

Desingularizations of
Conformally Kähler,
Einstein Orbifolds

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Joint work with

Joint work with

Tristan Ozuch

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Massachusetts Institute of Technology

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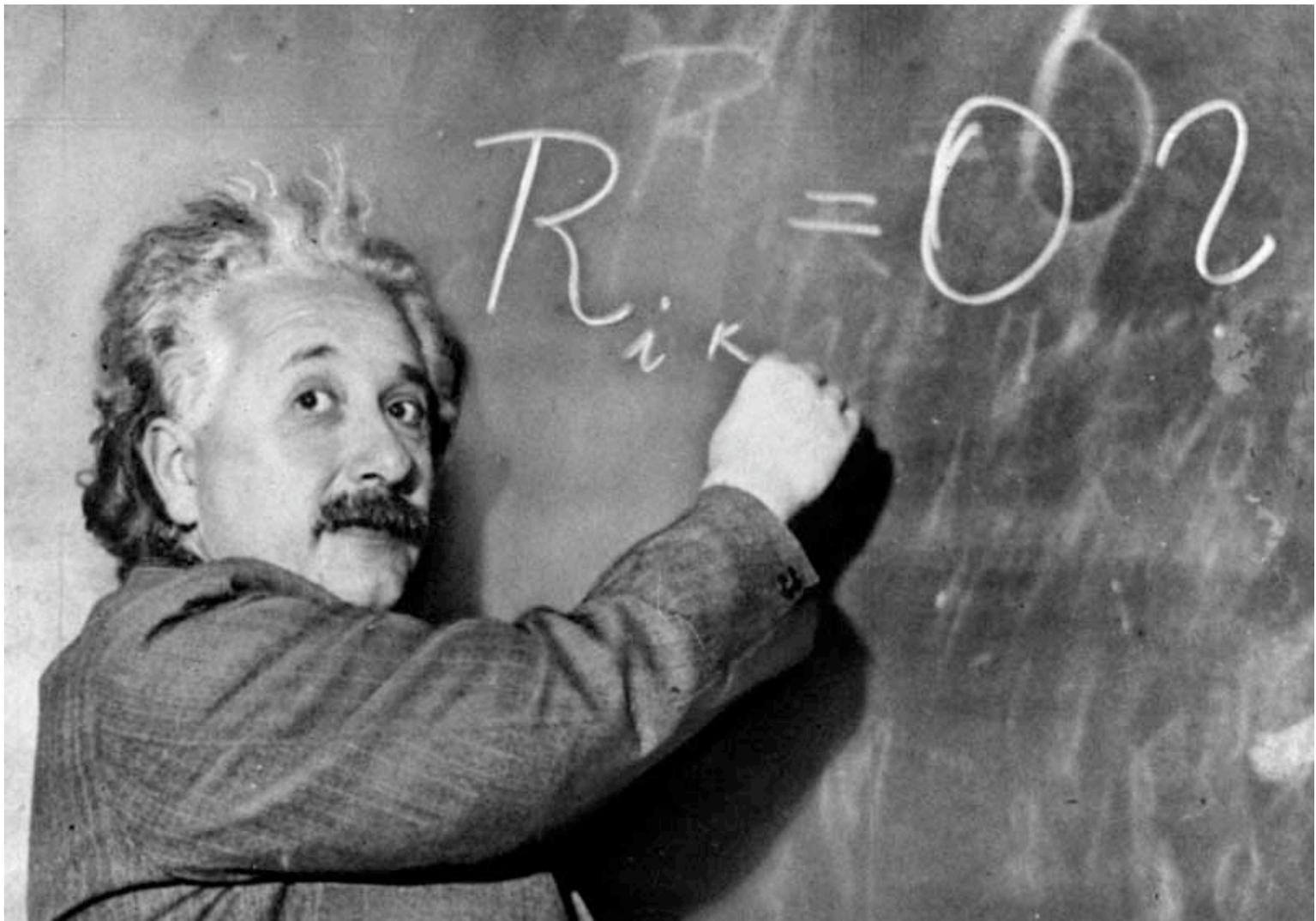
“... the greatest blunder of my life!”

— A. Einstein, to G. Gamow

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$$s = r_j^j = \mathcal{R}^{ij}_{ij}.$$

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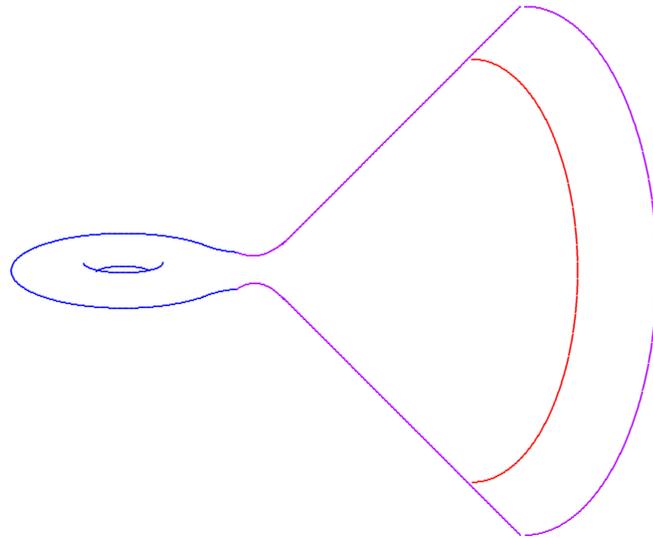
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- Iterate to eliminate all orbifold singularities.

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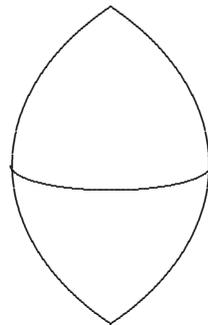
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Goal: Show that this doesn't change anything!

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All known examples all have $W_+ = 0 \dots$

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Reversing orientation interchanges $\Lambda^+ \leftrightarrow \Lambda^-$.

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where

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Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

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We avoid this question by means of a definition!

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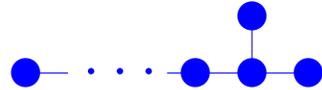
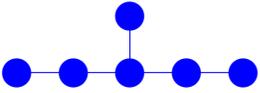
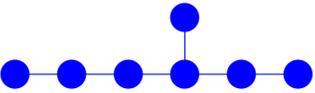
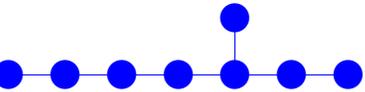
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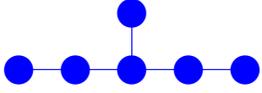
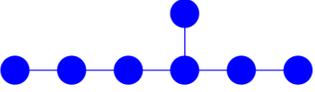
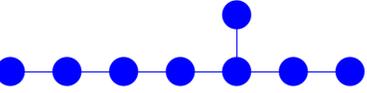
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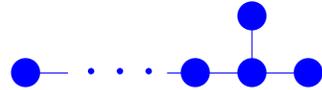
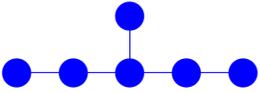
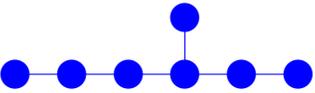
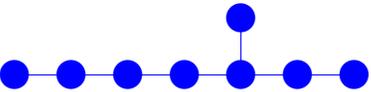
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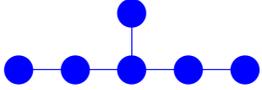
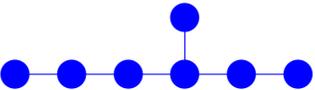
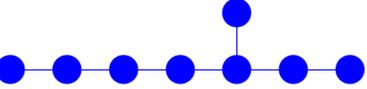
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The corresponding gravitational instantons are exactly the hyper-Kähler ALE manifolds.

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Standard shorthand: $\frac{1}{\ell m^2}(1, \ell mn - 1)$.

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The corresponding singularities \mathbb{C}^2/Γ are exactly the ones that algebraic geometers term singularities of **class \mathbf{T}** . For them, the defining property is that they have \mathbb{Q} -Gorenstein local smoothings...

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Proposition. *Either there are infinitely many topological types of compact **K-E** 4-dimensional orbifolds with only isolated singularities that cannot arise as Gromov-Hausdorff limits of smooth Einstein manifolds, or else there are infinitely many Ricci-flat ALE 4-manifolds that remain to be discovered.*

Example. $\mathbb{C}\mathbb{P}_2/\mathbb{Z}_p$, where the \mathbb{Z}_p action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

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These examples form part of a systematic picture...

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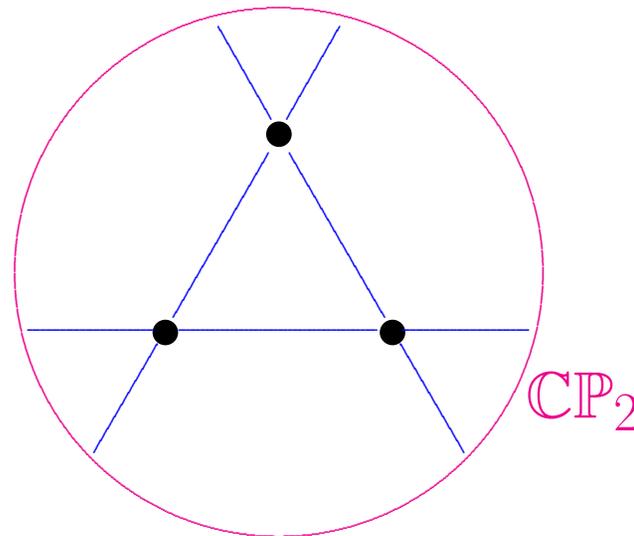
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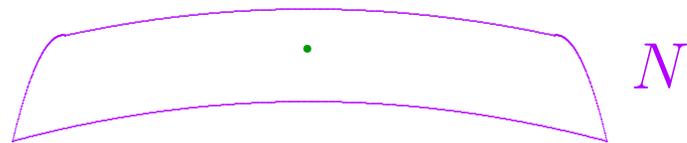
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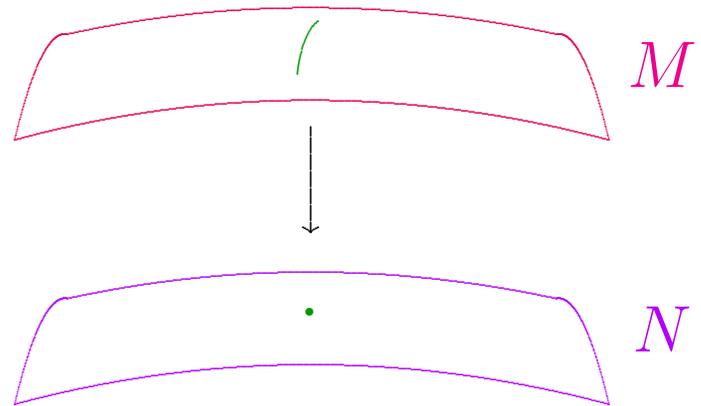
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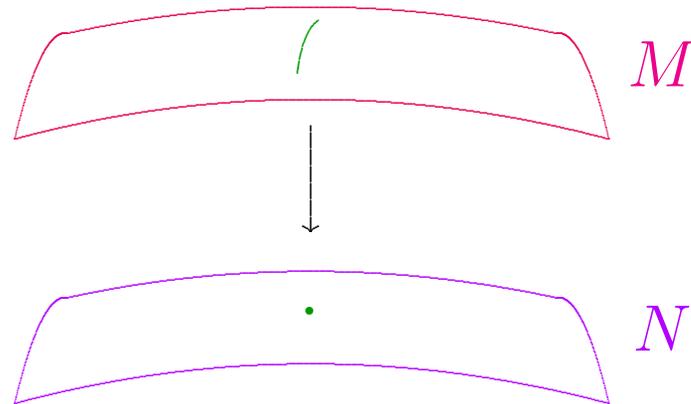
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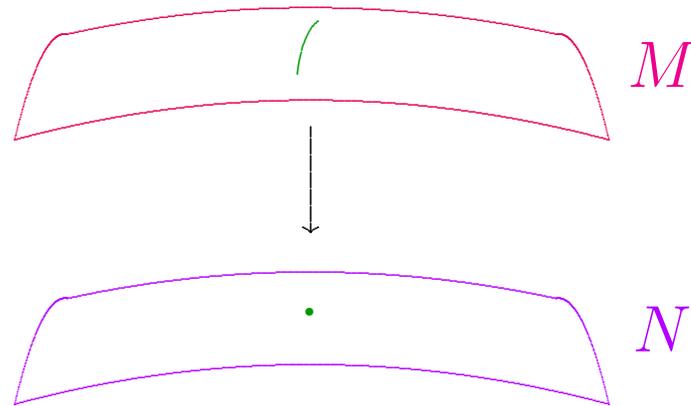


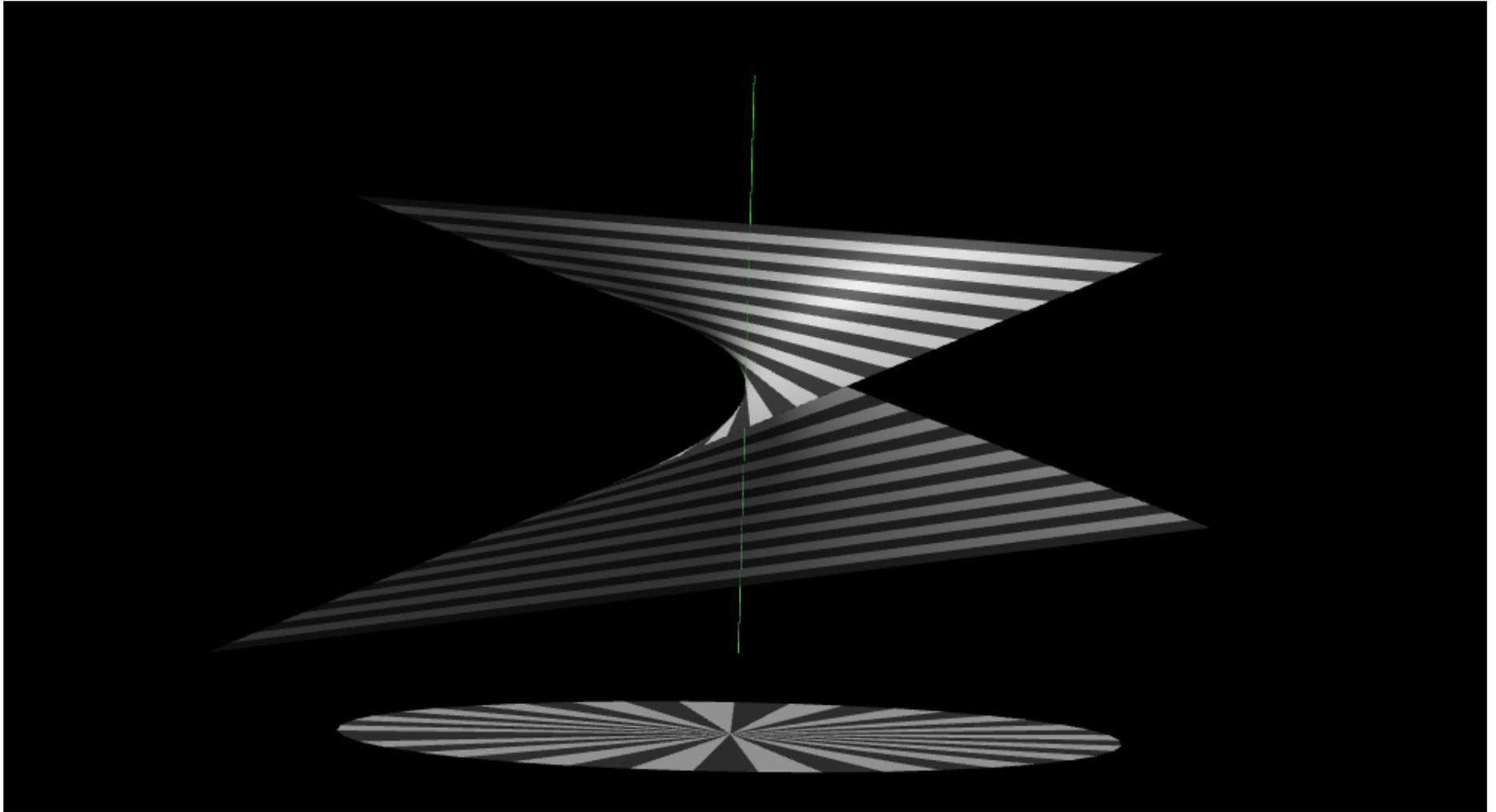
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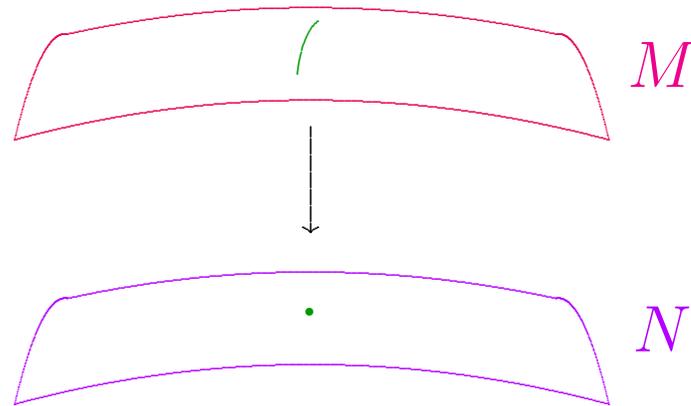


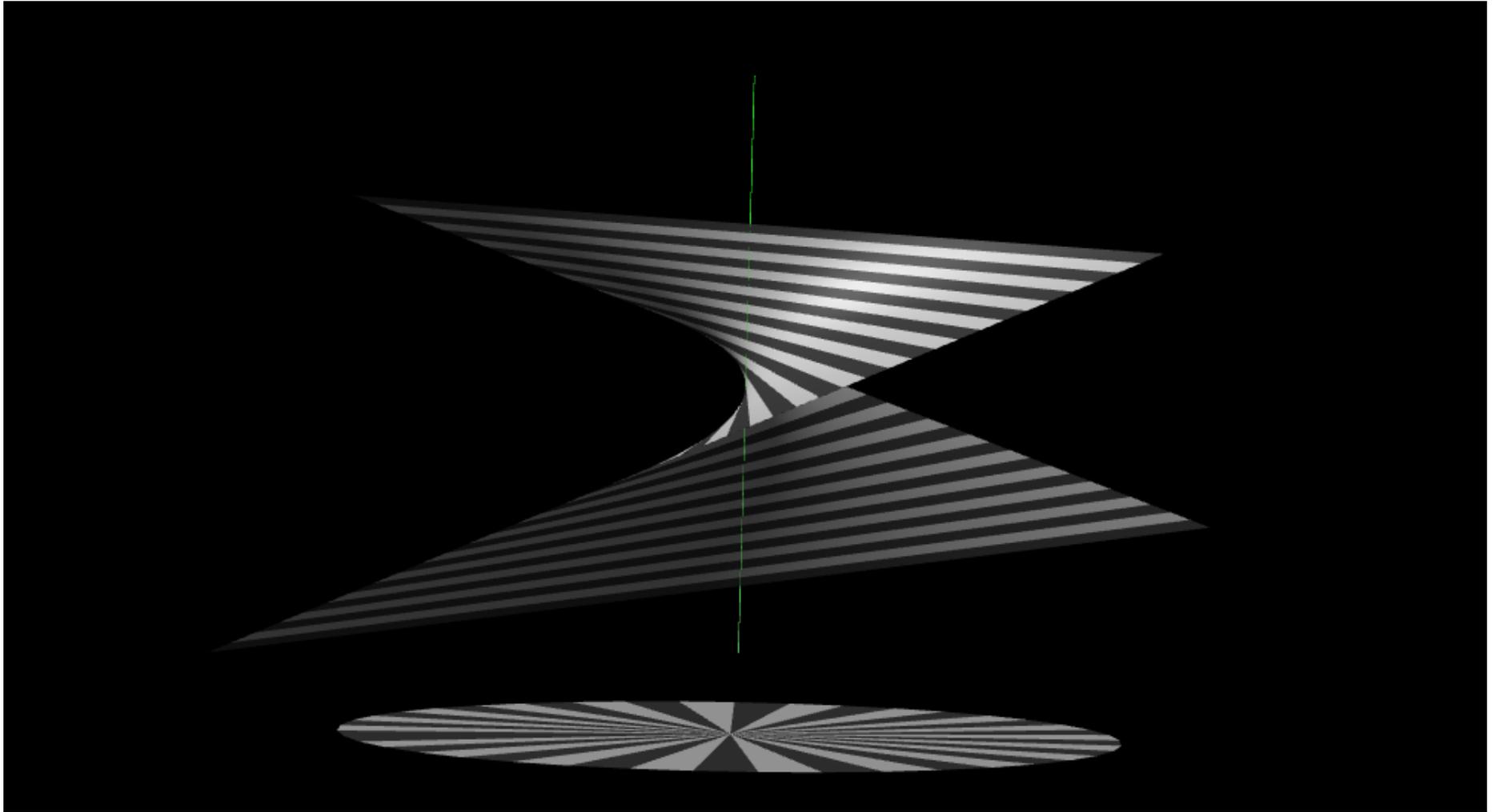
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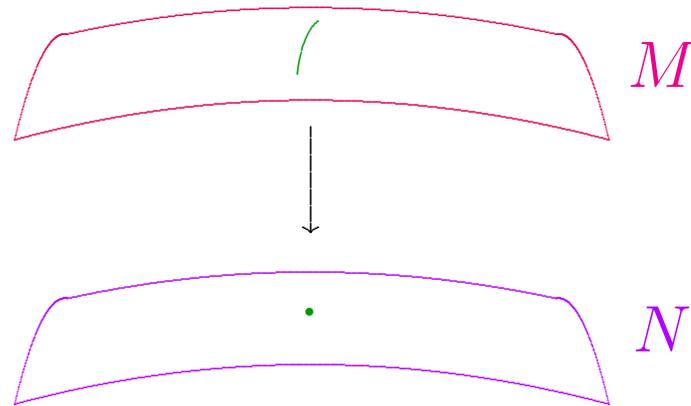


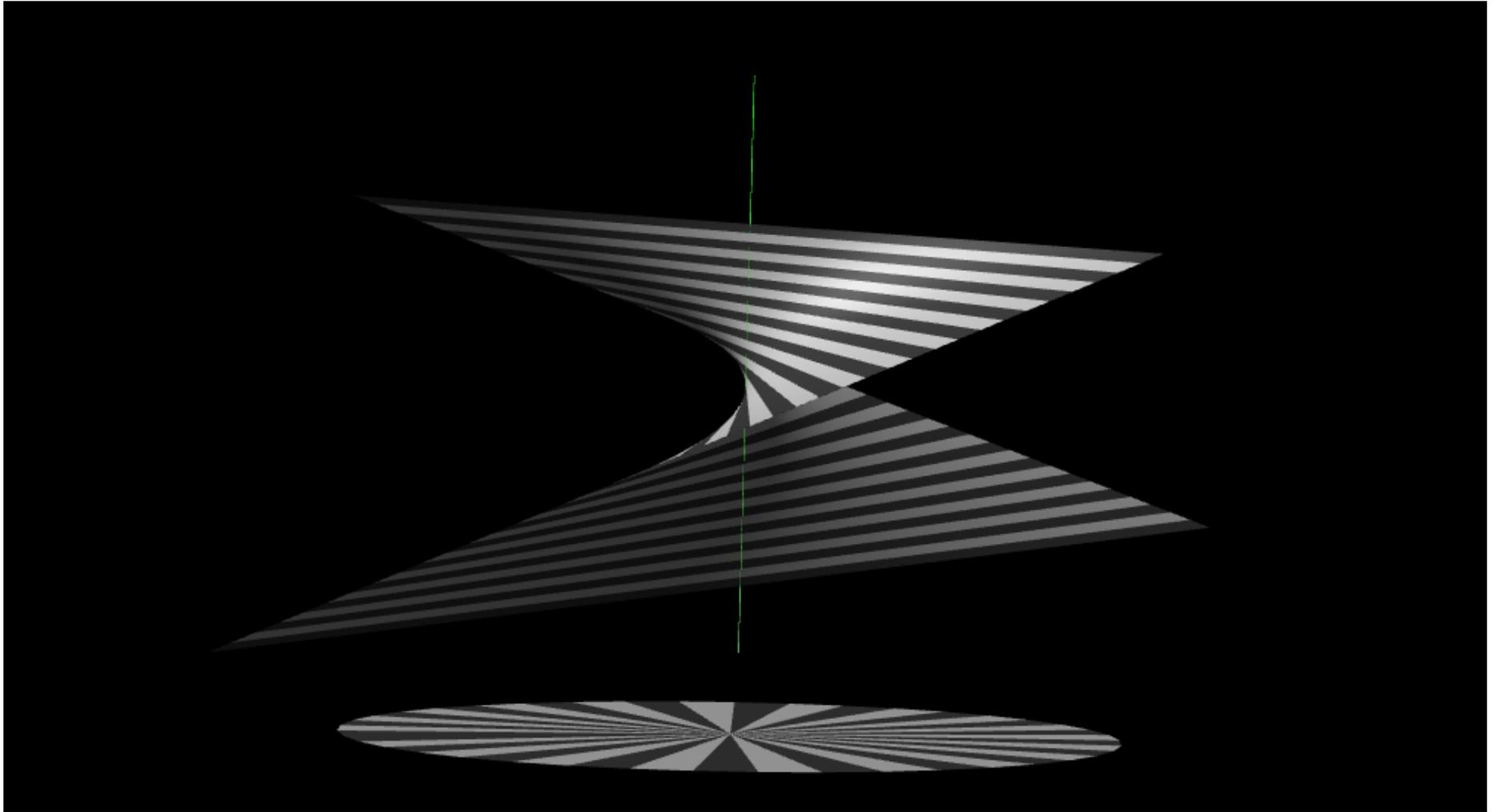
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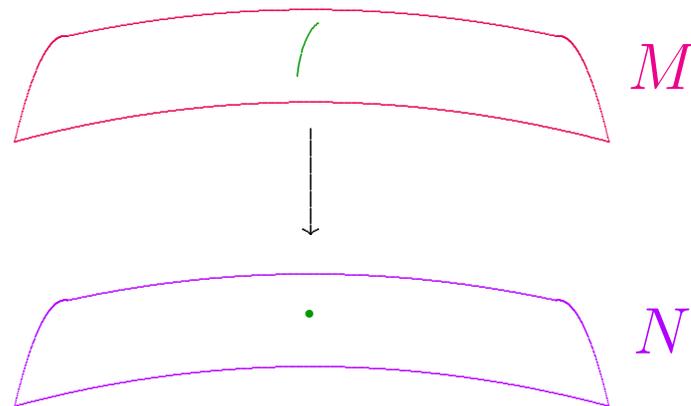


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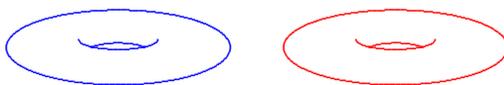
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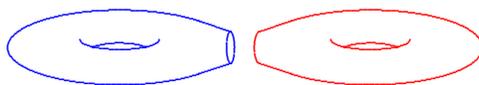
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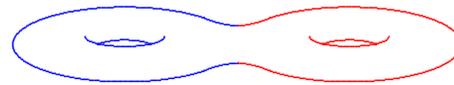
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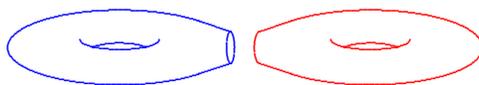
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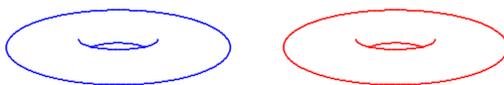
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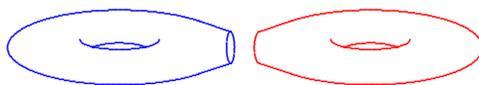
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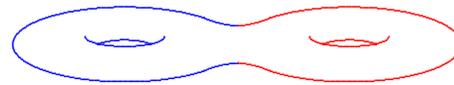
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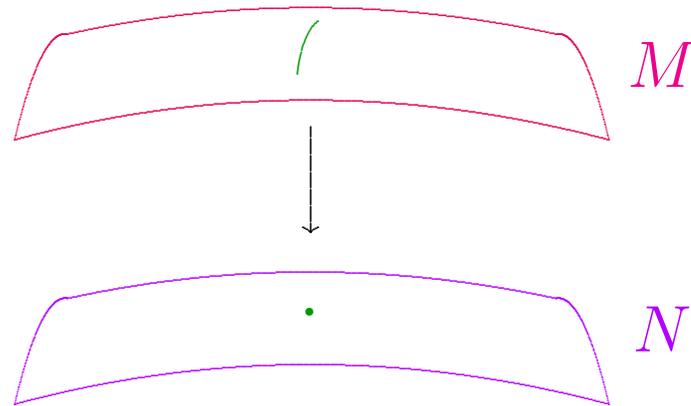


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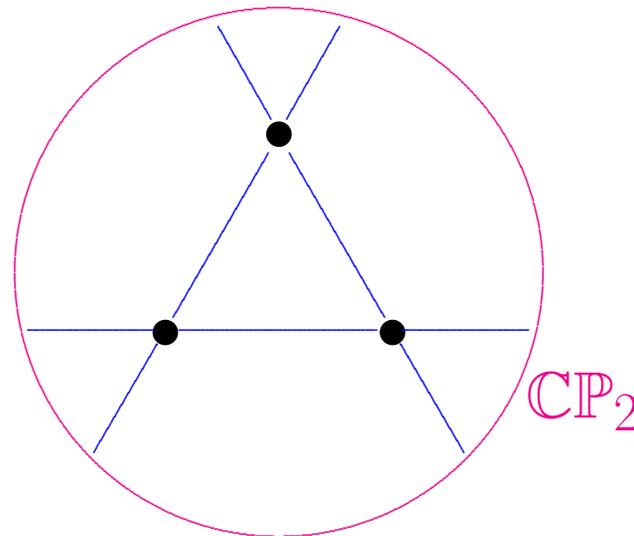
$$M \stackrel{\text{diff}}{\approx} \begin{cases} \mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, & 0 \leq k \leq 8, \\ \text{or} \\ S^2 \times S^2 \end{cases}$$

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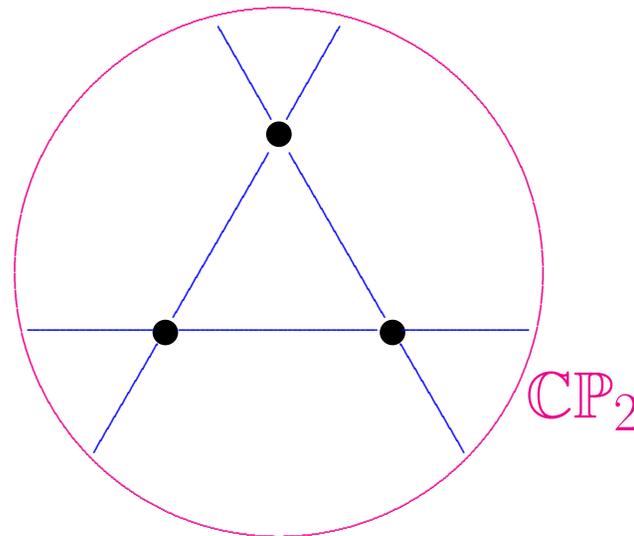
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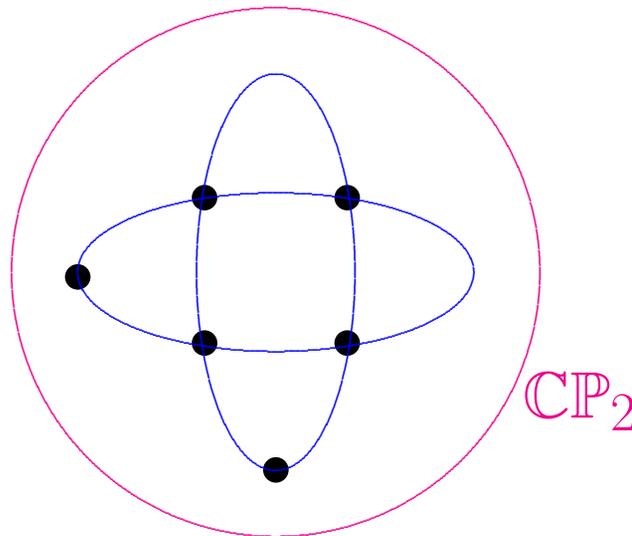


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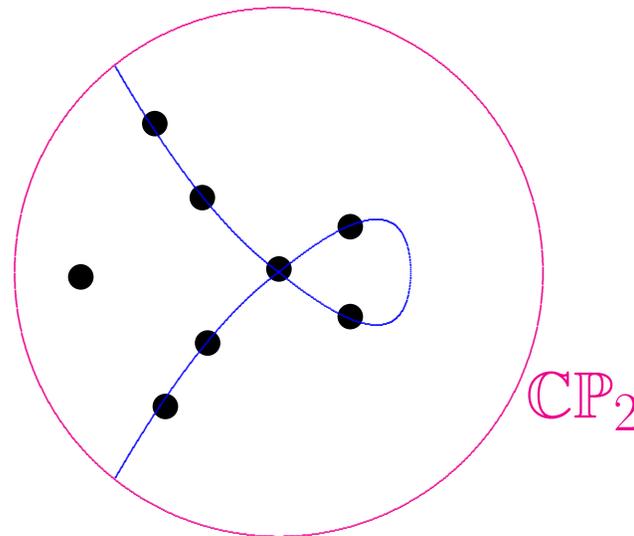


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No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Existence: Page

1978

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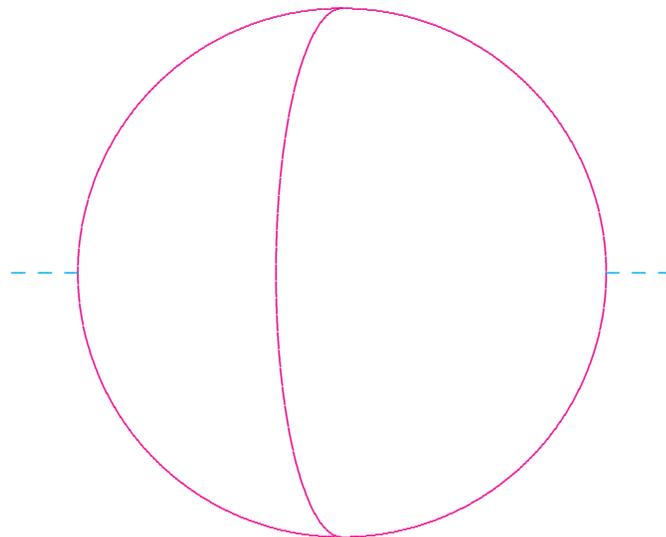
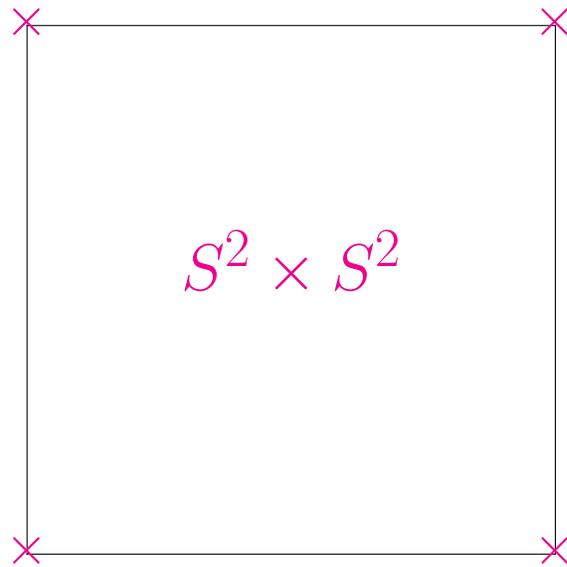
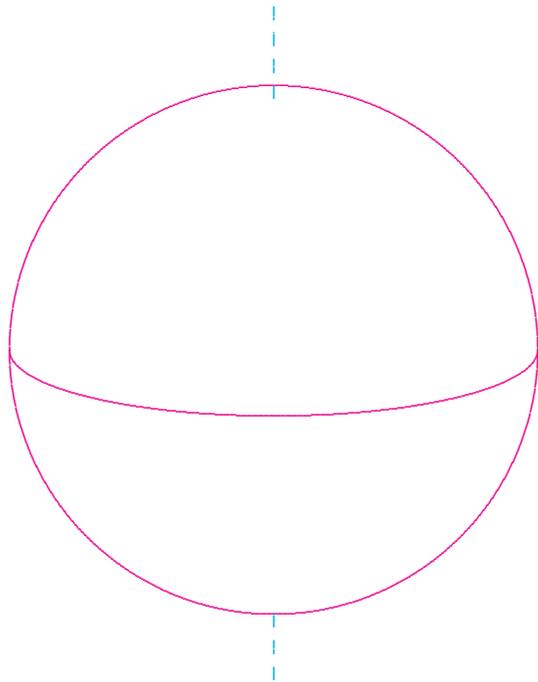
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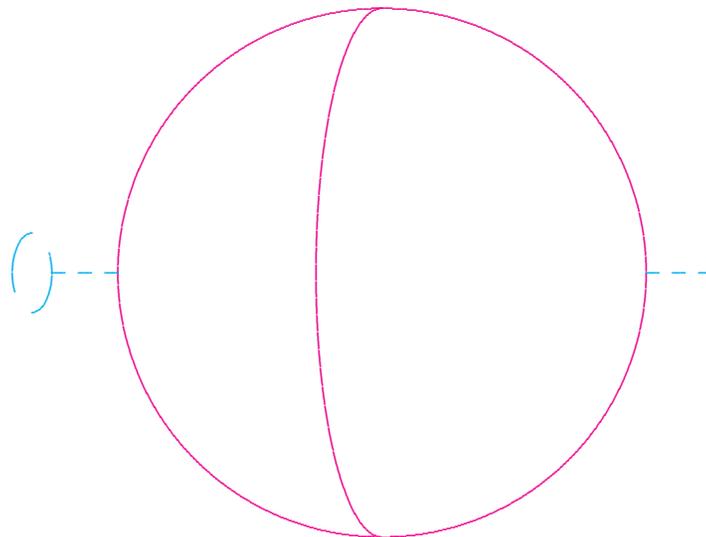
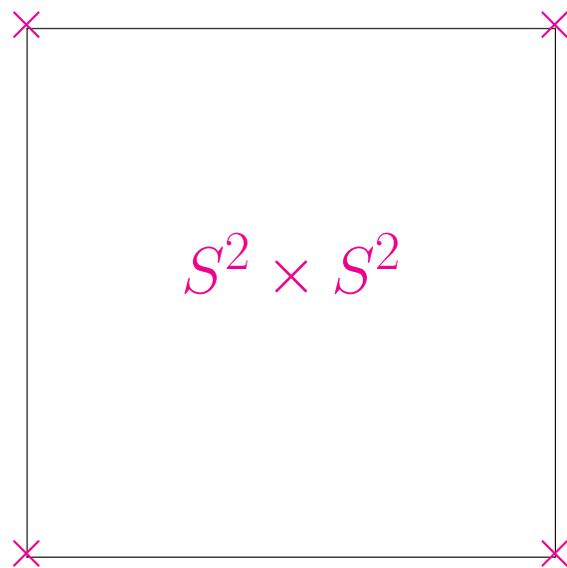
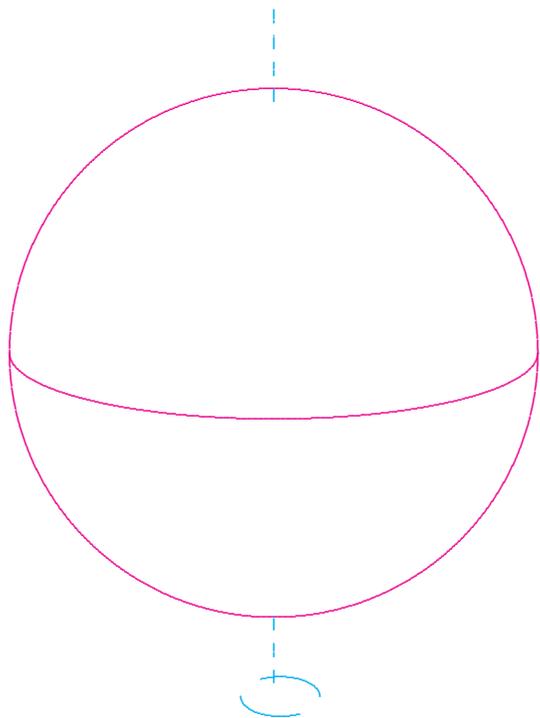
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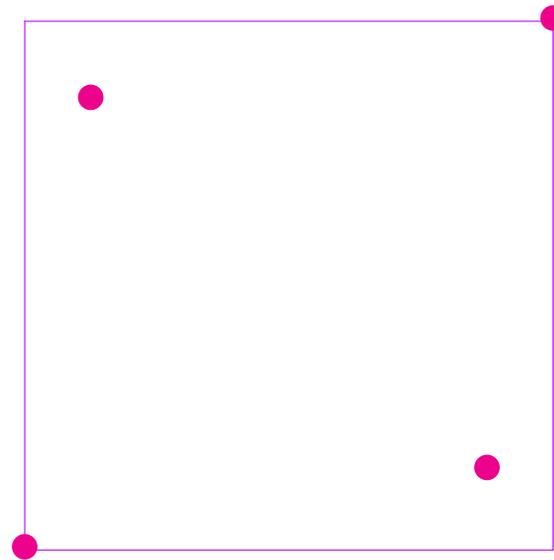
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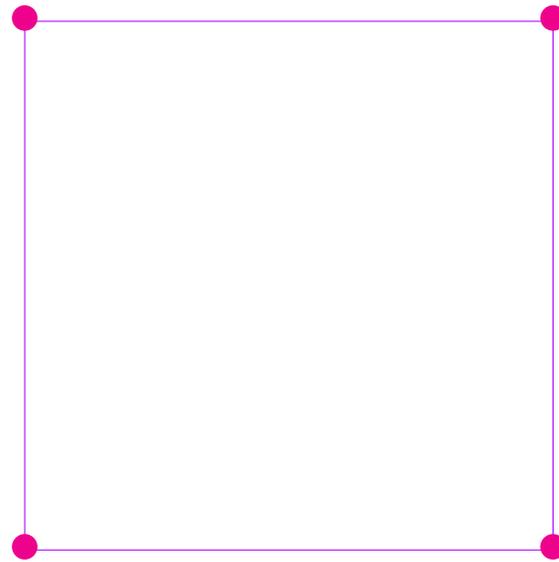


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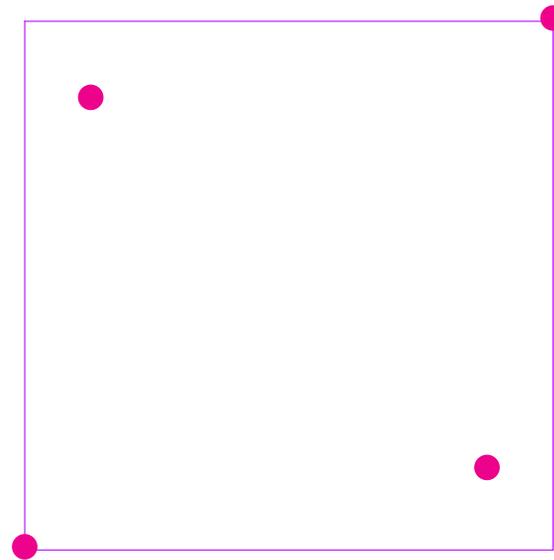


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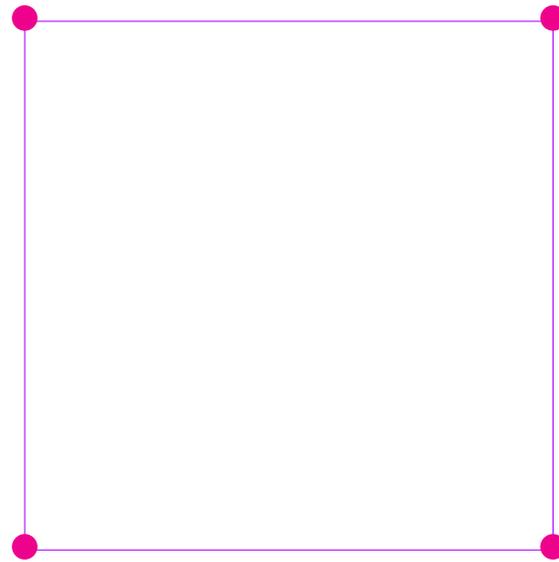


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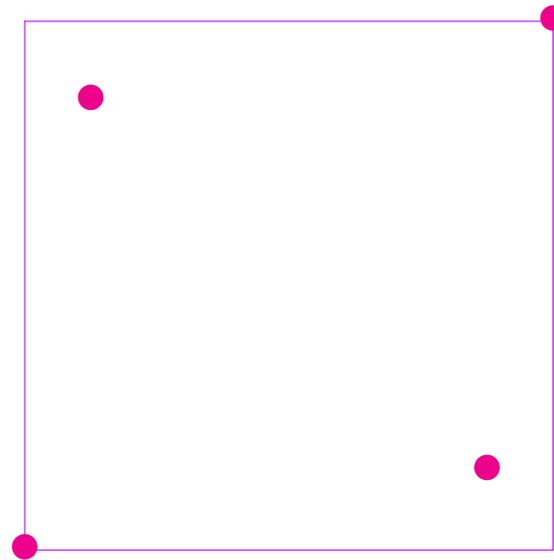


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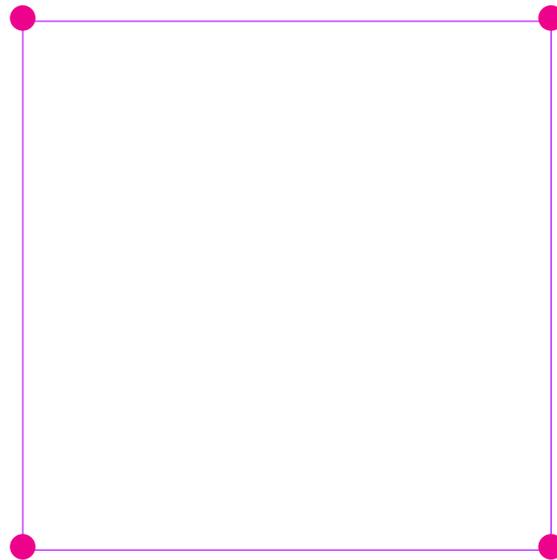


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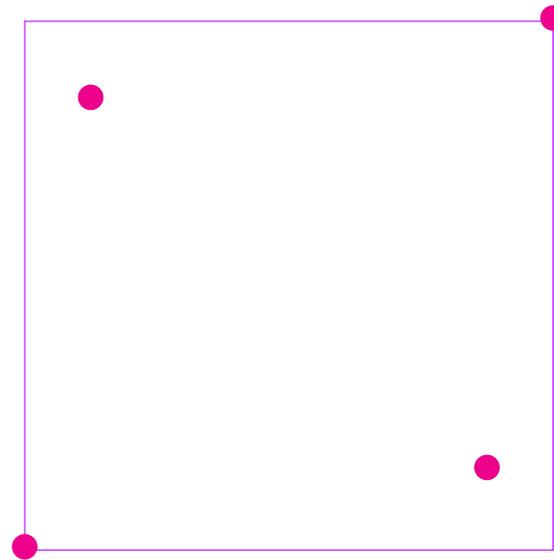


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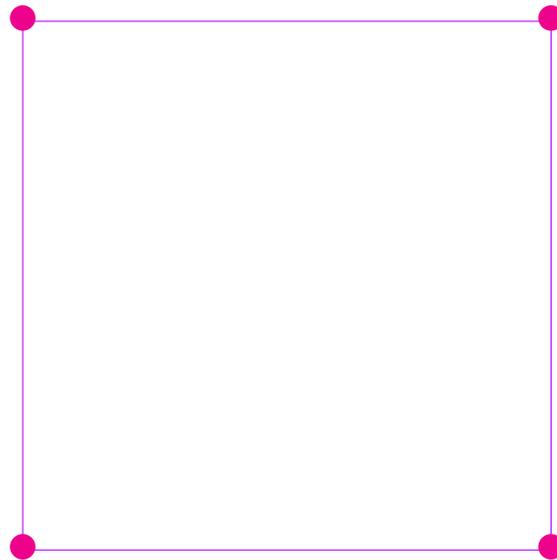


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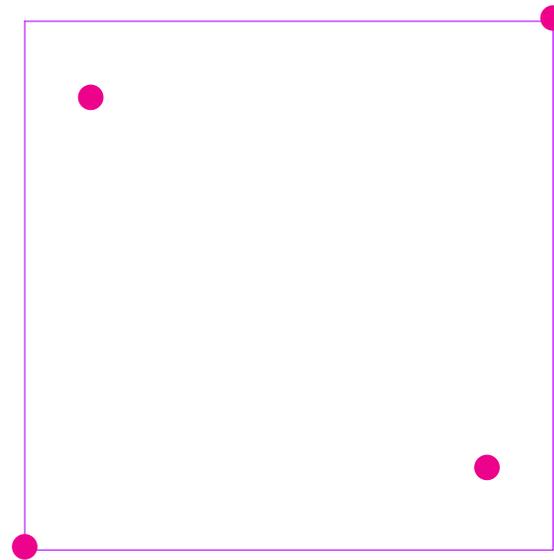


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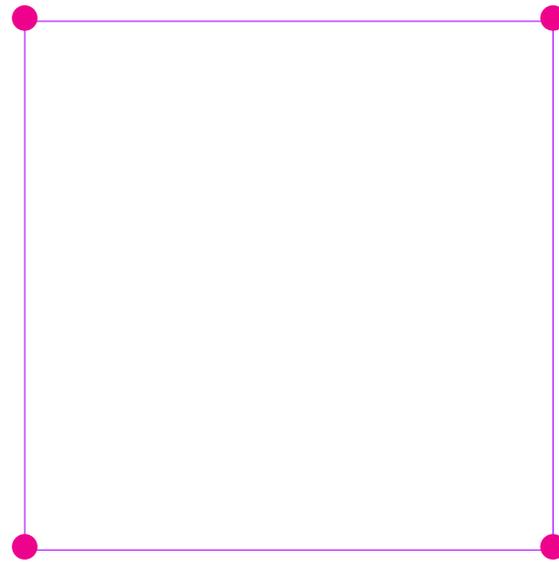


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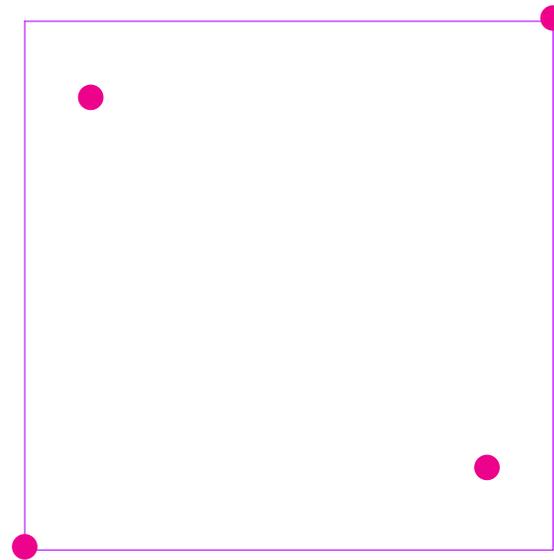


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But what about limits of general Einstein metrics?

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[A more transparent proof was then given in L '21](#).

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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And hence that they are actually conformally Kähler!

Merci de m'avoir invité!

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