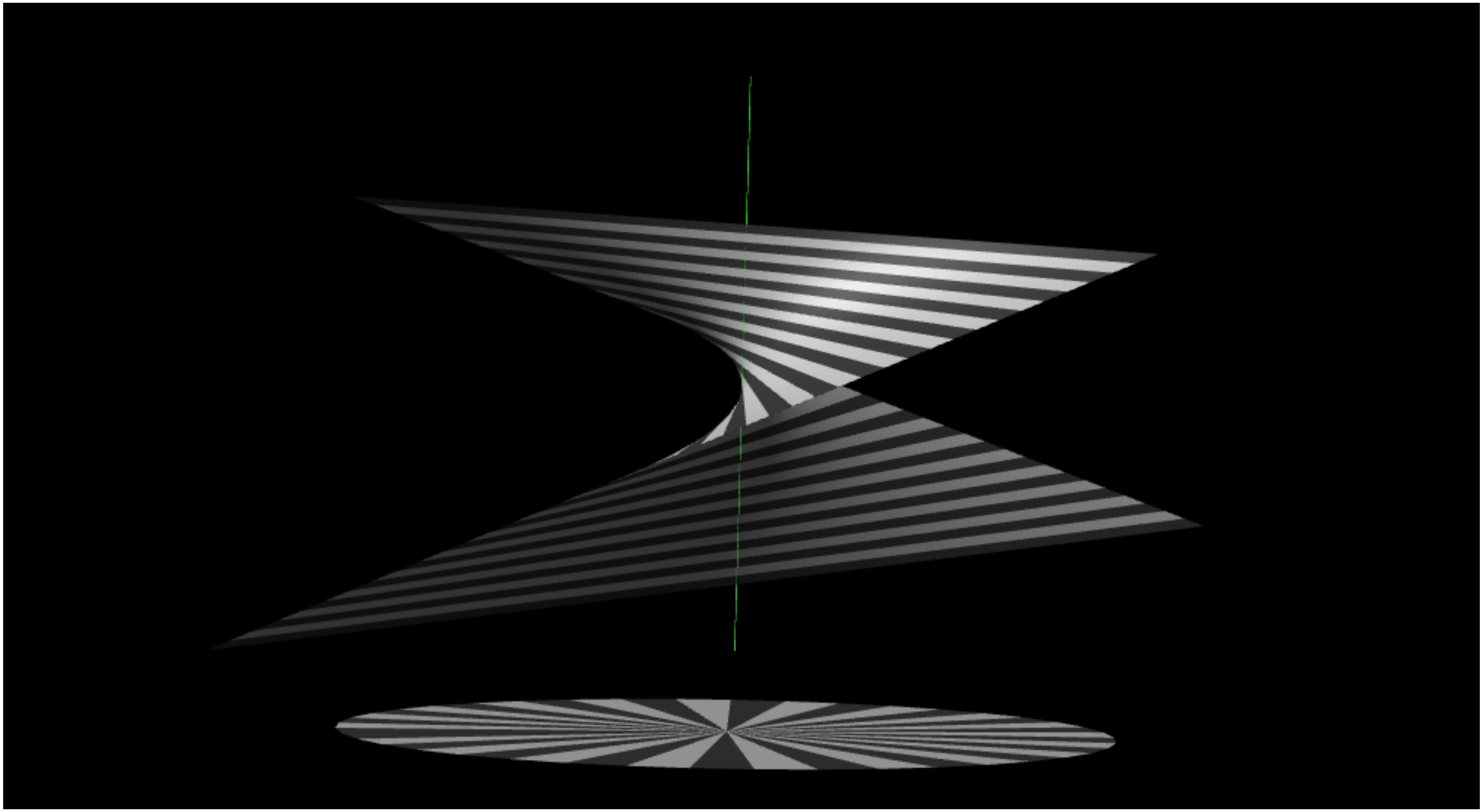

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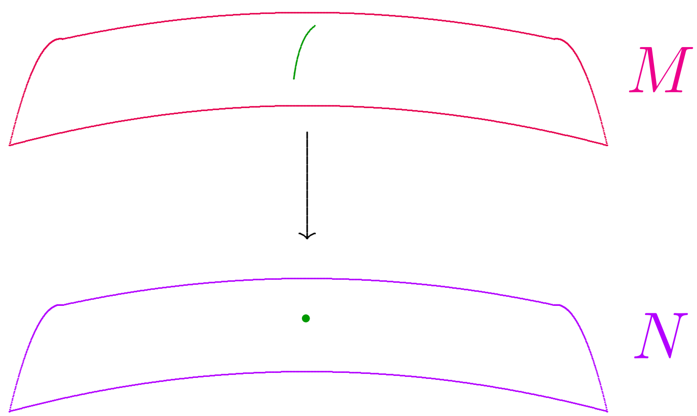

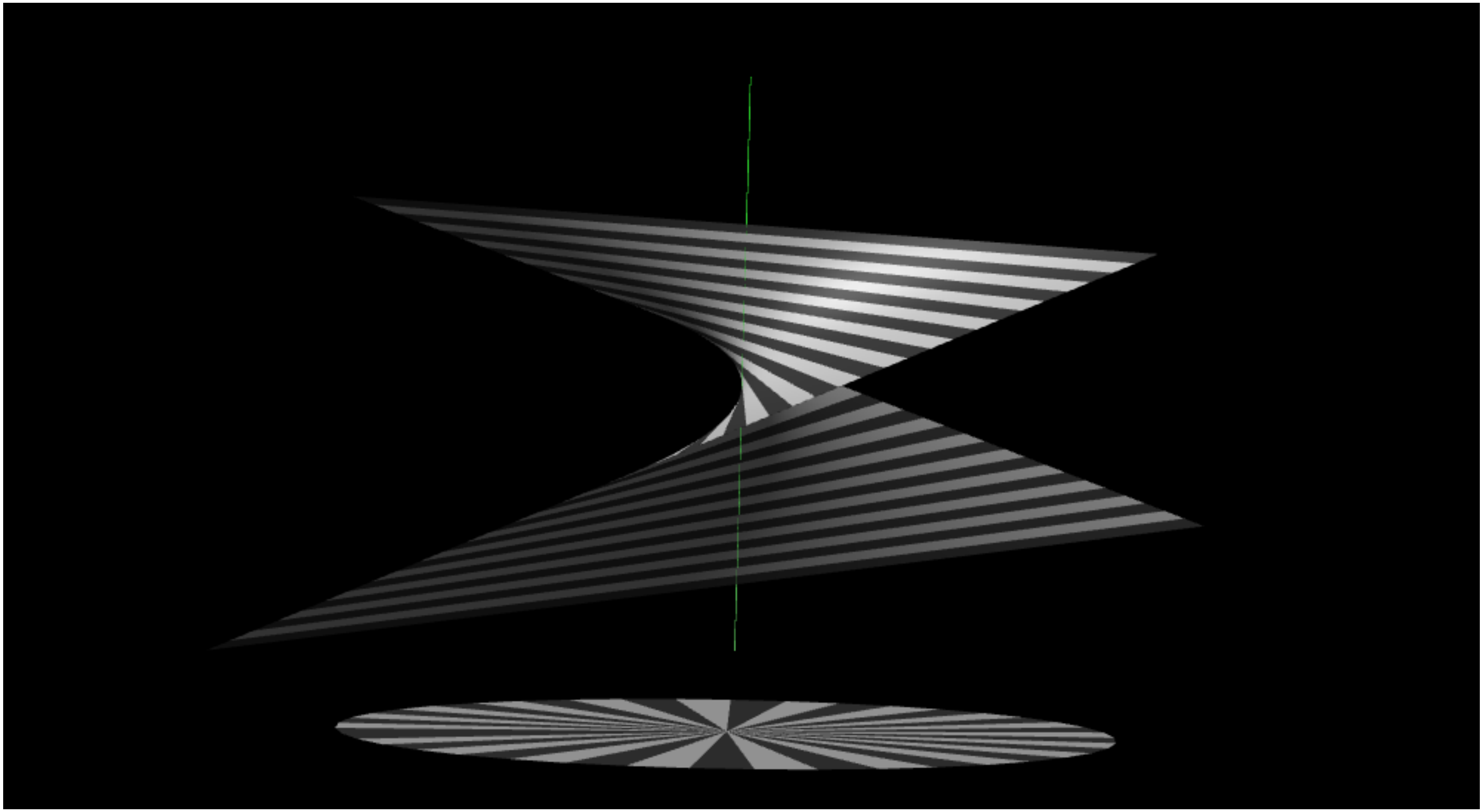

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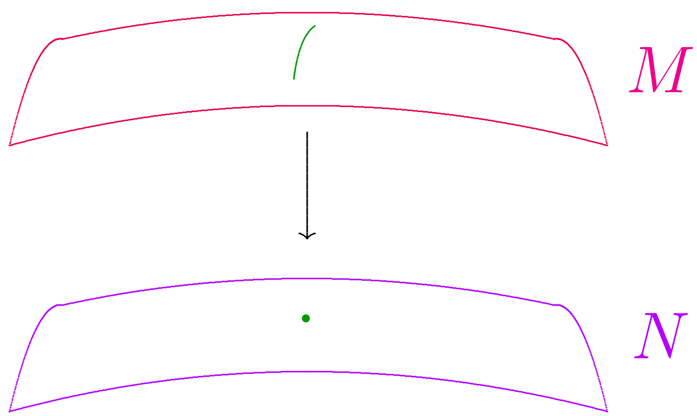

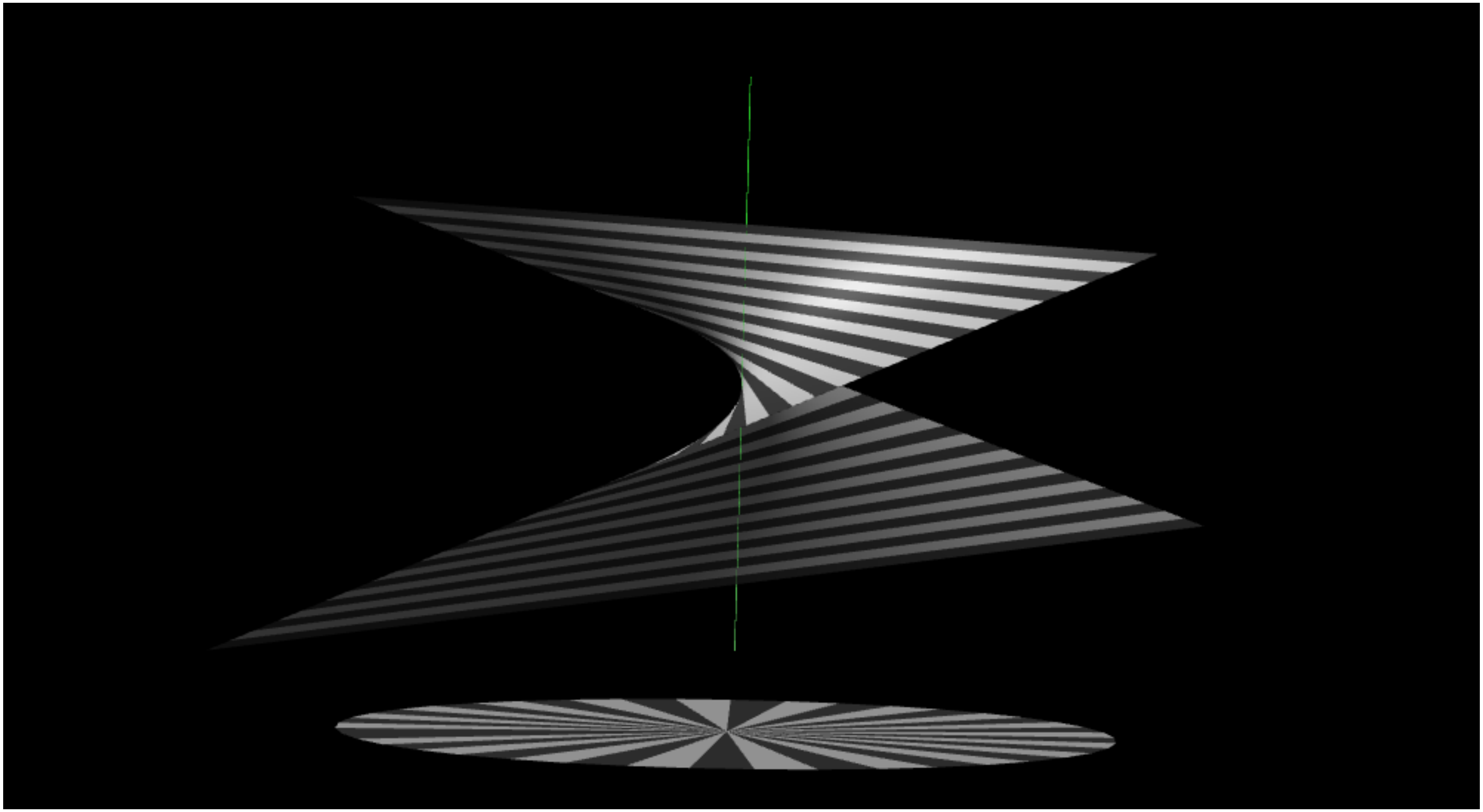

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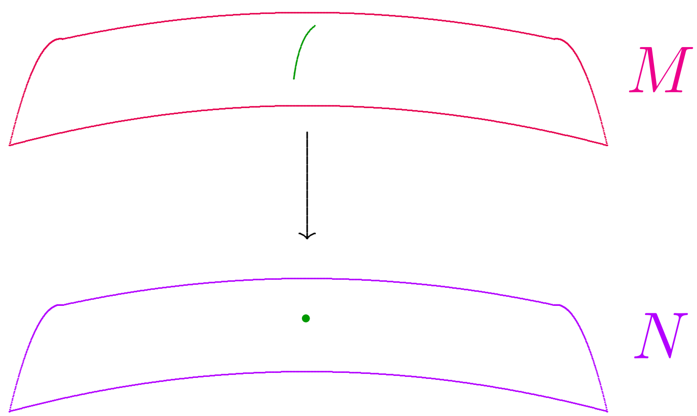

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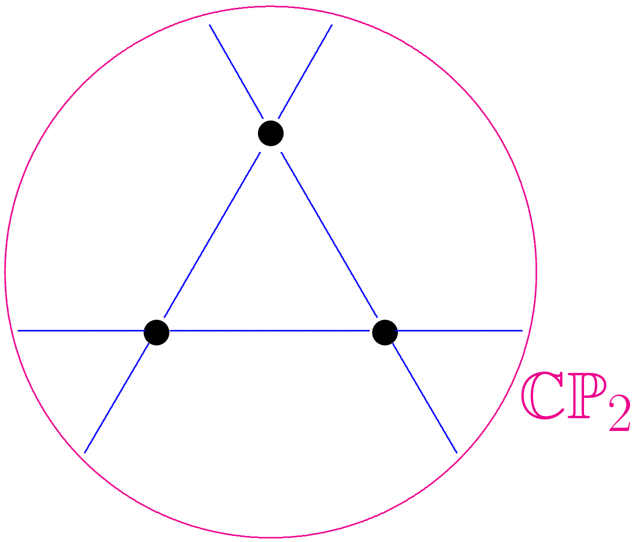
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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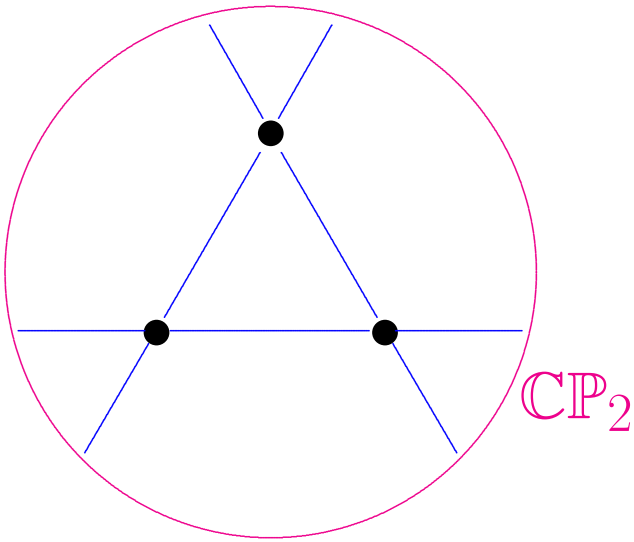

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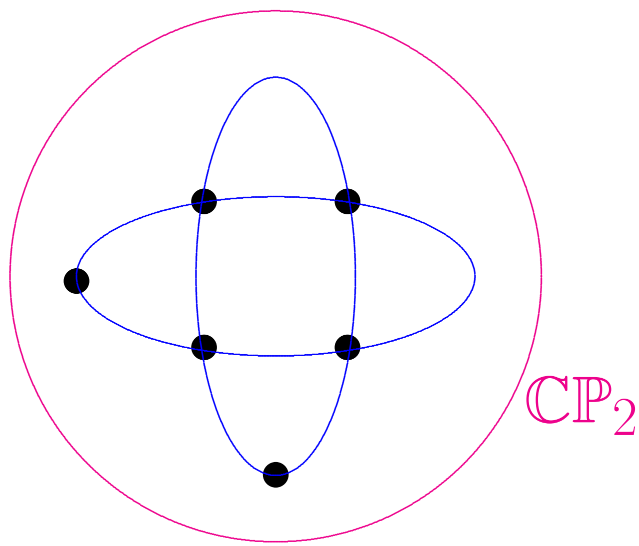

No 3 on a line, no 6 on conic,

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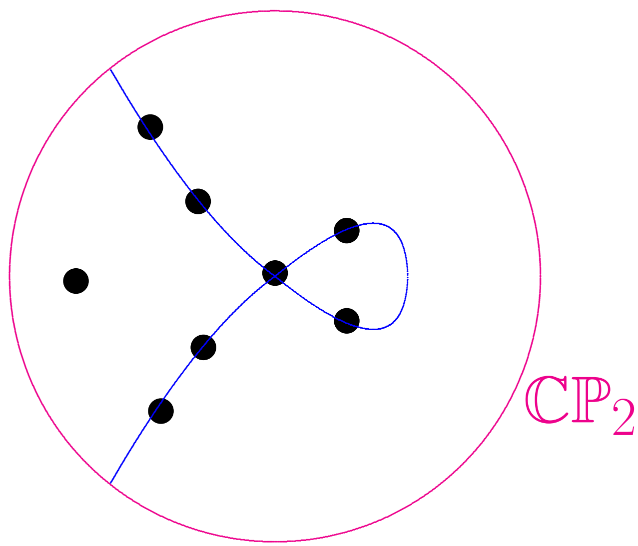

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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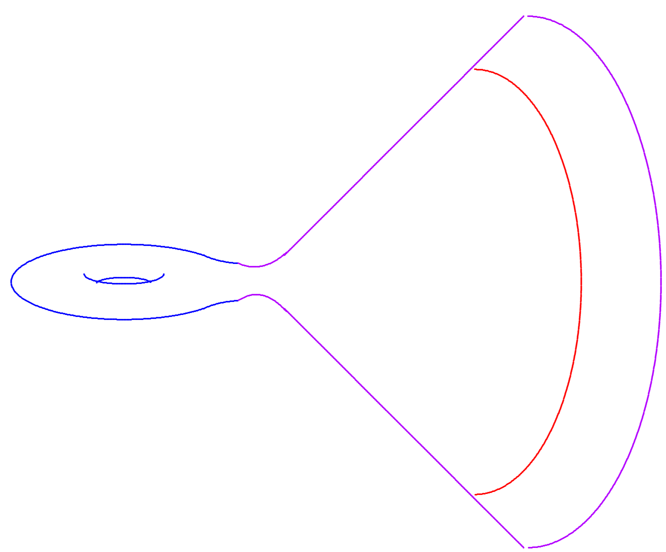

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- Iterate to elimate all orbifold singularities.

Theorem. *Let* $\{(M_j, g_j)\}$ *be a sequence of cpt connected Einstein* 4*-manifolds with fixed Einstein constant* $\lambda > 0$*, fixed Euler characteristic* χ*, and* $Vol(M_j, g_j)$ *bounded away from zero. Then there exists a subsequence* (M_{j_i}, g_{j_i}) *which converges in the Gromov-Hausdorff sense to cpt connected* 4*-dimensional Einstein* orbifold (X, g_∞) *with the same Einstein constant* λ*.*

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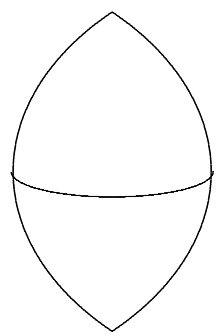

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Goal: Show that this doesn't change anything!

Technical Hitch!

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We avoid this question by means of a definition!

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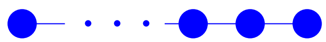
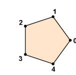
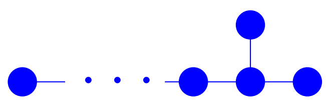
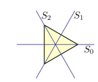
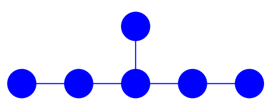
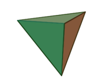
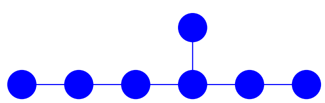
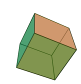
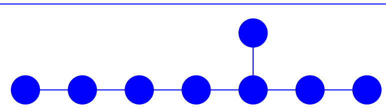
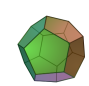
A_k		*Cyclic*	
D_k		*Binary Dihedral*	
E_6		*Binary Tetrahedral*	
E_7		*Binary Octahedral*	
E_8		*Binary Icosahedral*	

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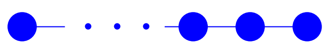
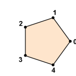
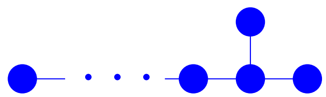
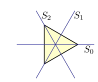
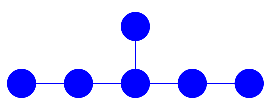
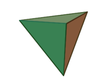
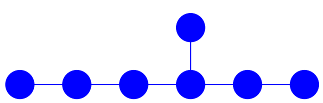
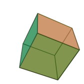
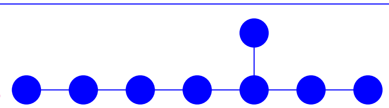
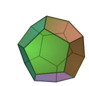
A_k 	$\mathbb{Z}_{k+1}$	
D_k 	Dih^*_{k-2}	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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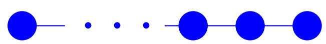
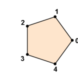
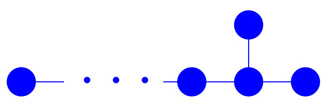
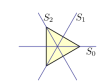
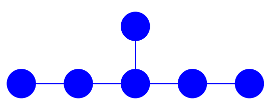
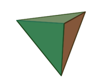
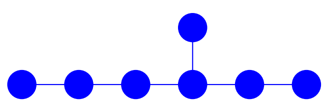
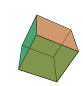
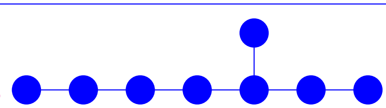
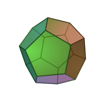
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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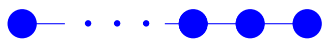
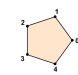
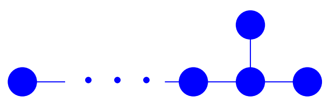
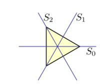
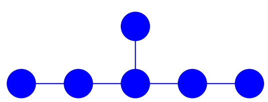
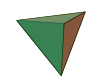
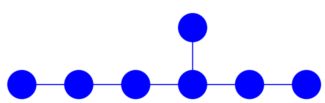
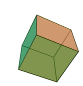
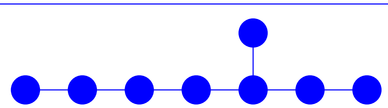
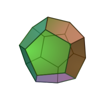
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

The corresponding gravitational instantons are exactly the hyper-Kähler ALE manifolds.

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Standard shorthand: $\frac{1}{\ell m^2}(1, \ell m n - 1)$.

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Proposition. *Either there are infinitely many topological types of compact K-E 4-dimensional orbifolds with only isolated singularities that cannot arise as Gromov-Hausdorff limits of smooth Einstein manifolds, or else there are infinitely many Ricci-flat ALE 4-manifolds that remain to be discovered.*

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These examples form part of a systematic picture. . .

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Classified Gromov-Hausdorff limits of Del Pezzos.

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- $\mathbb{CP}_2\#8\overline{\mathbb{CP}}_2$: only (!) $A_1, \ldots, A_{10}$, D_4, $\frac{1}{4}(1,1)$, $\frac{1}{8}(1,3)$, or $\frac{1}{9}(1,2)$ singularities arise.

Example.

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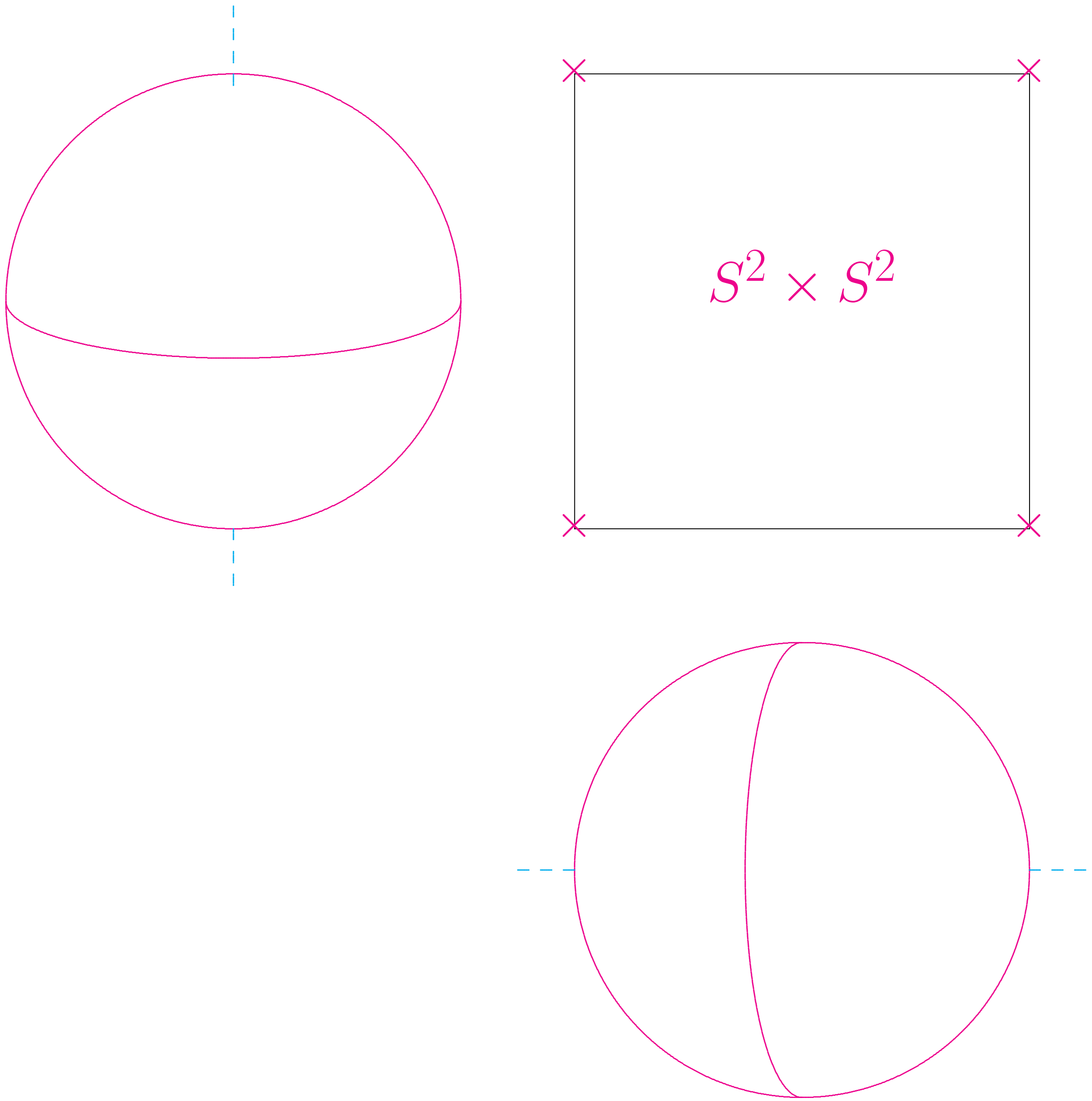

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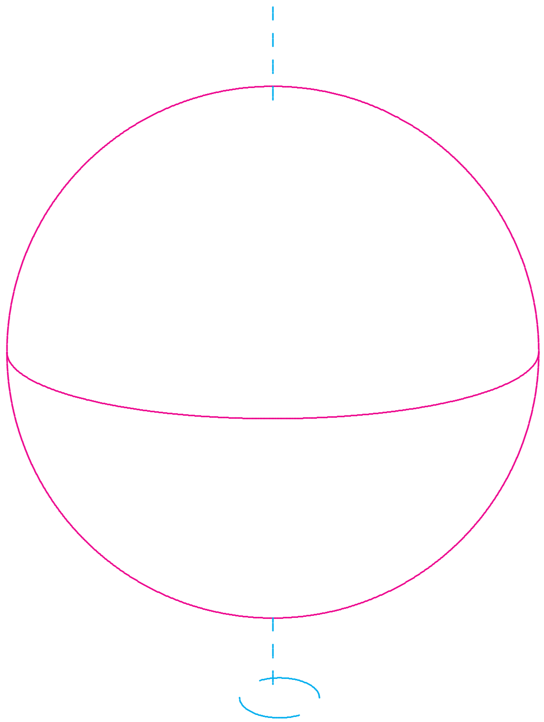

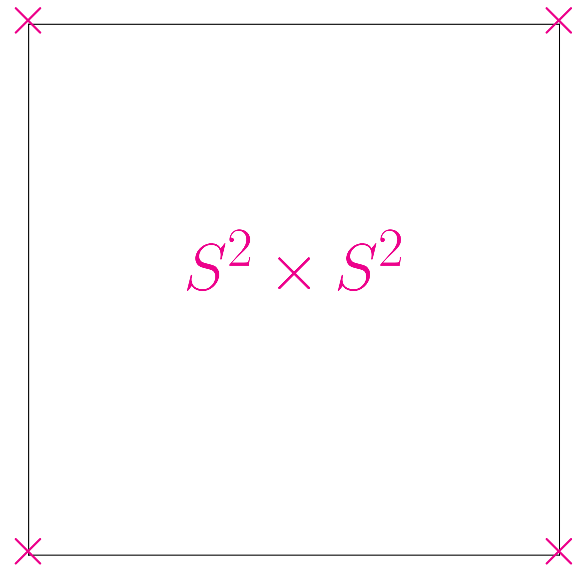

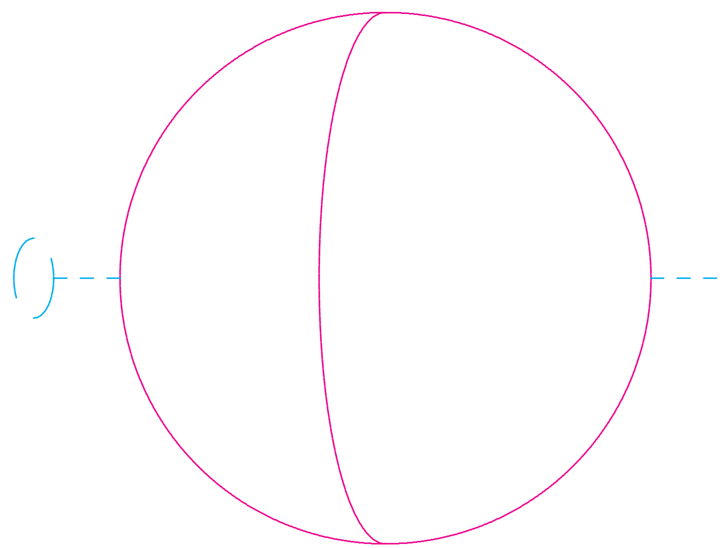

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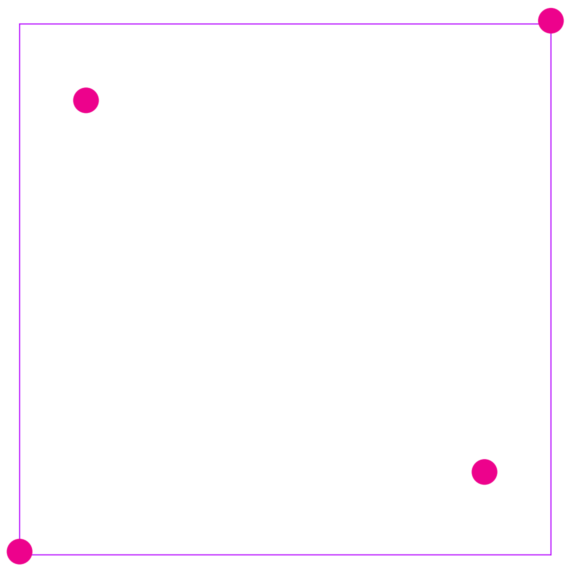

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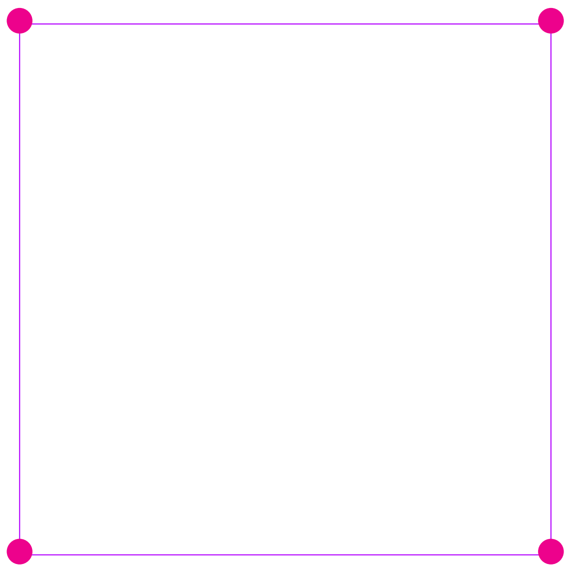

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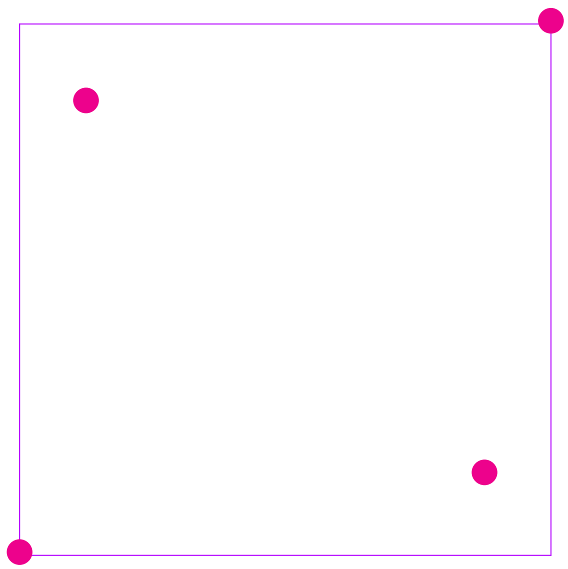

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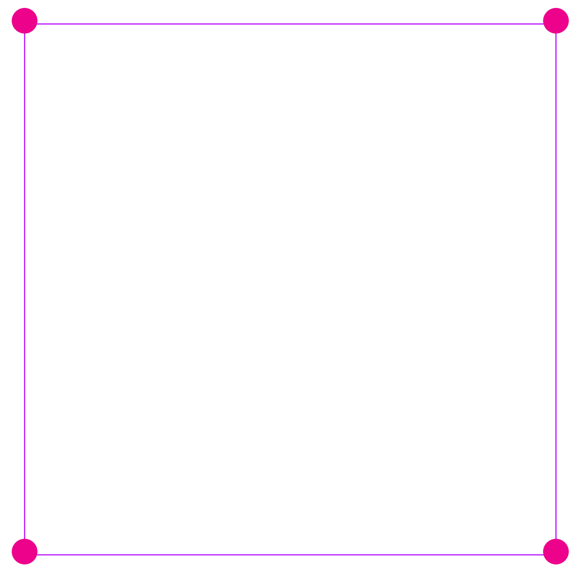

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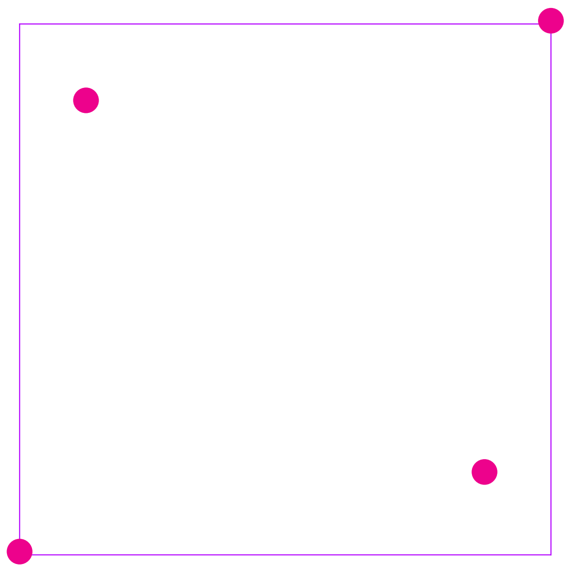

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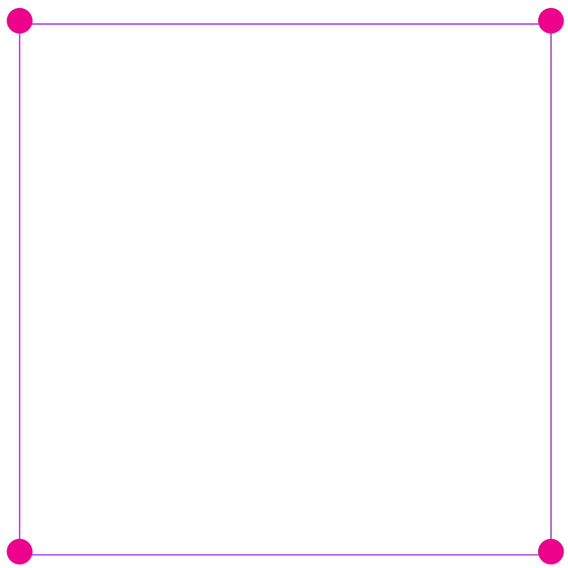

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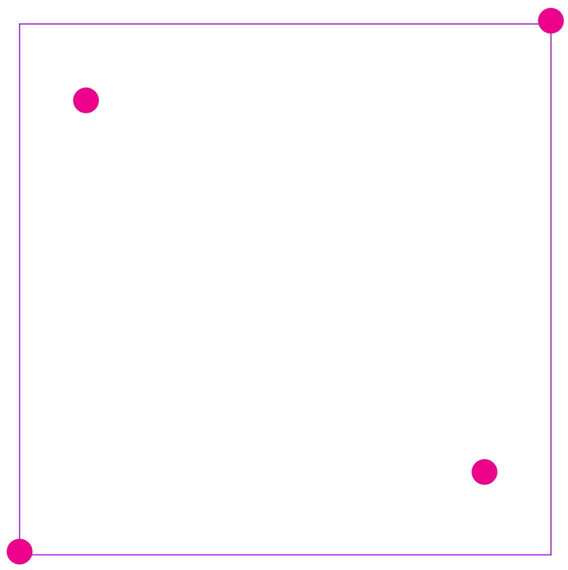

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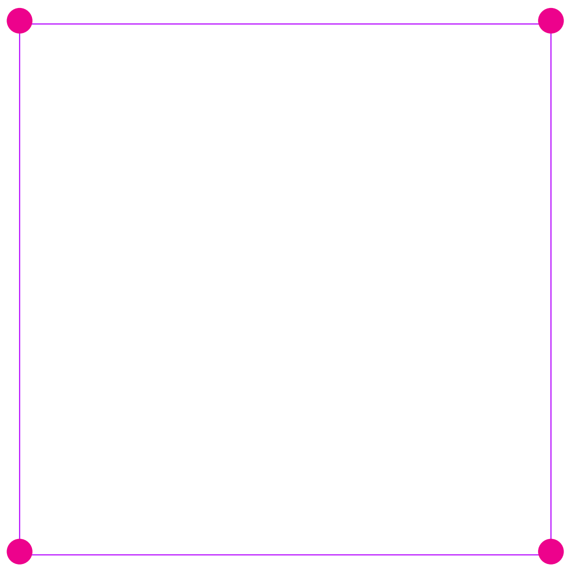

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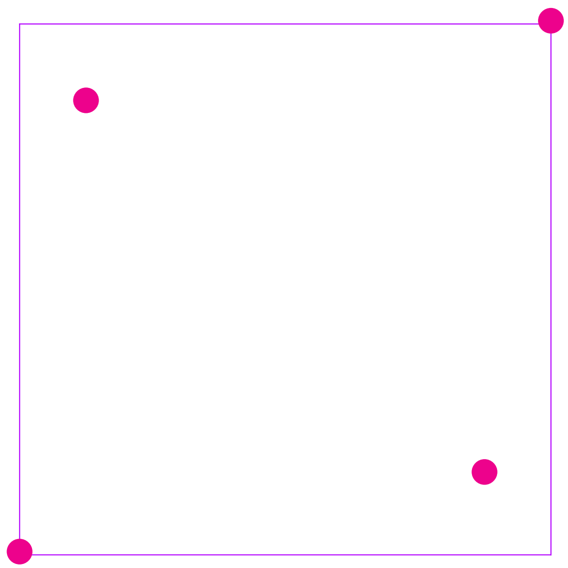

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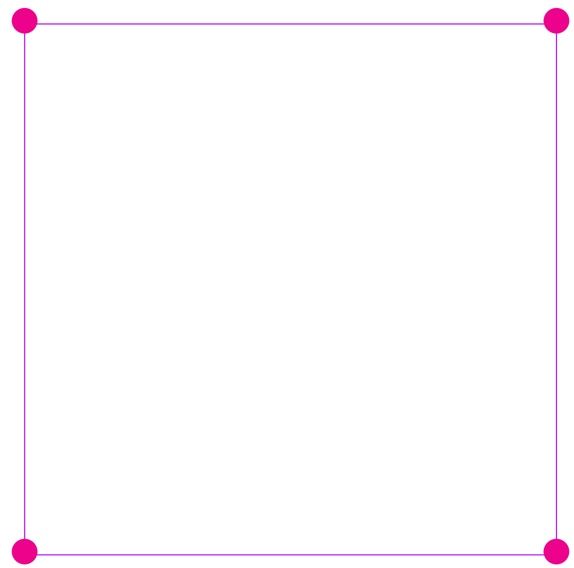

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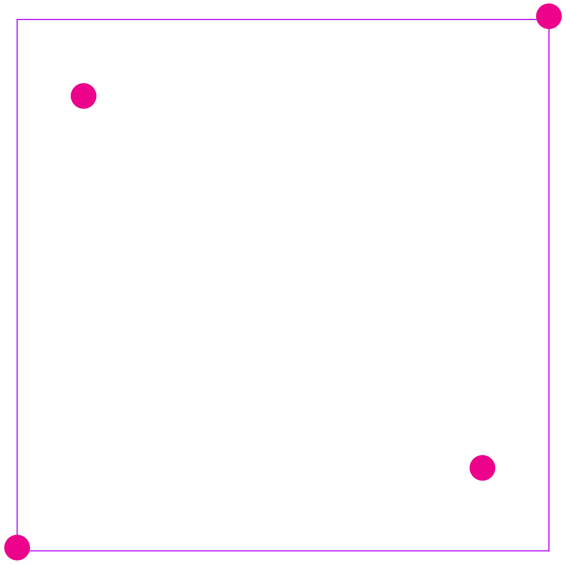

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But what about limits of general Einstein metrics?

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the orbifold Einstein metric g_∞ is conformally Kähler.

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It is false in all higher dimensions!

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A more transparent proof was then given in L '21.

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In most cases, this implies that g is Kähler-Einstein. There are two exceptions, but these are both rigid, and thus never lead to non-trivial G-H limits. Conversely, $\lambda > 0$ K-E $\Longrightarrow \det(W^+) > 0$.

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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And hence that they are actually conformally Kähler!

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