Solutions to Final Exam, MAT 125, Spring 2002

Question 1 Compute each of the following limits:

(a) $\displaystyle\lim_{x\to 2}\frac{x^2+x-6}{x^2-4}$

(b) $\displaystyle\lim_{x\to 2^-}\frac{\sqrt{x^2-4x+4}}{x^2+2x-8}$

(c) $\displaystyle\lim_{h\to 0}\sin\left(\frac{\pi}{3}+h\right)$

(d) $\displaystyle\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{3}+h\right)}{h}$

Answer 1

(a) If we simply plug in $x=2$ to the function $\frac{x^2+x-6}{x^2-4}$, we obtain $\frac{0}{0}$. Thus we must factor the polynomials first.

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+x-6}{x^2-4} &= \lim_{x\to 2}\frac{(x+3)(x-2)}{(x+2)(x-2)}\\ &= \lim_{x\to 2}\frac{x+3}{x+2}\\ &= \frac{2+3}{2+2}\\ &= \mathbf{\frac{5}{4}}\end{aligned}$$

(b) Again plugging in $x=2$ results in an answer of the form $\frac{0}{0}$, so we must simplify first.

$$\begin{aligned}\lim_{x\to 2^-}\frac{\sqrt{x^2-4x+4}}{x^2+2x-8} &= \lim_{x\to 2^-}\frac{\sqrt{(x-2)^2}}{(x+4)(x-2)}\\ &= \lim_{x\to 2^-}\frac{|x-2|}{(x+4)(x-2)}\\ &= \lim_{x\to 2^-}\frac{1}{x+4}\cdot\lim_{x\to 2^-}\frac{|x-2|}{x-2}\\ &= \frac{1}{6}\lim_{x\to 2^-}\frac{|x-2|}{x-2}\end{aligned}$$

Now if $x\to 2^-$, then $x<2$ so $x-2<0$. Therefore $|x-2|=-(x-2)$, and so $\frac{|x-2|}{x-2}=-1$ for $x<2$. So in the end we have

$$\lim_{x\to 2^-}\frac{\sqrt{x^2-4x+4}}{x^2+2x-8}=\frac{1}{6}\lim_{x\to 2^-}\frac{|x-2|}{x-2}=\mathbf{-\frac{1}{6}}$$

(c) This one is easy. Since the function $f(x)=\sin x$ is continuous, we can pull the limit inside the function. So

$$\lim_{h\to 0}\sin\left(\frac{\pi}{3}+h\right)=\sin\left(\lim_{h\to 0}\left[\frac{\pi}{3}+h\right]\right)=\sin\left(\frac{\pi}{3}\right)=\mathbf{\frac{\sqrt{3}}{2}}$$

(d) We recognize this formula as the definition of the derivative. In general, the derivative is

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

and the limit

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+h\right)-\frac{\sqrt{3}}{2}}{h}$$

is the same thing if we let $f(x)=\sin x$ and $a=\frac{\pi}{3}$.

Since $f'(x)=\cos x$, we have

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+h\right)-\frac{\sqrt{3}}{2}}{h} = \cos\left(\frac{\pi}{3}\right) = \mathbf{\frac{1}{2}}$$

Question 2 Compute each of the following derivatives:

(a) $\dfrac{d}{dt}\sqrt[3]{t-2}$

(b) $\dfrac{d}{du}\ln\left(\dfrac{1+u}{1-u}\right)$

(c) $\dfrac{d^2}{dx^2}\big[\cos(3x)\big]$

(d) $\dfrac{d^3}{dx^3}(e^x)^2$

Answer 2

(a) To compute the derivative, we first rewrite the root as a power, and then use the Chain Rule. We have

$$\frac{d}{dt}\sqrt[3]{t-2} = \frac{d}{dt}(t-2)^{1/3} = \tfrac{1}{3}(t-2)^{-2/3}\frac{d}{dt}(t-2) = \tfrac{1}{3}(t-2)^{-2/3}\cdot 1 = \boldsymbol{\tfrac{1}{3}(t-2)^{-2/3}}$$

(b) Although we could use the Chain Rule directly on this function, it is much easier to simplify algebraically first, using the general formula

$$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$$

So we have

$$\begin{aligned}
\frac{d}{du}\ln\left(\frac{1+u}{1-u}\right) &= \frac{d}{du}\big[\ln(1+u)-\ln(1-u)\big] \\
&= \frac{d}{du}\ln(1+u) - \frac{d}{du}\ln(1-u) \\
&= \frac{1}{1+u}\frac{d}{du}(1+u) - \frac{1}{1-u}\frac{d}{du}(1-u) \\
&= \frac{1}{1+u}\cdot 1 - \frac{1}{1-u}\cdot(-1) \\
&= \frac{1}{1+u} + \frac{1}{1-u} \\
&= \boldsymbol{\frac{2}{1-u^2}}
\end{aligned}$$

(c) The first derivative of $\cos(3x)$ is

$$\frac{d}{dx}\big[\cos(3x)\big] = -\sin(3x)\frac{d}{dx}(3x) = -3\sin(3x)$$

Therefore the second derivative of $\cos(3x)$ is the derivative of $-3\sin(3x)$, which is

$$\begin{aligned}\frac{d^2}{dx^2}\big[\cos(3x)\big] &= \frac{d}{dx}\big[-3\sin(3x)\big] \\ &= -3\frac{d}{dx}\big[\sin(3x)\big] \\ &= -3\cos(3x)\frac{d}{dx}(3x) \\ &= \boldsymbol{-9\cos(3x)}\end{aligned}$$

(d) This problem is really difficult if you don't simplify algebraically first. One should first rewrite the function as

$$(e^x)^2 = e^{2x}$$

since a power raised to a power is the product of the powers. Then we have

$$\frac{d^3}{dx^3}(e^x)^2 = \frac{d^3}{dx^3}(e^{2x})$$

The first derivative is

$$\frac{d}{dx}(e^{2x}) = 2e^{2x},$$

the second is

$$\frac{d^2}{dx^2}(e^{2x}) = \frac{d}{dx}(2e^{2x}) = 4e^{2x},$$

and the third is

$$\frac{d^3}{dx^3}(e^{2x}) = \frac{d}{dx}(4e^{2x}) = \boldsymbol{8e^{2x}}.$$

Question 3 Consider the function $f(x) = \dfrac{x^4}{4} + x^3 - 2x^2$.

(a) Locate all the relative maxima and relative minima of $f(x)$.

(b) Locate the absolute maximum and absolute minimum of $f(x)$ on the interval $0 \le x \le 2$.

Answer 3

(a) To find relative (local) maxima and minima, we must first compute the critical points. To do this, we compute the derivative:

$$f'(x) = x^3 + 3x^2 - 4x$$

The derivative exists everywhere, so the critical points are those values of x for which $f'(x) = 0$. We solve this equation to find

$$\begin{aligned}f'(x) &= 0 \\ x^3 + 3x^2 - 4x &= 0 \\ x(x^2 + 3x - 4) &= 0 \\ x(x+4)(x-1) &= 0\end{aligned}$$

So the critical points are $x = 0$, $x = -4$, and $x = 1$.

To determine whether these are local maxima, local minima, or stationary points, we can use either the First Derivative Test or the Second Derivative Test. For the First Derivative Test, we check the sign of the derivative at

– points to the left of $x = -4$, for example $x = -5$. Then

$$f'(-5) = (-5)^3 + 3(-5)^2 - 4(-5) = -125 + 75 + 20 = -5 < 0.$$

Therefore the derivative is negative everywhere to the left of $x = -4$, and so the function is decreasing on $(-\infty, -4)$.

– points between $x = -4$ and $x = 0$, for example $x = -1$. Then

$$f'(-1) = (-1)^3 + 3(-1)^2 - 4(-1) = -1 + 3 + 4 = 6 > 0.$$

So the derivative is positive everywhere between $x = -4$ and $x = 0$, and the function is increasing on $(-4, 0)$.

– points between $x = 0$ and $x = 1$, for example $x = \frac{1}{2}$. Then

$$f'\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{3}{4} - 2 = -\frac{9}{16} < 0.$$

So the derivative is negative everywhere between $x = 0$ and $x = 1$, and thus the function is decreasing on $(0, 1)$.

– points to the right of $x = 1$, for example $x = 2$. Then

$$f'(2) = 2^3 + 3(2)^2 - 4(2) = 8 + 12 - 8 = 12 > 0.$$

So the derivative is positive for $x > 1$, and therefore the function is increasing on $(1, \infty)$.

Since the function goes from decreasing to increasing at $x = -4$, $x = -4$ is a local minimum. Since the function goes from increasing to decreasing at $x = 0$, $x = 0$ is a local maximum. Since the function goes from decreasing to increasing at $x = 1$, $x = 1$ is a local minimum. So

relative maxima at $\boldsymbol{x = -4, 1}$

relative minimum at $\boldsymbol{x = 0}$

The easier method is to use the Second Derivative Test: first compute $f''(x) = 3x^2 + 6x - 4$. Since $f''(-4) = 3(-4)^2 + 6(-4) - 4 = 20 > 0$, $x = -4$ is a local minimum. Since $f''(0) = -4 < 0$, $x = 0$ is a local maximum. Since $f''(1) = 3 + 6 - 4 = 5 > 0$, $x = 1$ is a local minimum. Clearly the result is the same either way.

(b) Absolute maxima and minima are easier. We have already computed that the critical points are $x = -4$, $x = 0$, and $x = 1$. The only ones we care about are the ones in the interval $[0, 2]$, that is, $x = 0$ and $x = 1$. So we list the values of $f(x)$ at the critical points and the endpoints:

$$f(0) = \frac{0^4}{4} + 0^3 - 2(0^2) = 0$$
$$f(1) = \frac{1^4}{4} + 1^3 - 2(1^2) = -0.75$$
$$f(2) = \frac{2^4}{4} + 2^3 - 2(2^2) = 4$$

Since $f(2) = 4$ is the largest of these values, $x = 2$ is the absolute maximum. Since $f(1) = -0.75$ is the smallest of these values, $x = 1$ is the absolute minimum. So the answers are

absolute maximum at $\boldsymbol{x = 2}$

absolute minimum at $\boldsymbol{x = 1}$

Question 4 Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy $PV = C$, where C is a constant. At a certain instant, suppose that the volume of a given sample is 900 cm^3, the pressure is 300 kilopascals, and the pressure is increasing at a rate of 20 kilopascals per minute. At what rate is the volume decreasing at this moment?

Answer 4 This is a related rates problem; we are trying to find $\frac{dV}{dt}$ given $P = 300$, $V = 900$, and $\frac{dP}{dt} = 20$. The easiest way to do this is to simply compute the time derivative of the equation $PV = C$.

$$\frac{d}{dt}(PV) = \frac{dC}{dt}$$
$$P\frac{dV}{dt} + V\frac{dP}{dt} = 0$$

using the Product Rule and the fact that the derivative of a constant is zero. Solving this equation for $\frac{dV}{dt}$, we get

$$\frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt}$$

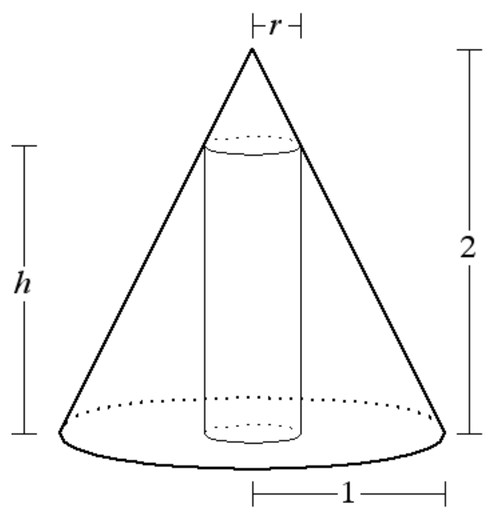
Now we can plug in numbers and obtain

$$\frac{dV}{dt} = -\frac{900}{300}(20) = -60 \text{ cm}^3 \text{ per second}$$

So the volume is *decreasing* at a rate of **60** cubic centimeters per second.

Students did *not* get full credit for obtaining this answer by dimensional analysis or by randomly multiplying and dividing the given numbers; such methods do not work in general and will usually give the wrong answer.

Question 5 A cylinder is inscribed inside a cone of height 2 meters and radius 1 meter. What is the maximum possible volume of the cylinder?



Answer 5 The variables we will need to find the volume of the cylinder are its height h and radius r. These are labeled in the picture as shown.

The geometry in this situation comes about because the full cone and the little cone at the top (between the top of the cylinder and the top of the cone) are proportional in size. This means that the ratio of height to radius of the two cones are equal. Since the height of the small cone is $2 - h$, the equation we have is

$$\frac{2-h}{r} = \frac{2}{1}$$

so that $2 - h = 2r$ or
$$h = 2 - 2r$$
The volume of the cylinder is
$$V = \pi r^2 h$$
and since we know $h = 2 - 2r$, we can plug in to get a function of r alone:
$$V(r) = \pi r^2(2 - 2r) = \pi(2r^2 - 2r^3)$$

To find the maximum volume of the cylinder, we compute the derivative $V'(r)$ and set it equal to zero. The derivative is computed as
$$V'(r) = \pi(4r - 6r^2) = 2\pi r(2 - 3r)$$
so that
$$V'(r) = 0 \qquad \Longrightarrow \qquad r = 0 \text{ or } r = \frac{2}{3}$$
Since $r = 0$ corresponds to a degenerate situation, the actual answer must be
$$r = \frac{2}{3}$$
It is intuitively clear that this must be the radius for which the volume is maximized.

Now we obtain h from the equation $h = 2 - 2r$, and get $h = \frac{2}{3}$ as well. So the volume of the maximum cylinder is
$$V = \pi r^2 h = \pi \left(\frac{2}{3}\right)^2 \cdot \frac{2}{3} = \boldsymbol{\frac{8}{27}\pi}$$