

*Twistors,*  
*Quasi-Fuchsian 3-Manifolds, &*  
*Almost-Kähler Geometry*

Claude LeBrun  
Stony Brook University

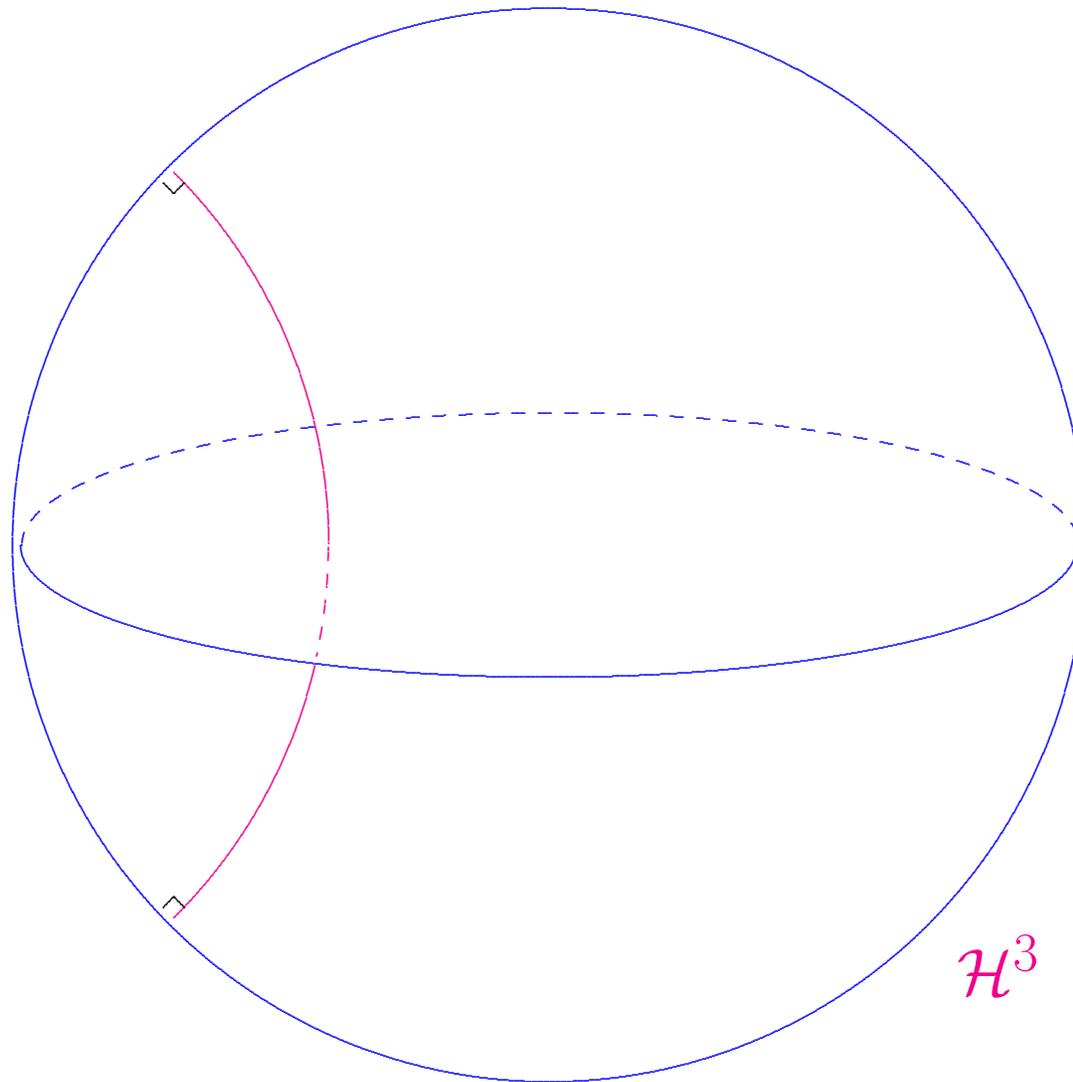
Twistors in Geometry and Physics:  
Isaac Newton Institute Workshop.  
Cambridge, UK. September 10, 2024.

# Poincaré Model of Hyperbolic 3-Space

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$$h = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - \rho^2)^2}$$

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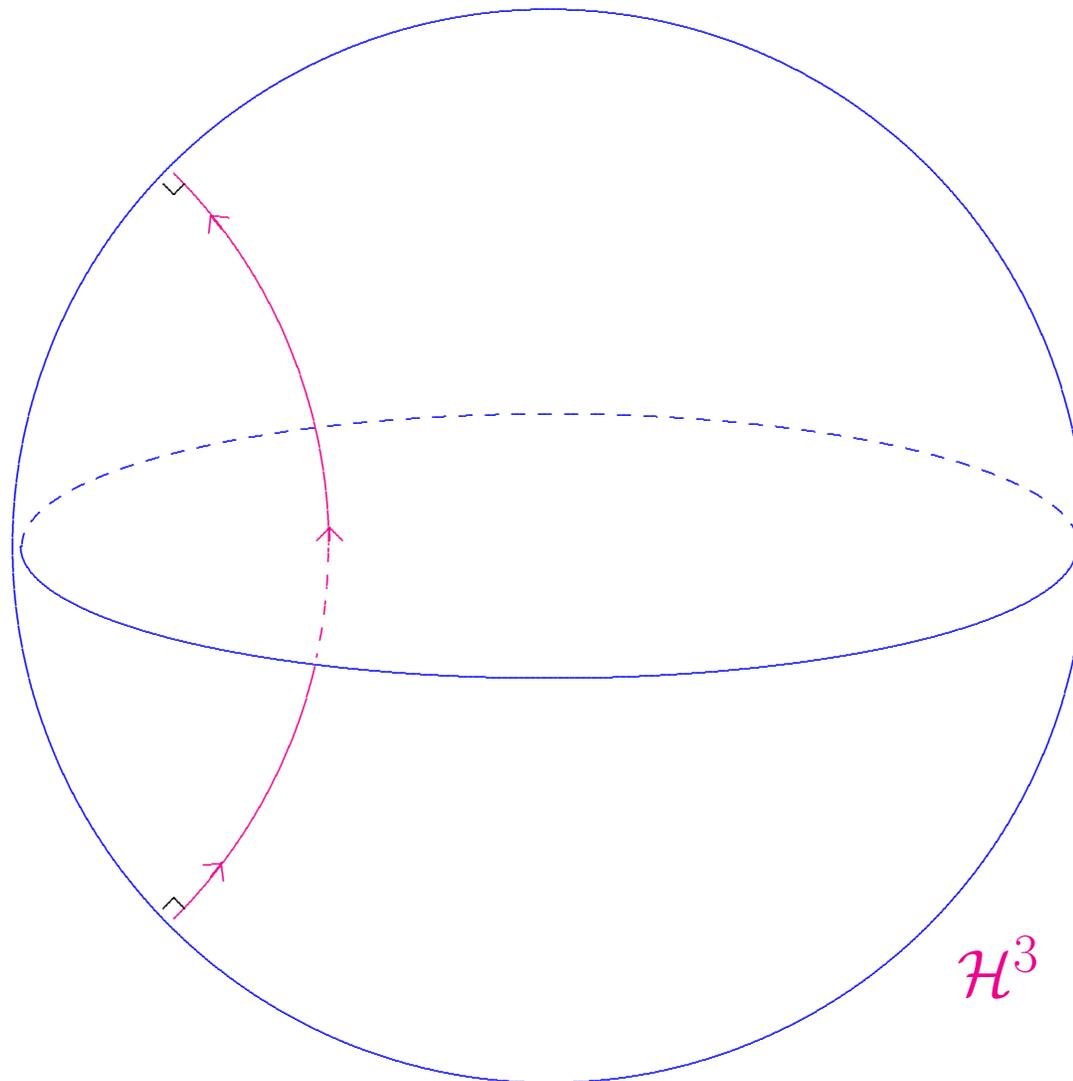


# Mini-Twistor Space of Hyperbolic 3-Space

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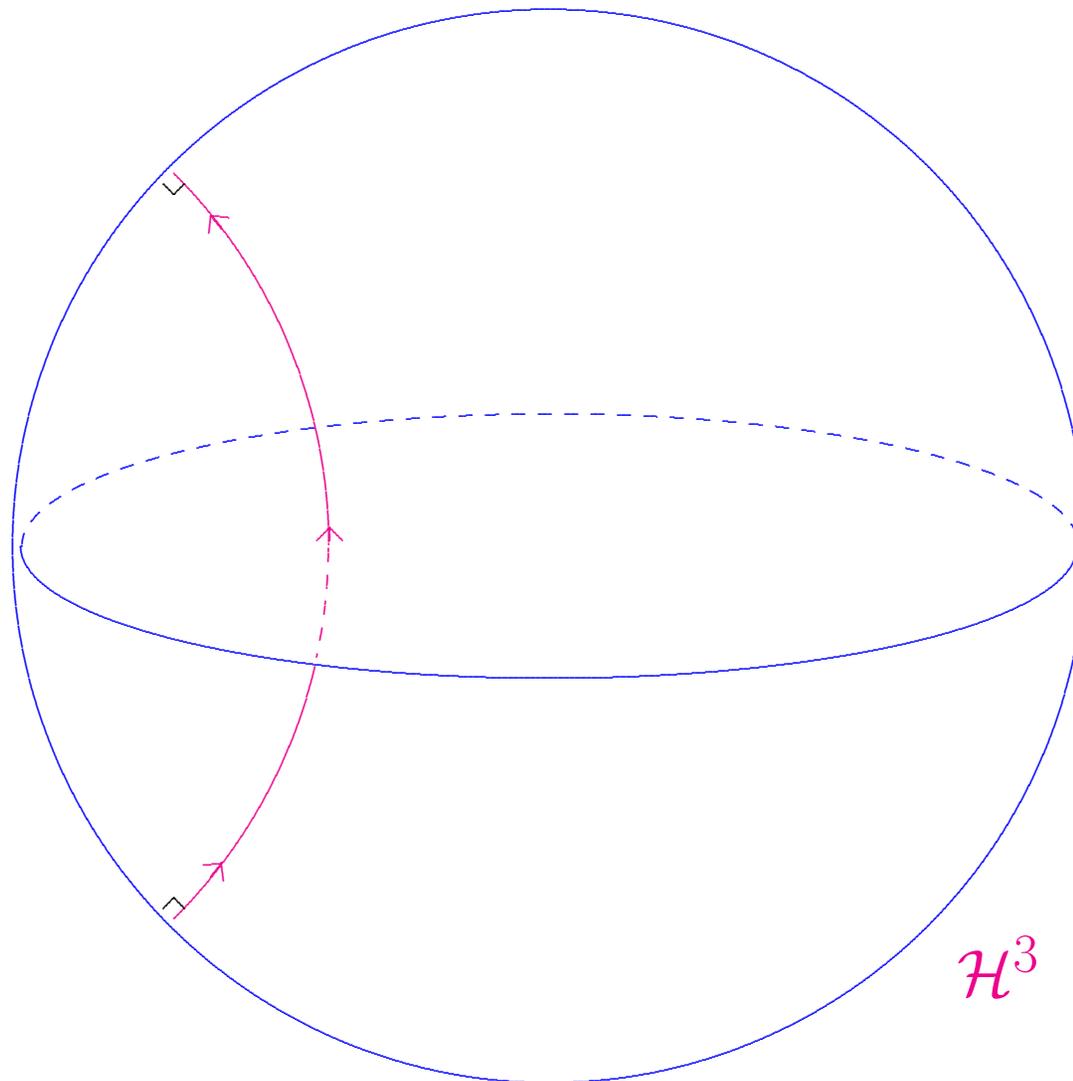
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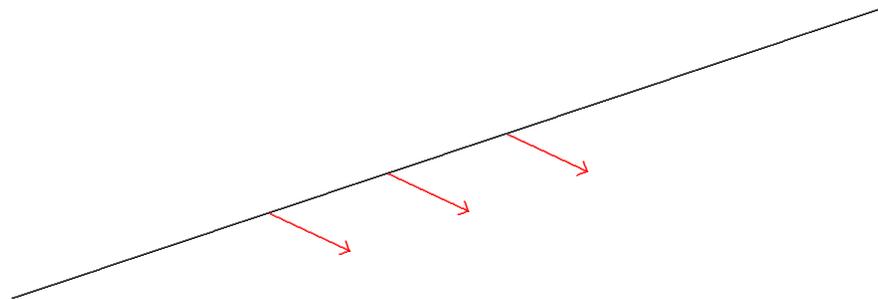
Out-pointing orientation for sphere of final points.

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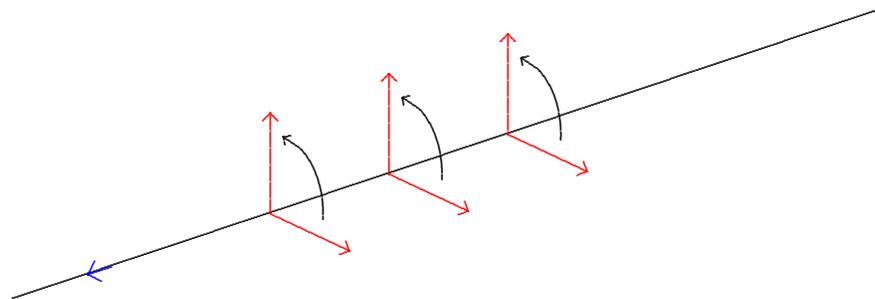


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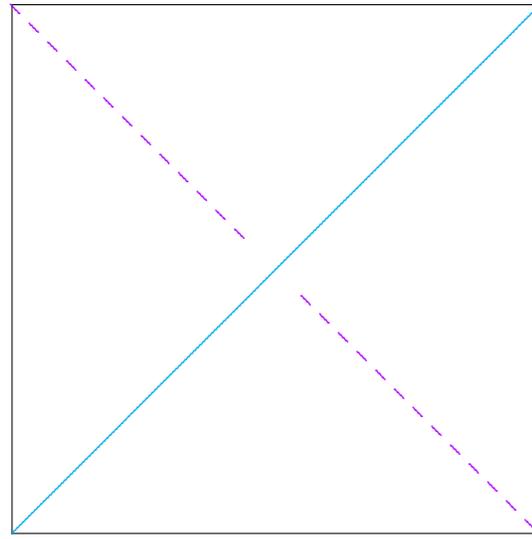
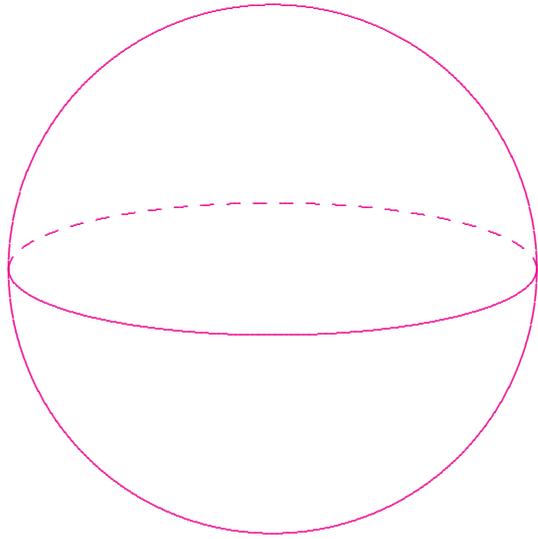
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Points of  $\mathcal{H}^3$   $\longleftrightarrow$  holomorphic curves  $\mathcal{C} \cong \mathbb{C}\mathbb{P}_1$  s.t.

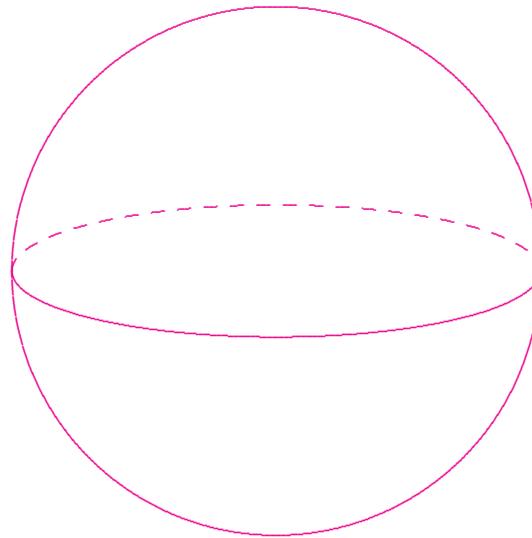
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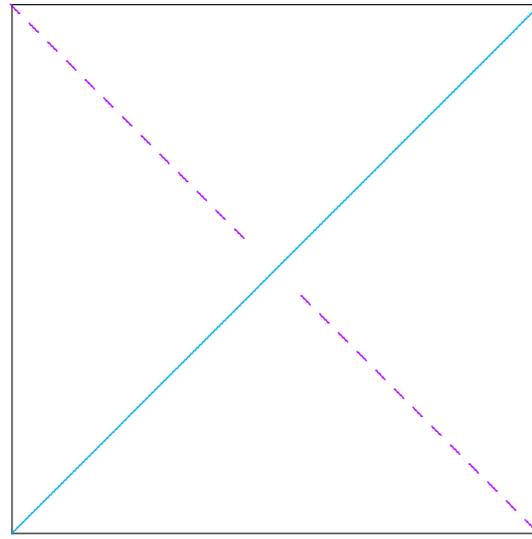
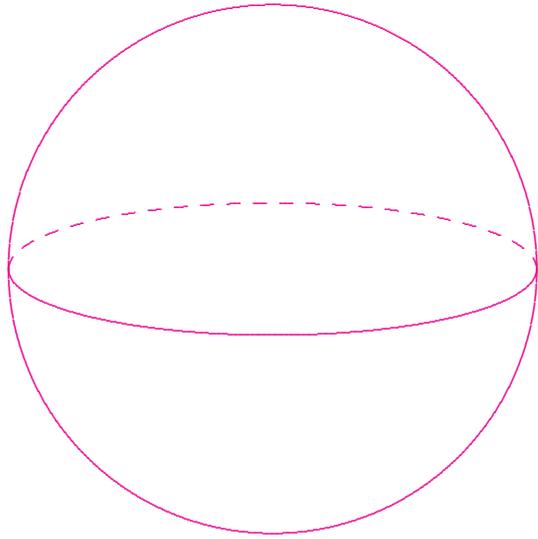
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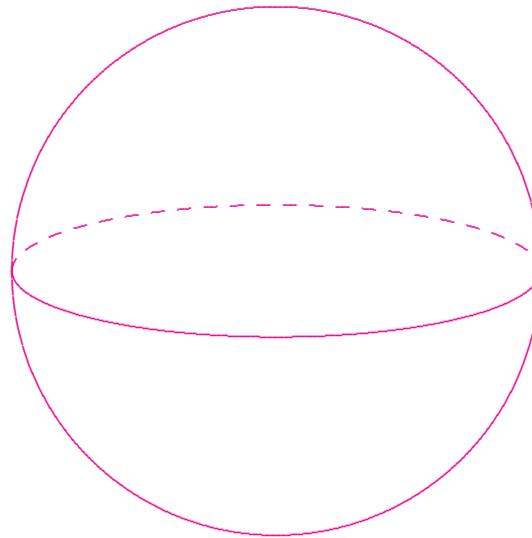
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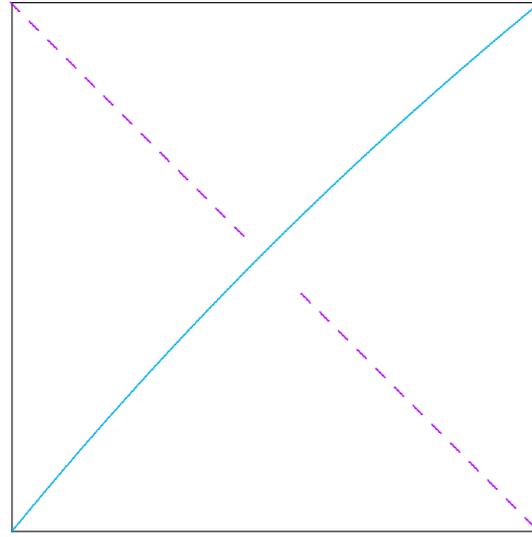
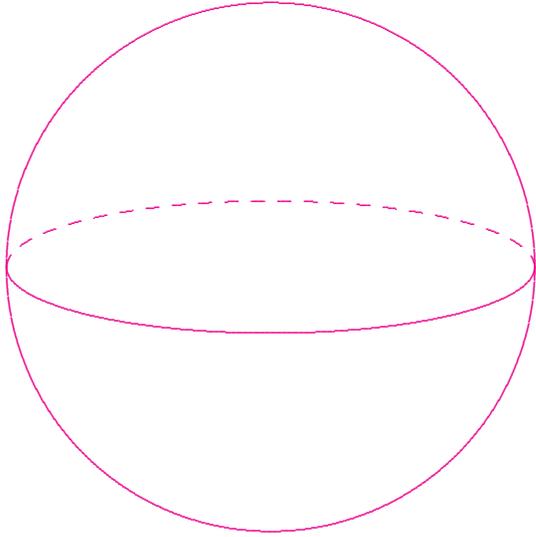
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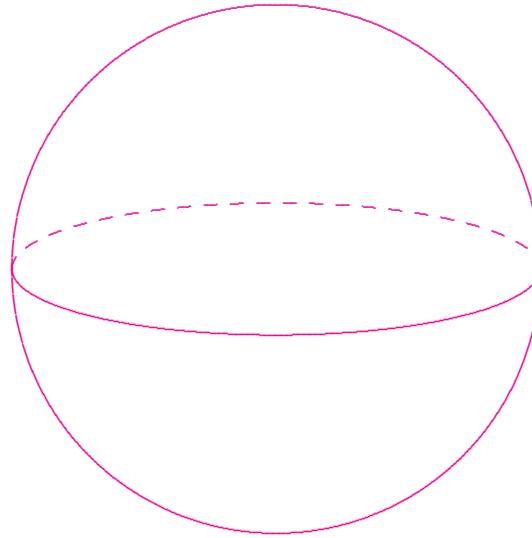


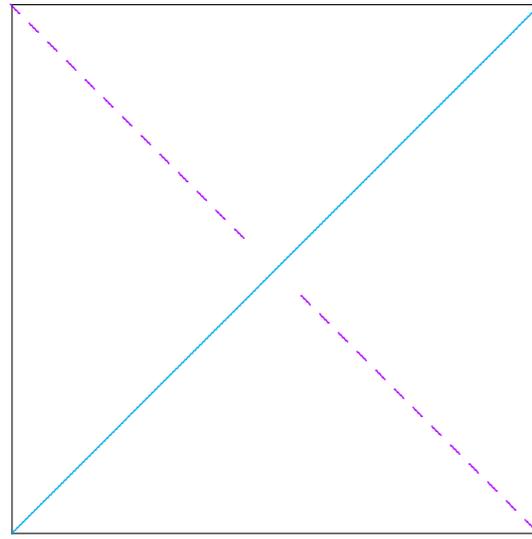
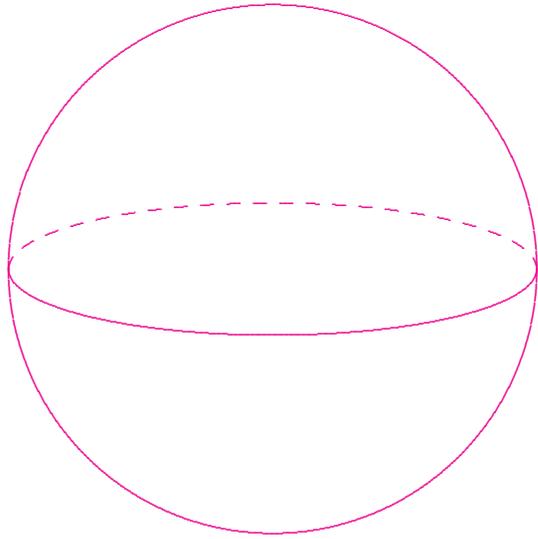
$\mathbb{C}P_1 \times \mathbb{C}P_1$



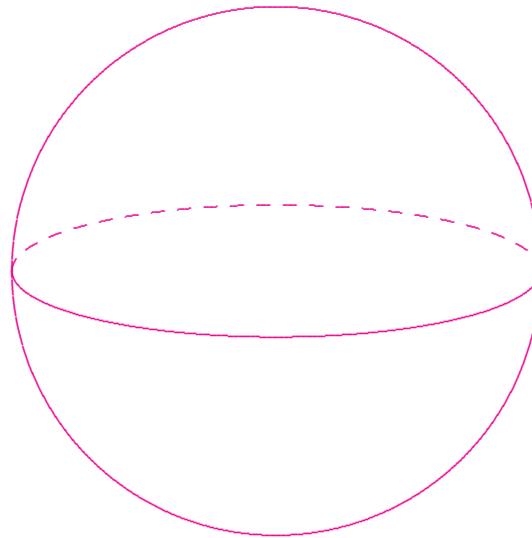


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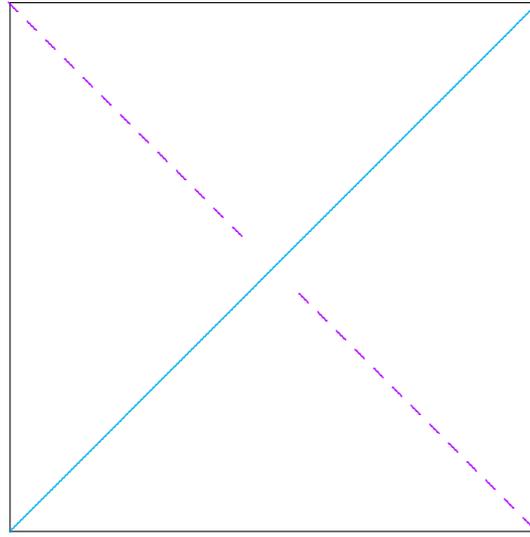
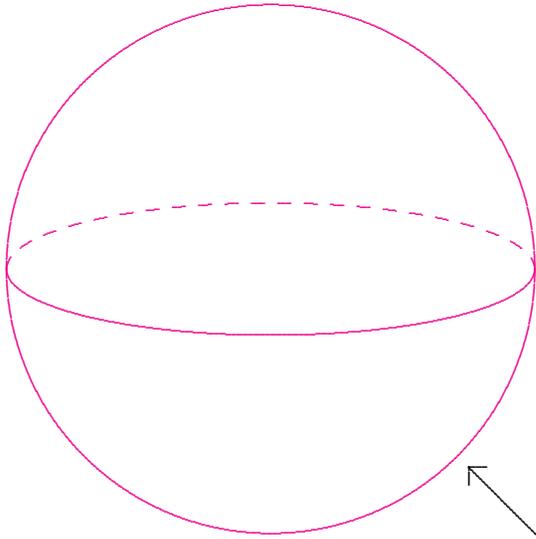
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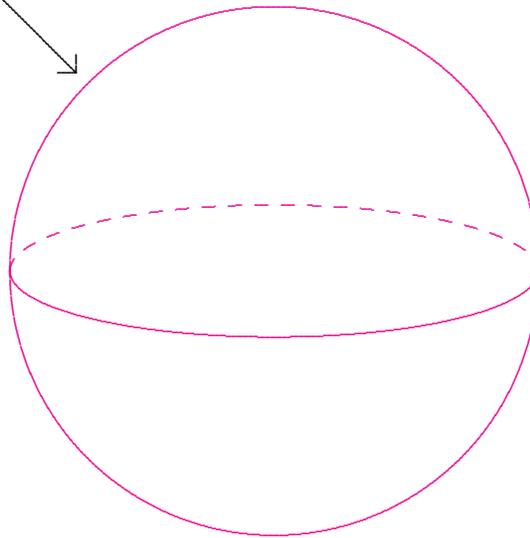
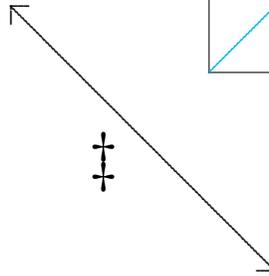
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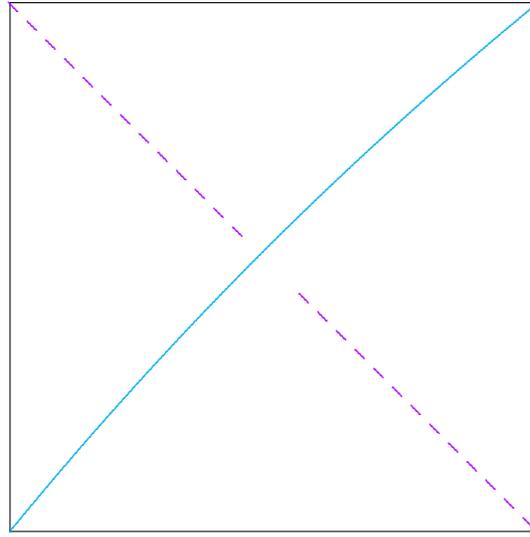
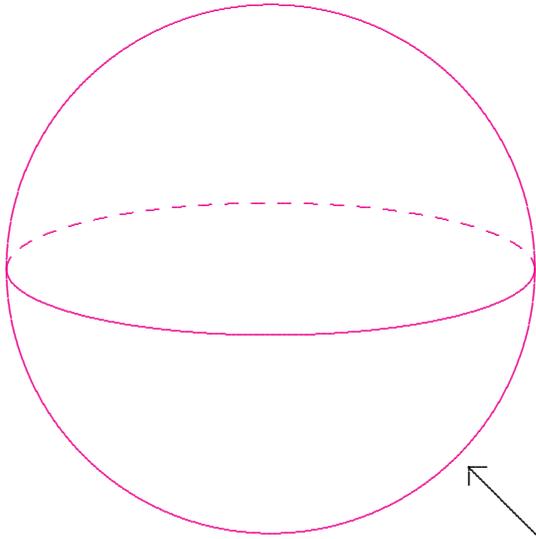
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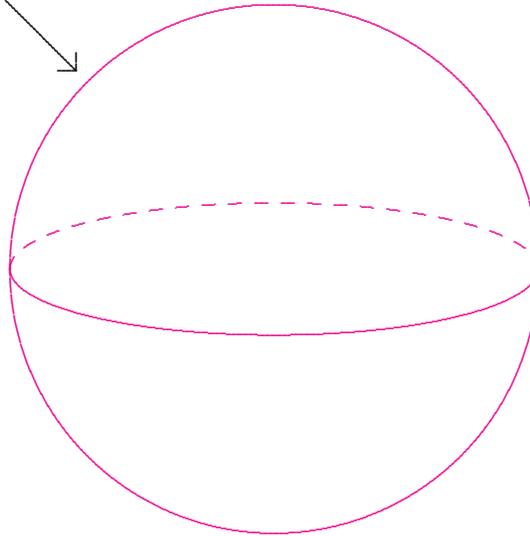
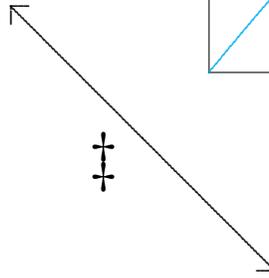


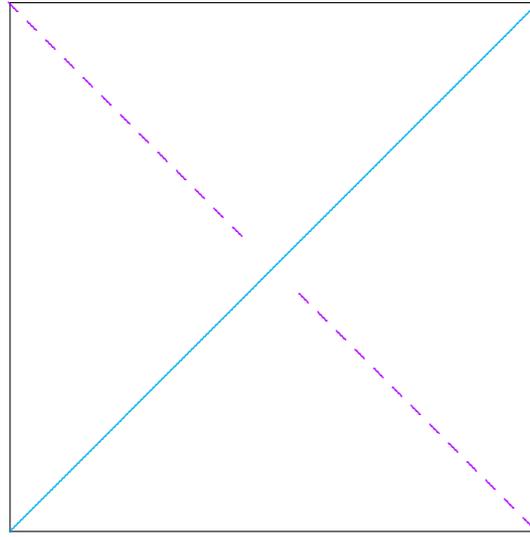
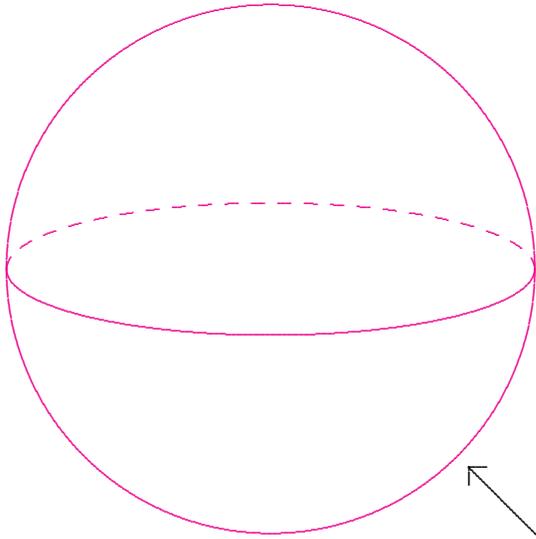
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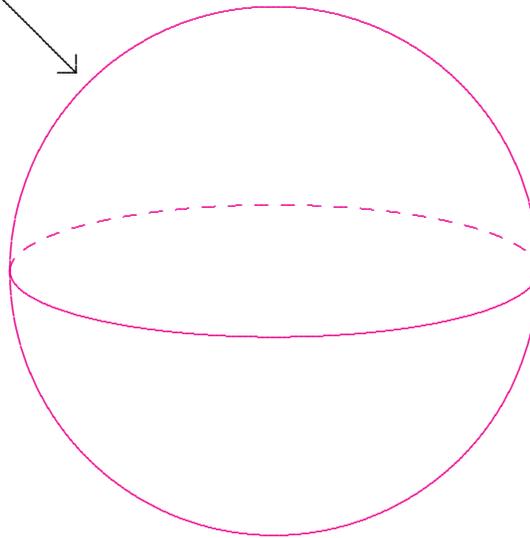
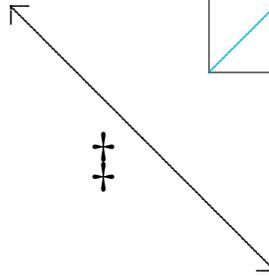


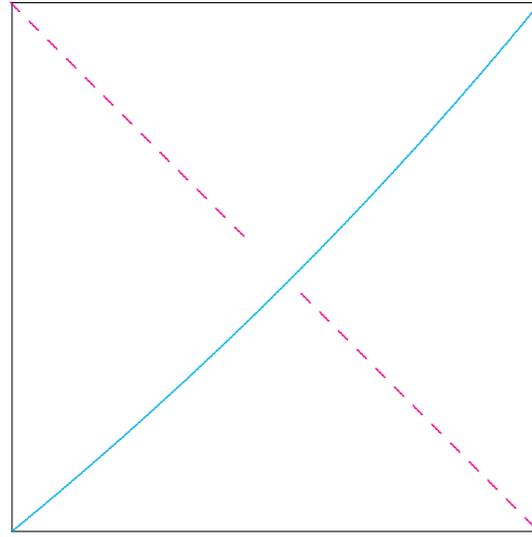
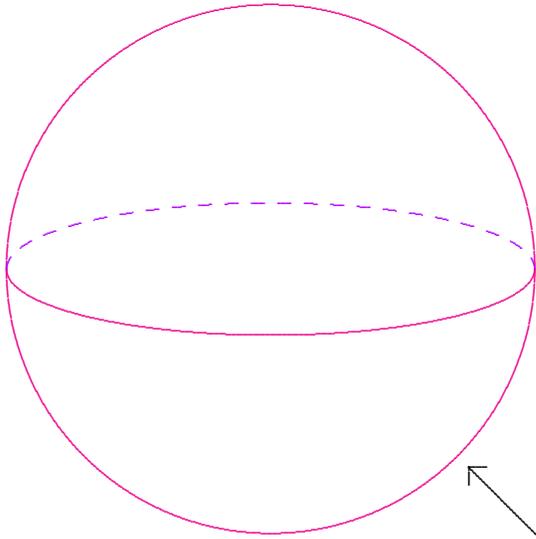
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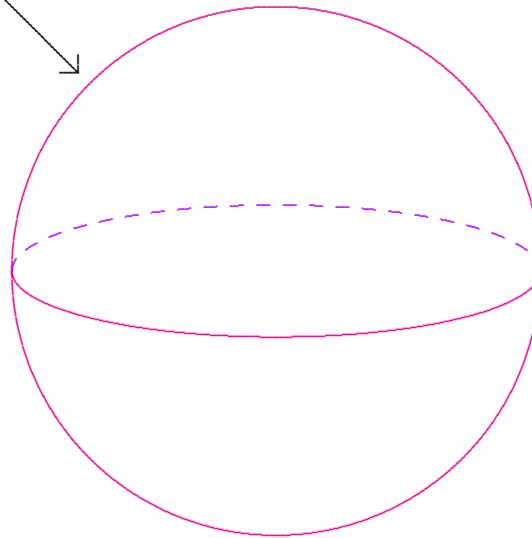
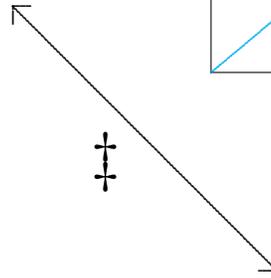


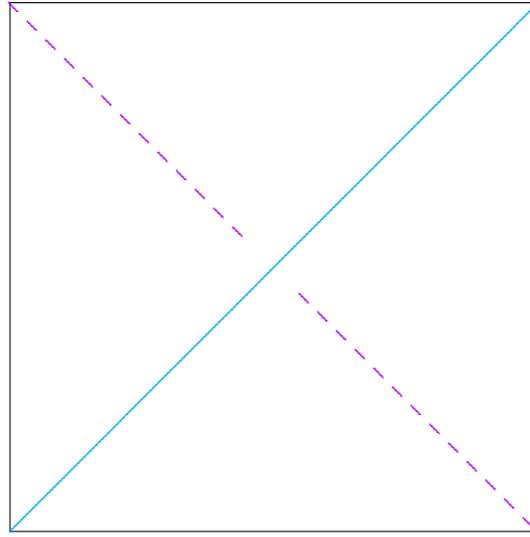
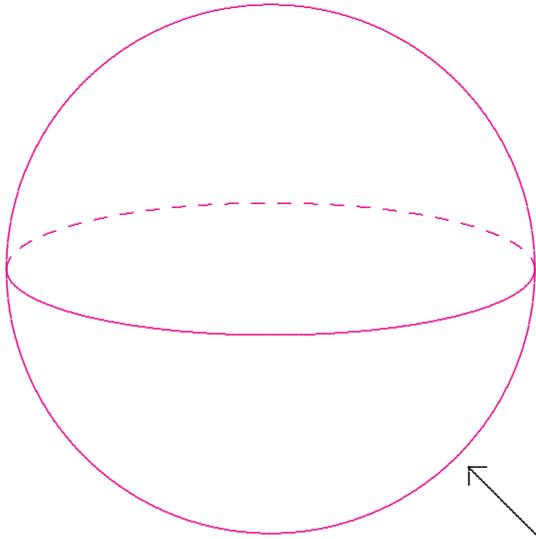
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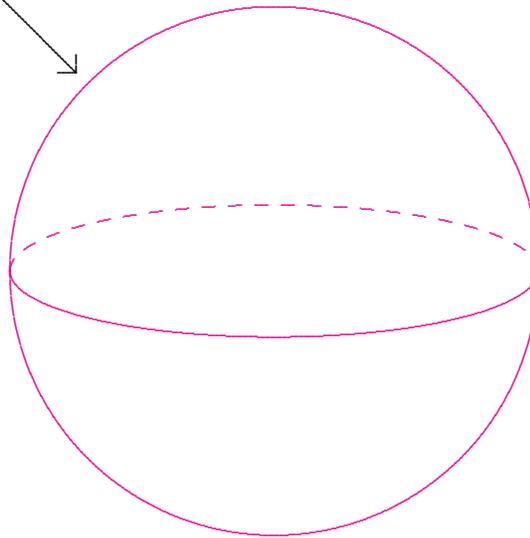
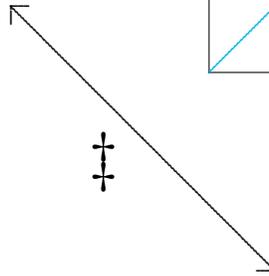


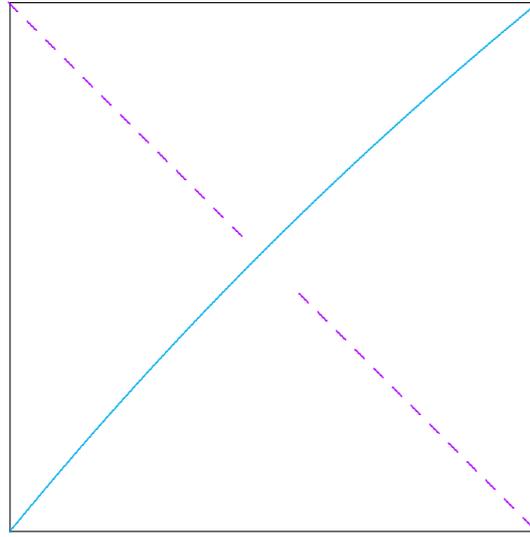
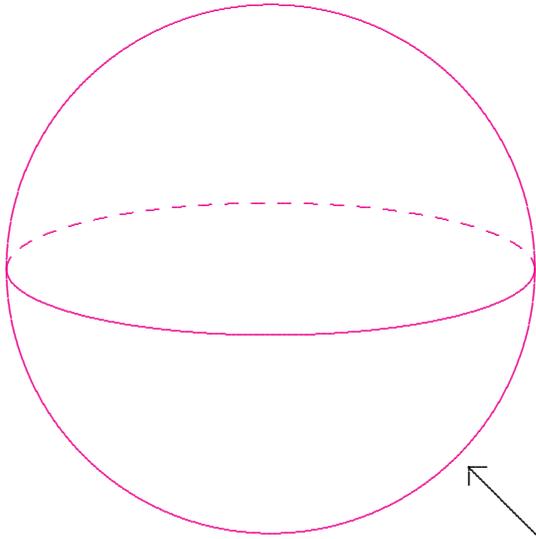
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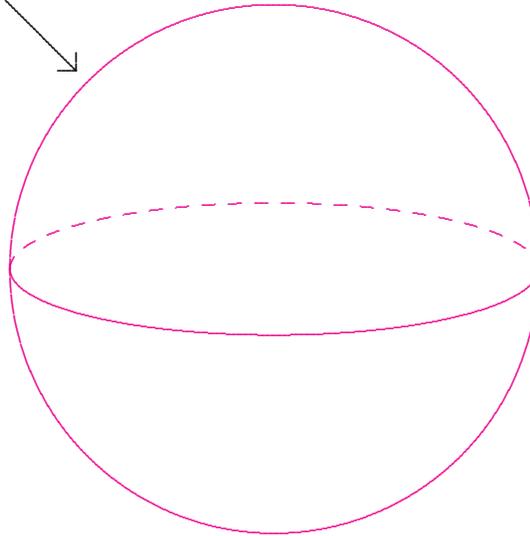
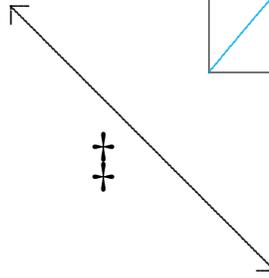


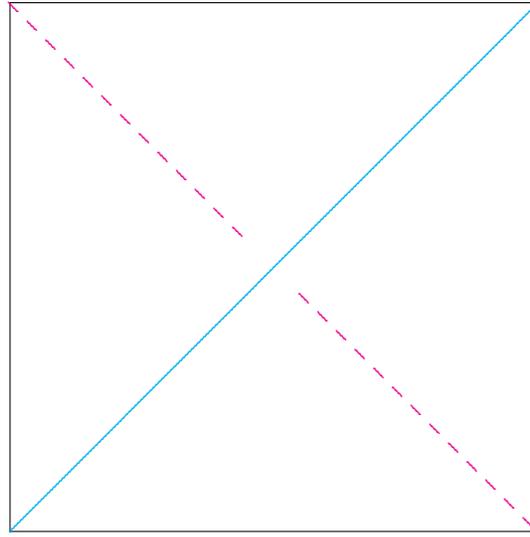
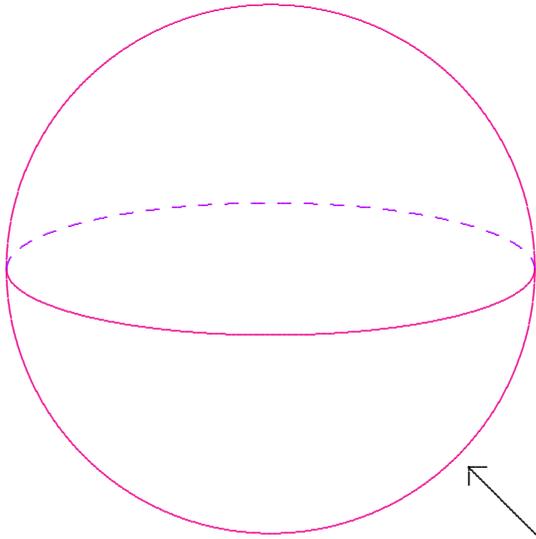
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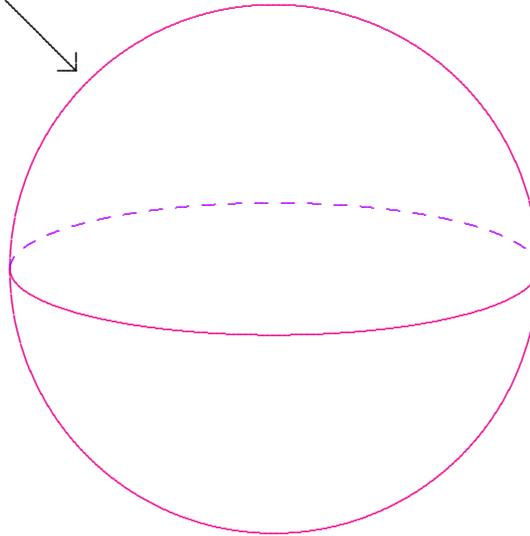
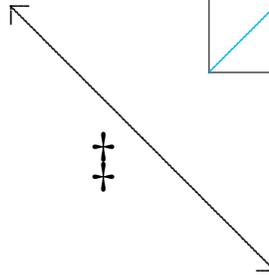


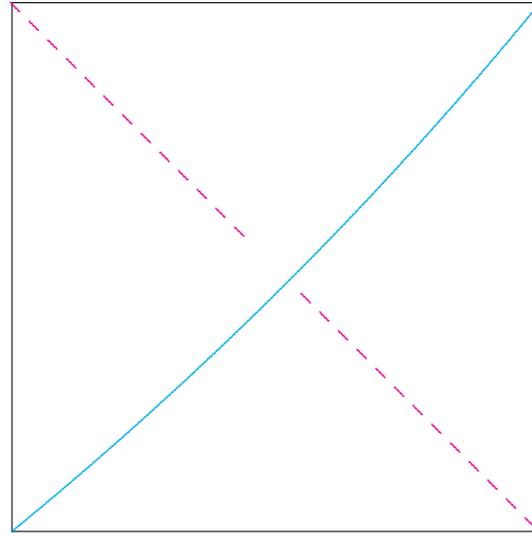
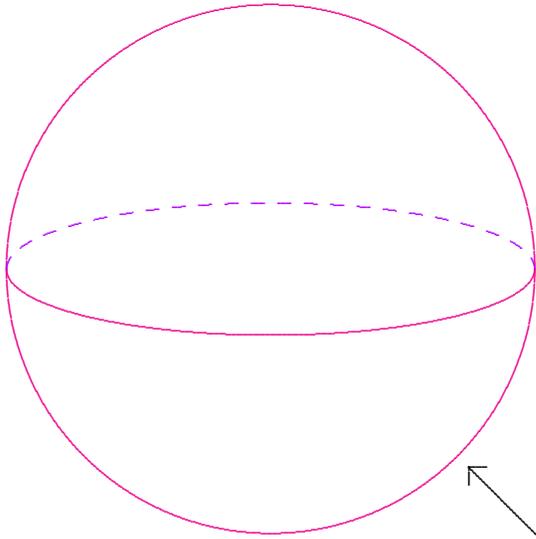
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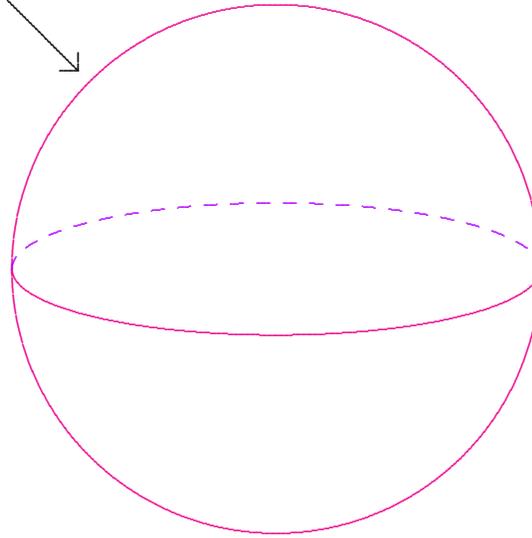
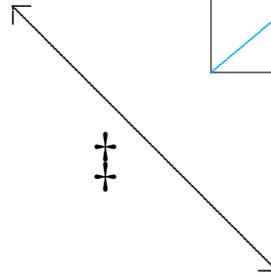


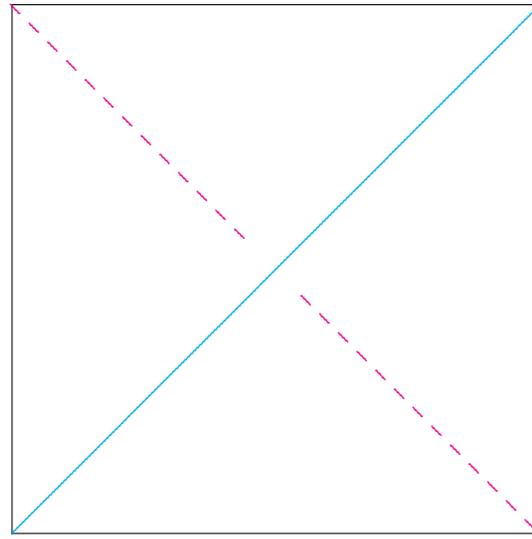
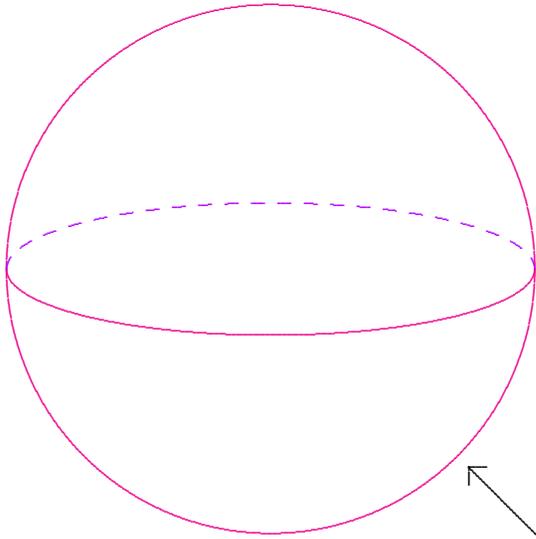
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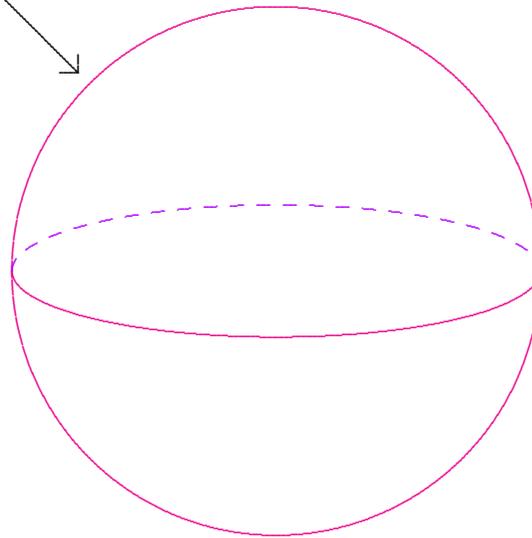
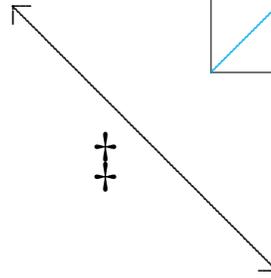


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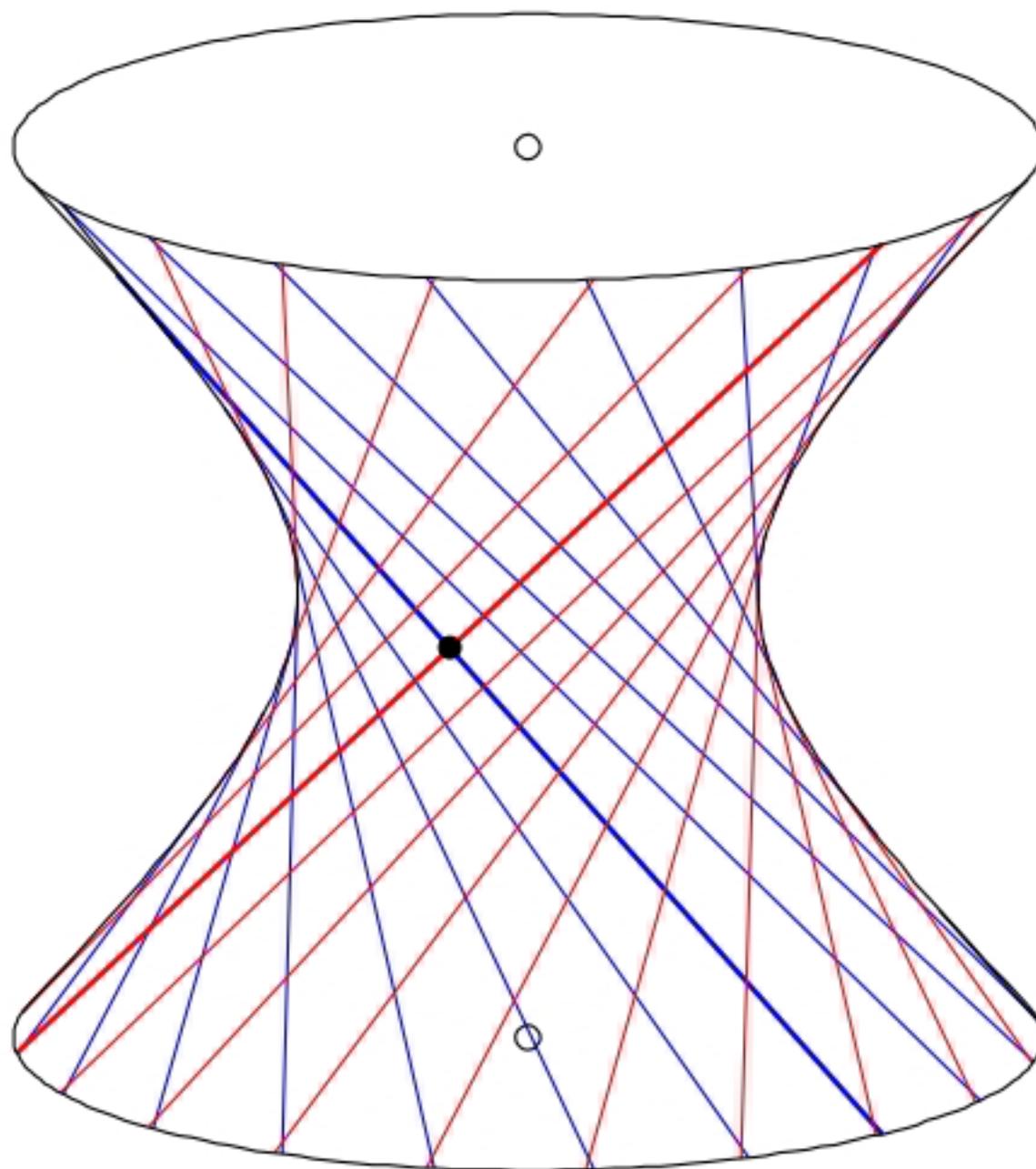
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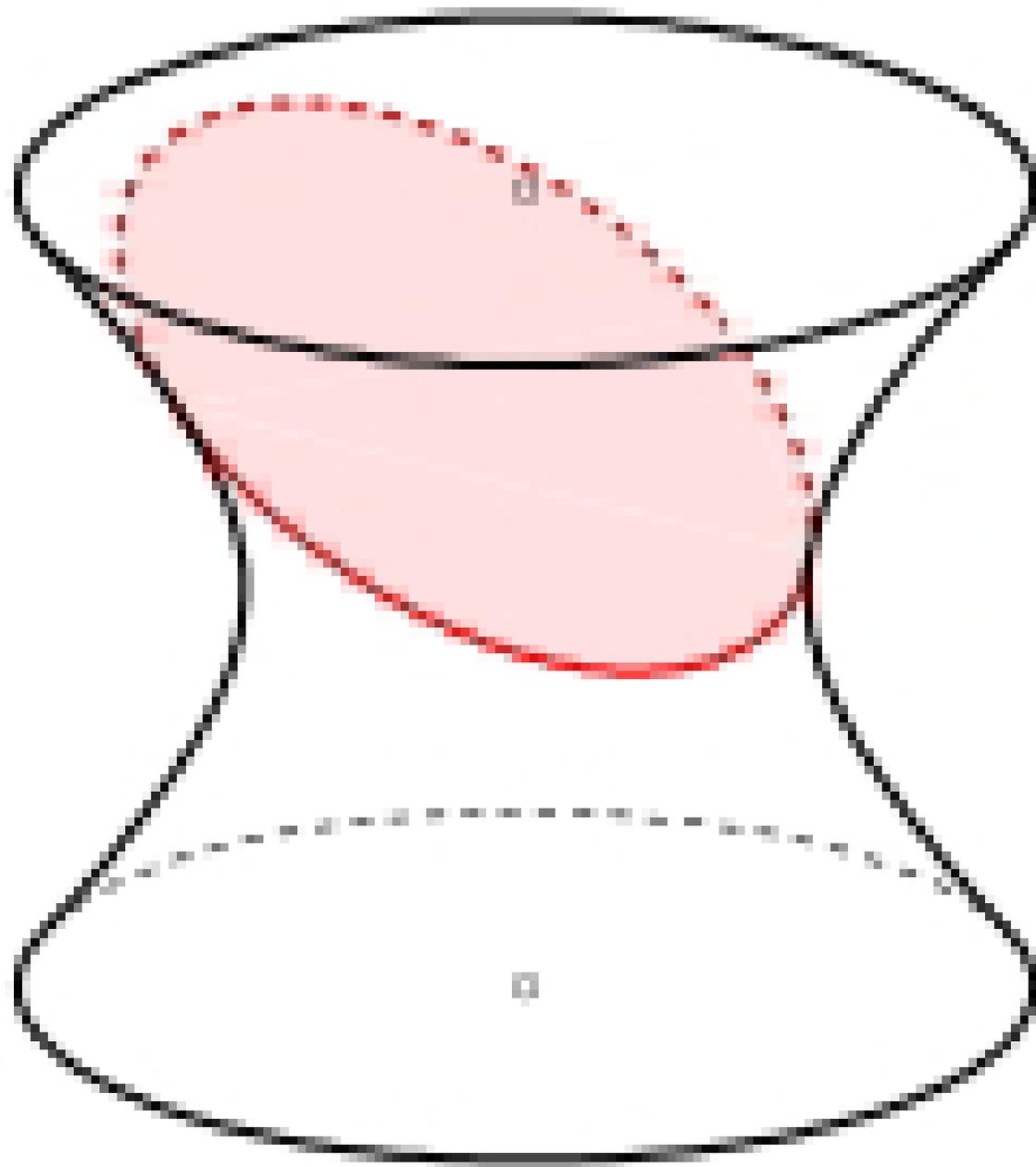


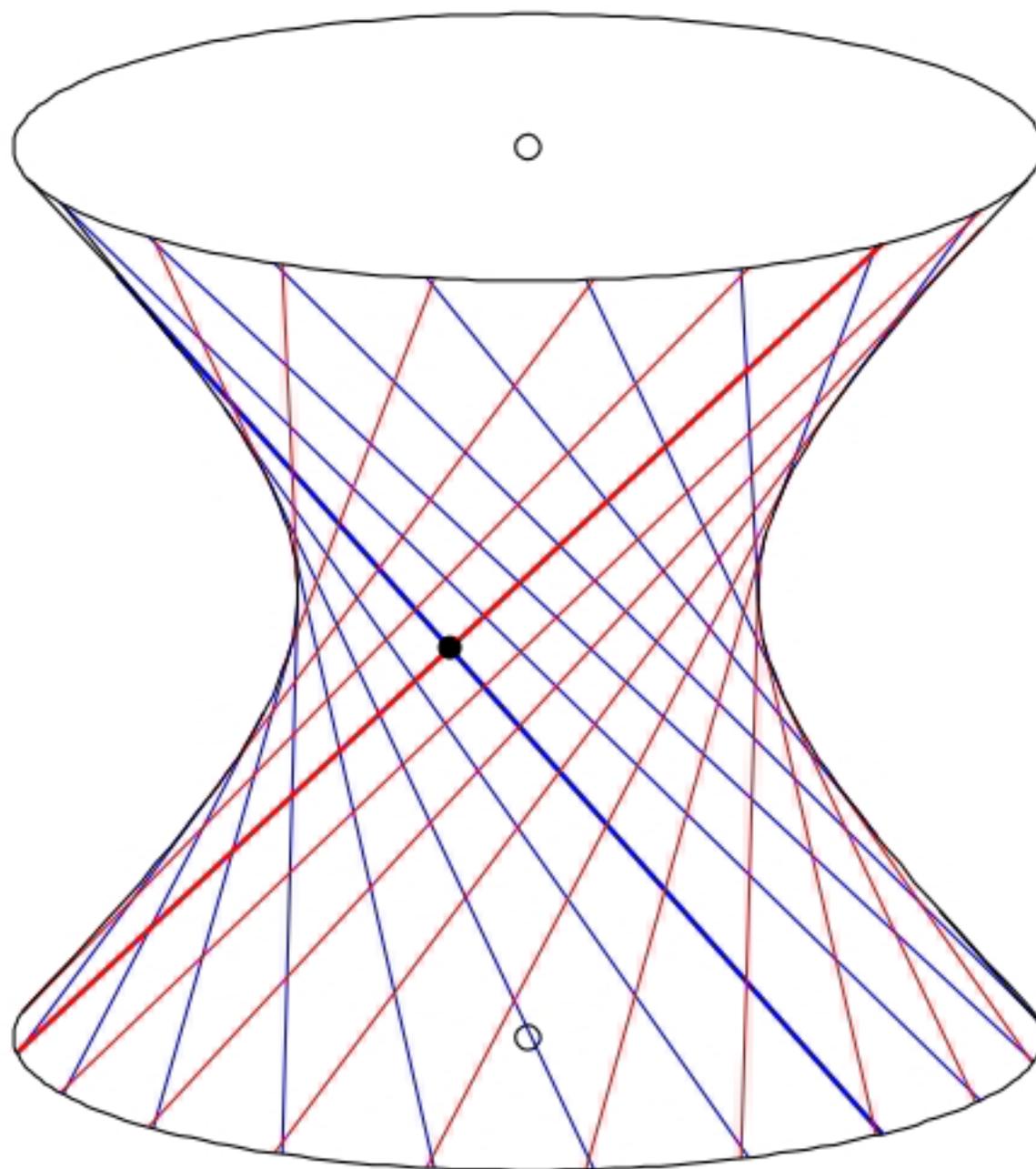
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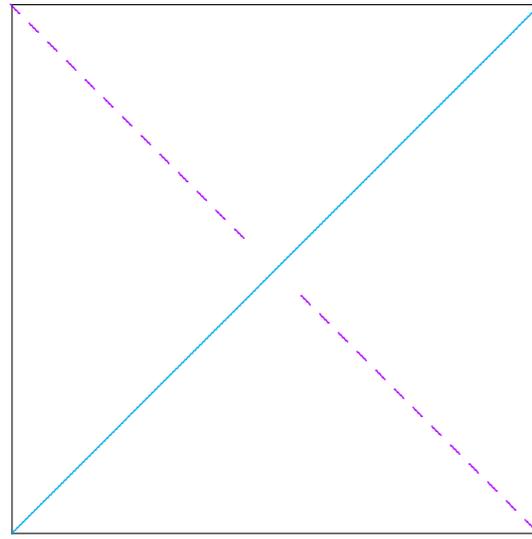
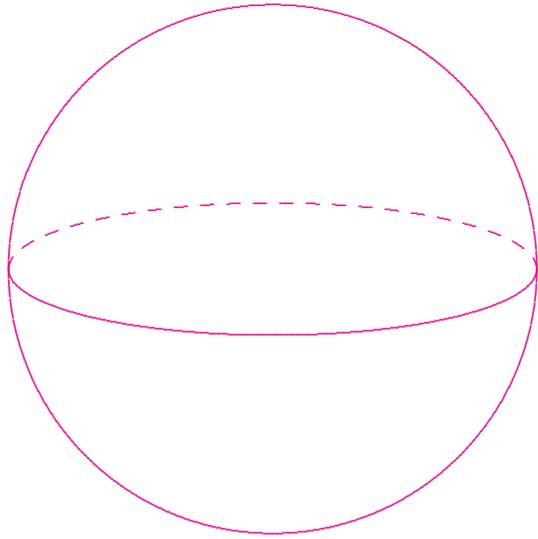
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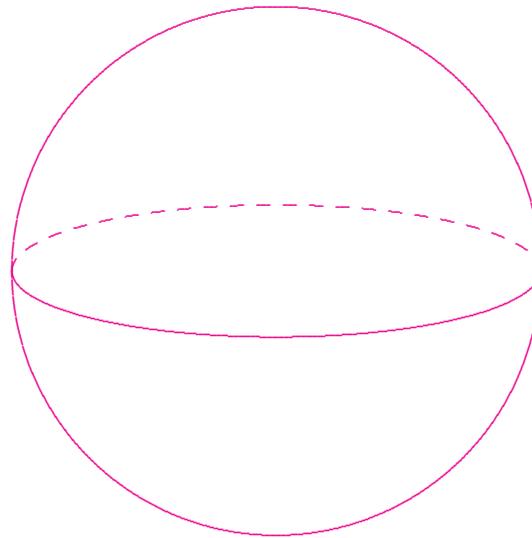








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- $\mathcal{C} \xrightarrow{\mathcal{O}} (\mathbb{CP}_1 \times \mathbb{CP}_1) - S$
- $[\mathcal{C}] = [\text{Diagonal}] \in H_2((\mathbb{CP}_1 \times \mathbb{CP}_1) - S)$
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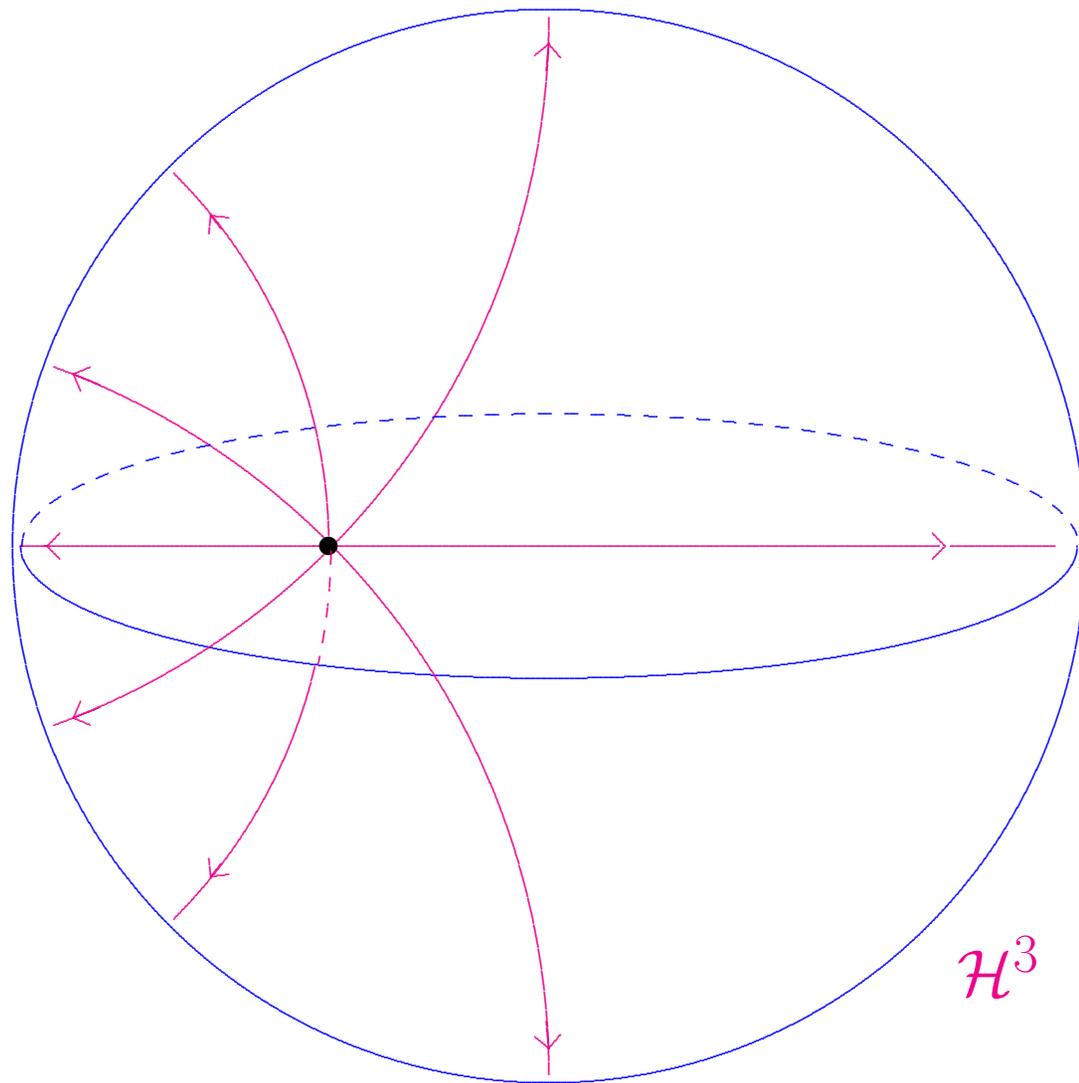
- $\mathcal{C} \xrightarrow{\mathcal{O}} \mathbb{C}\mathbb{P}_1 \times \mathbb{C}\mathbb{P}_1 - (\text{anti-diagonal})$
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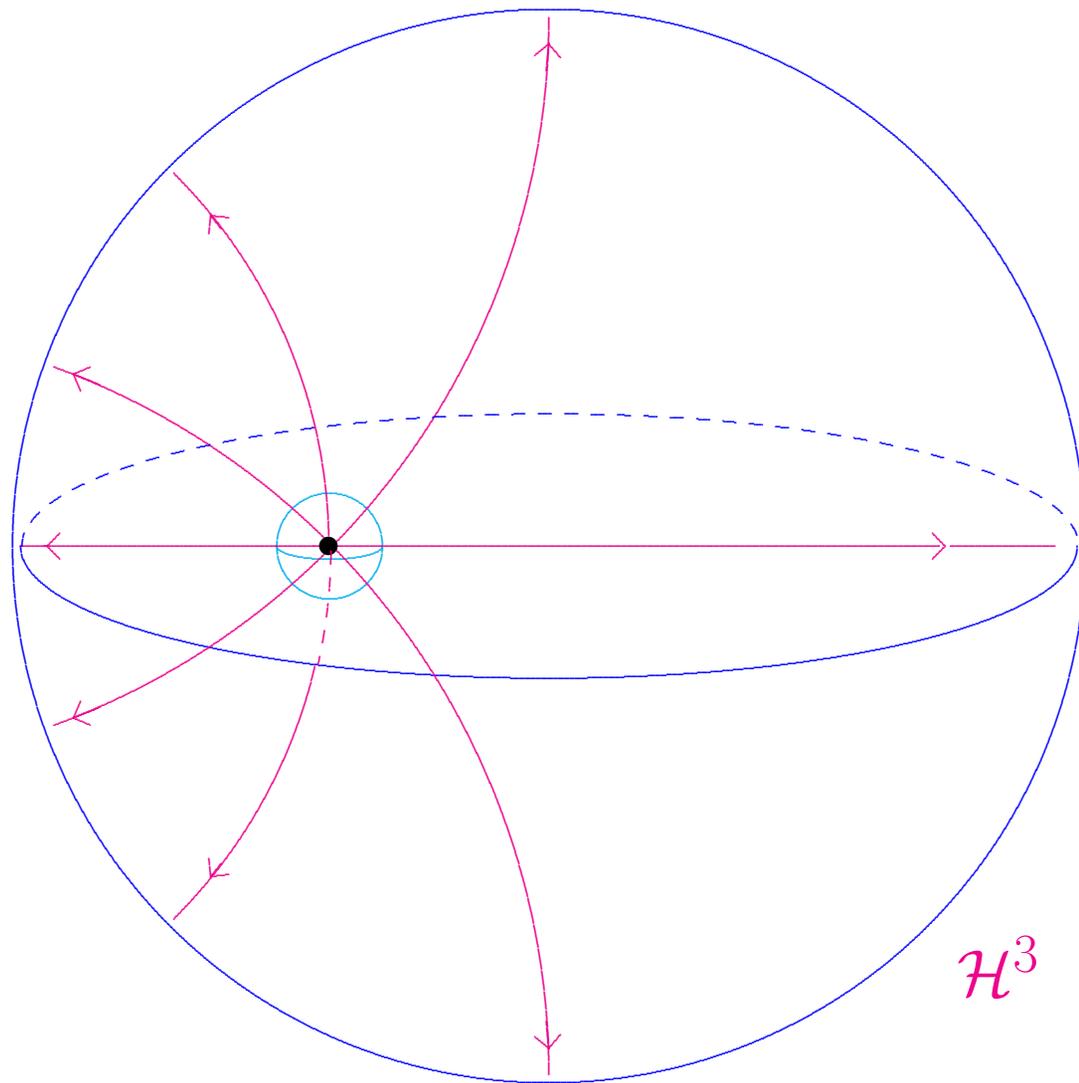
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Hitchin '82, Jones-Tod '85

“Hyperbolic Ansatz”

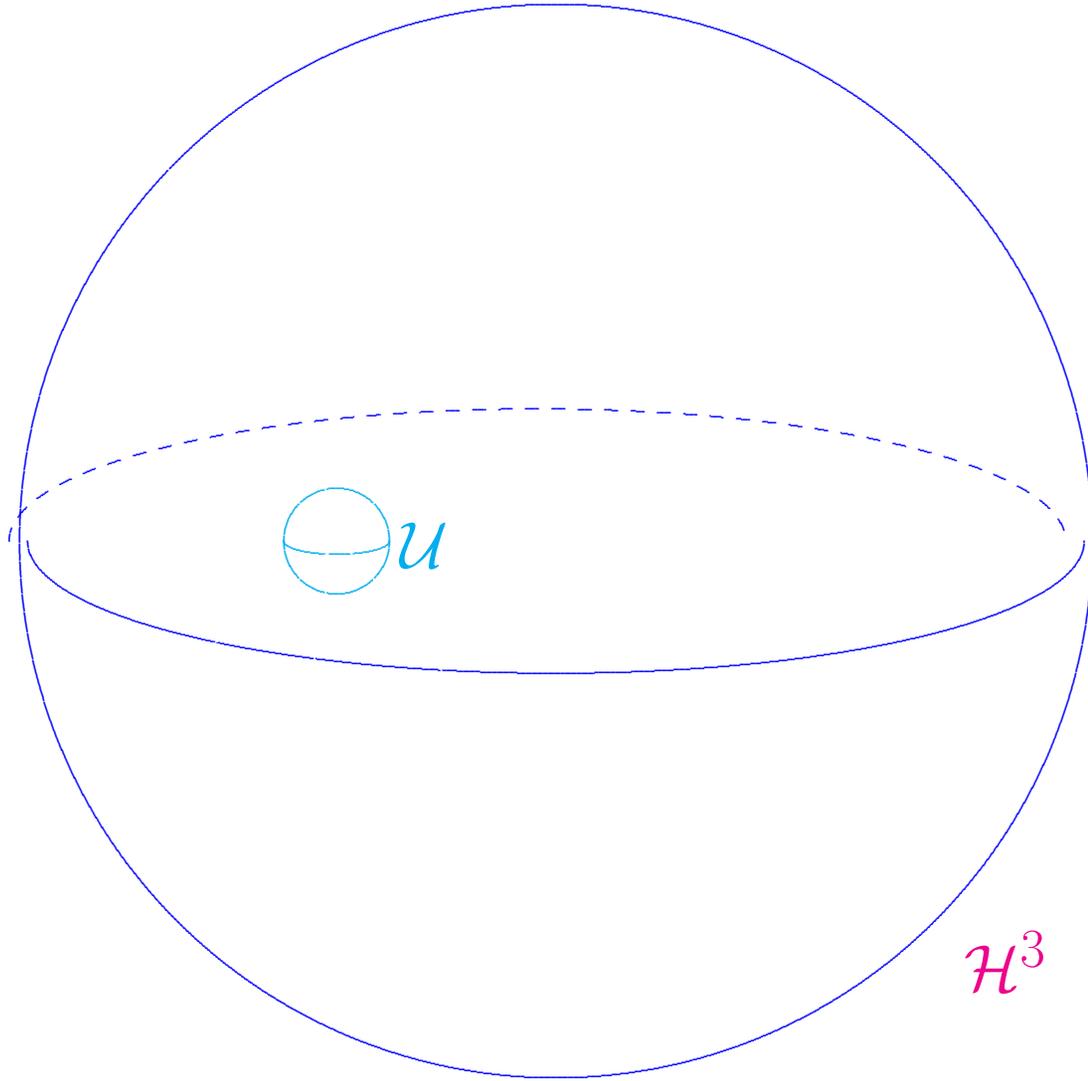
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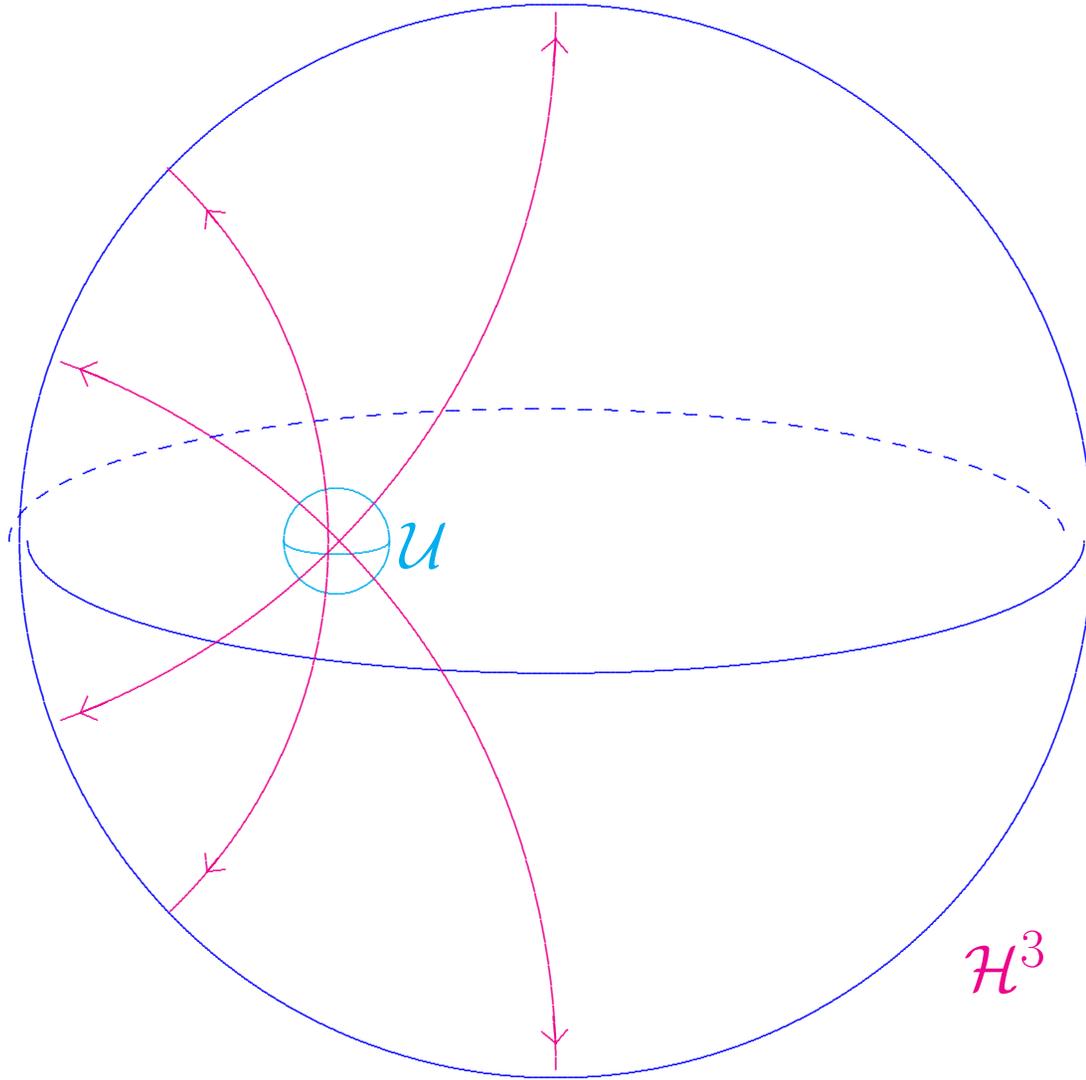


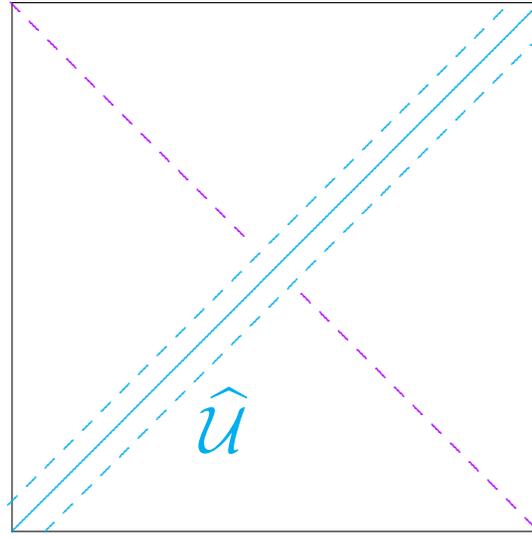
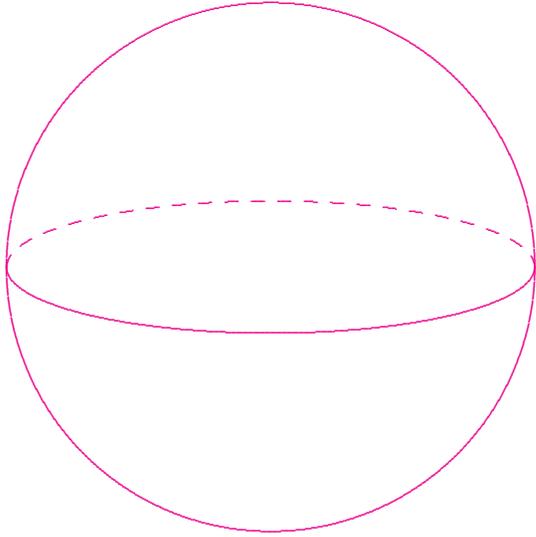
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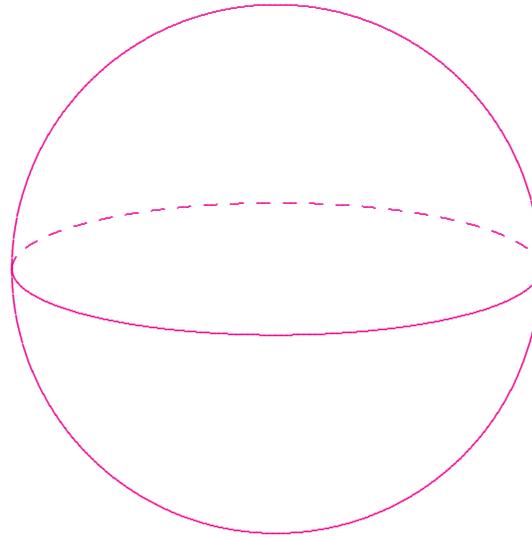
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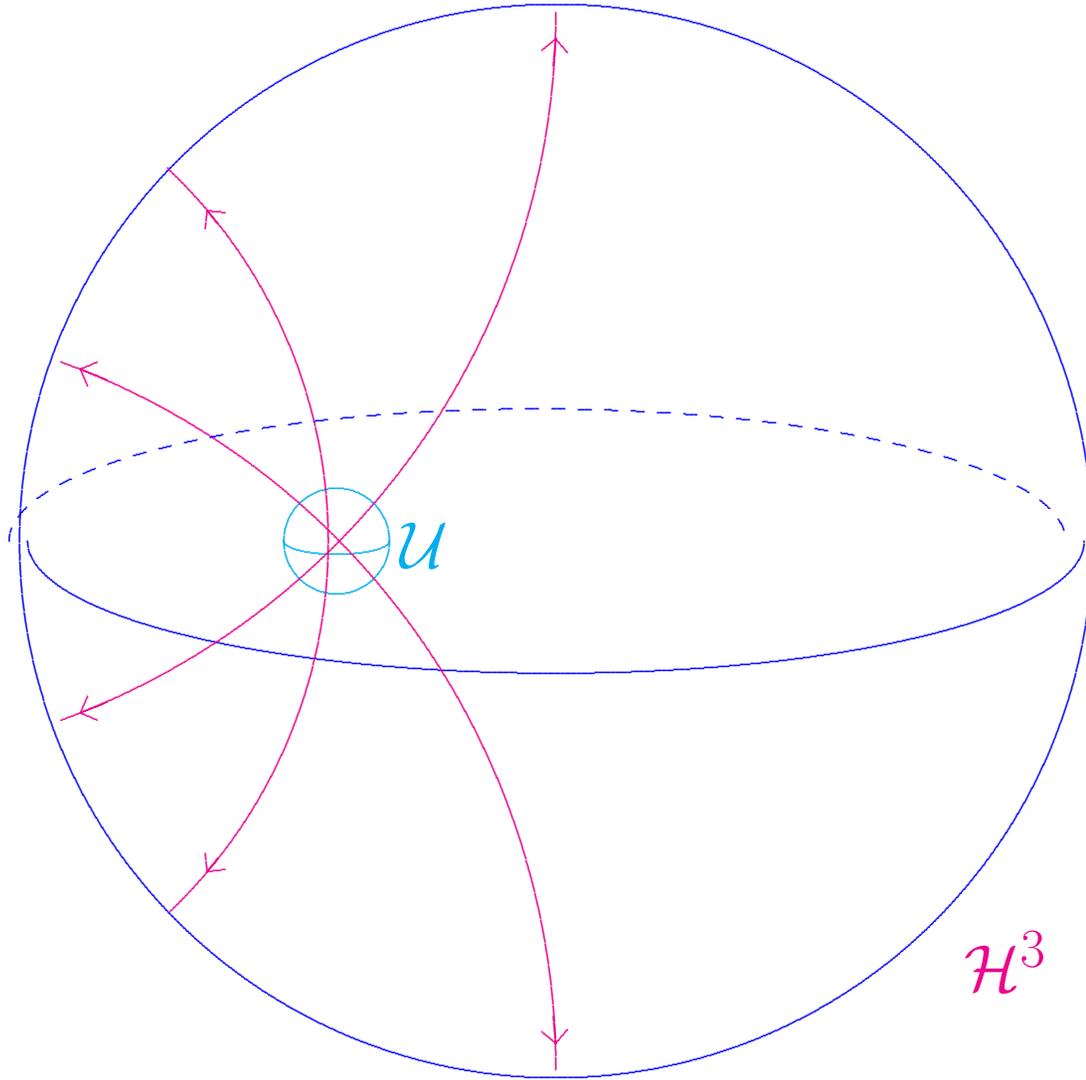
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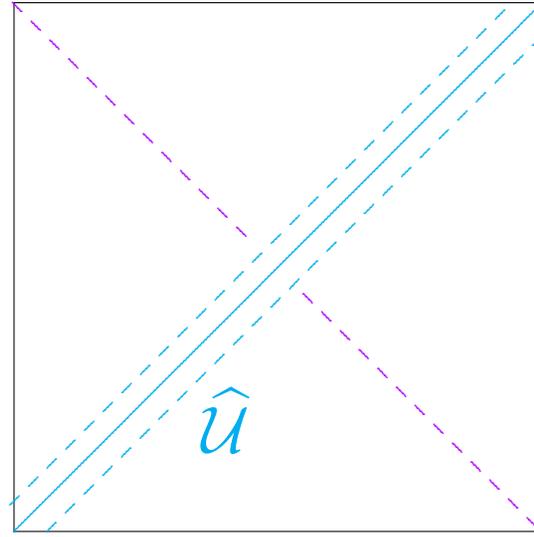
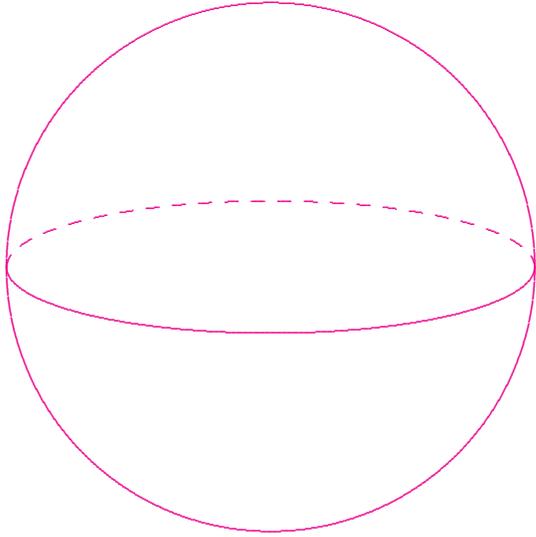




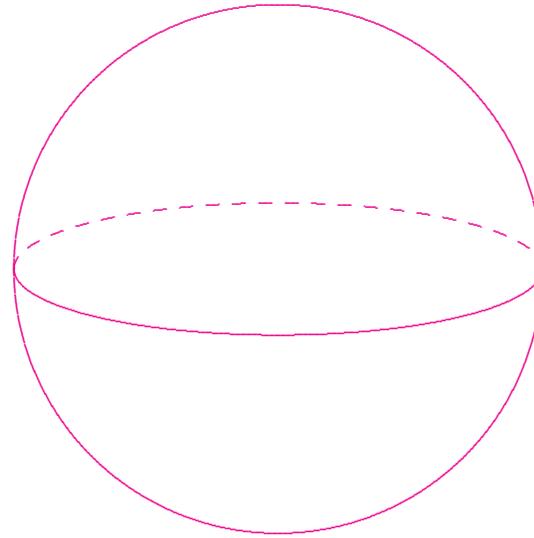
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cf. Ward’s twisted photon construction

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Each  $\mathcal{C}$  has  $\mathbb{C}^\times$ -family of lifts  $\tilde{\mathcal{C}} \subset \mathcal{L}^\times$ .

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and represents an anti-self-dual 4-manifold.

**Prototypical Example:**

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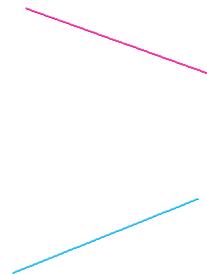
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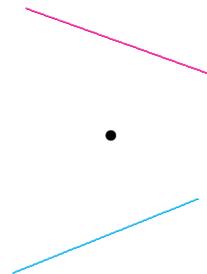
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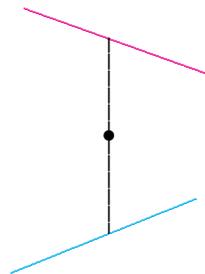
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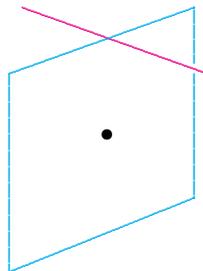
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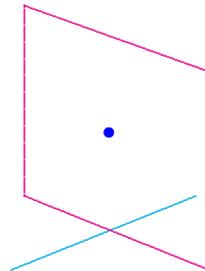
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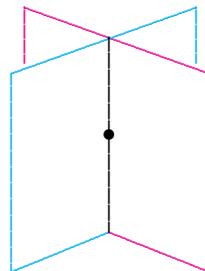
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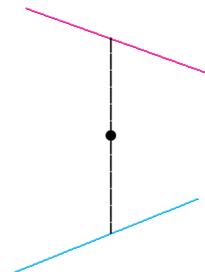
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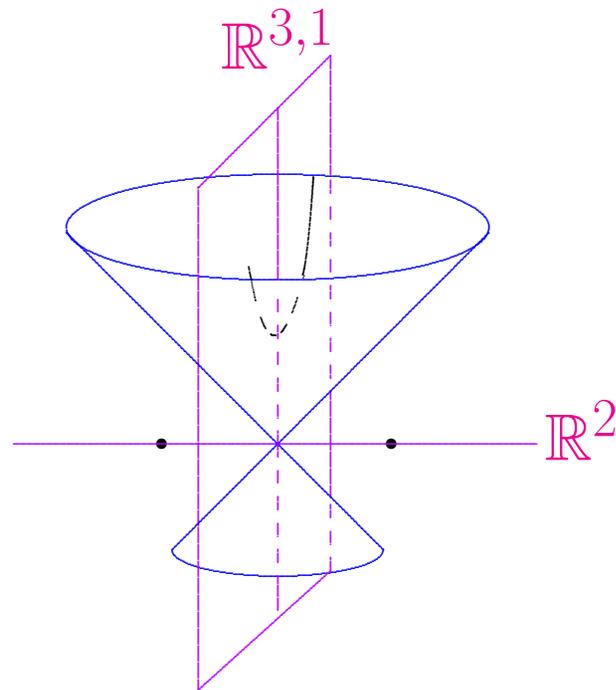
$\therefore \mathcal{H}^3 \times S^1 \subset \mathbb{R}^{5,1}$  embedded in null cone

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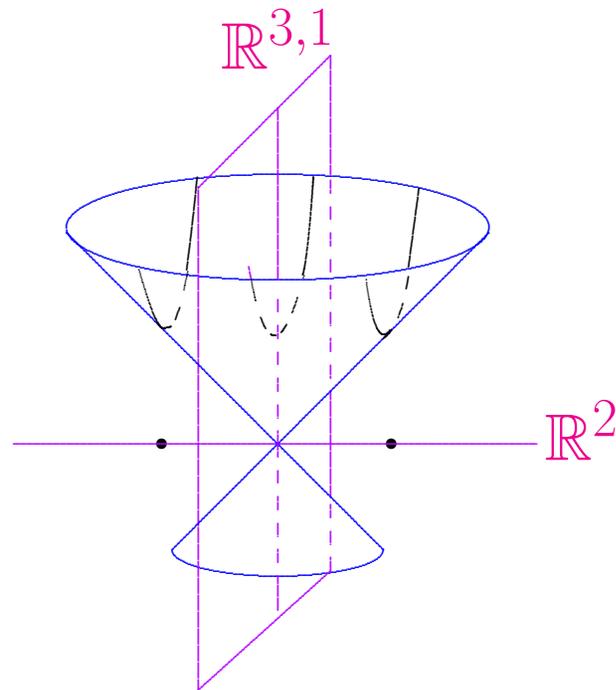


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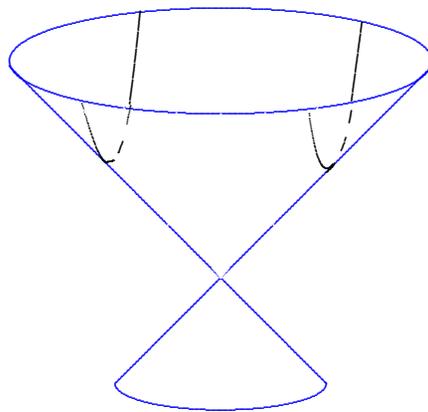


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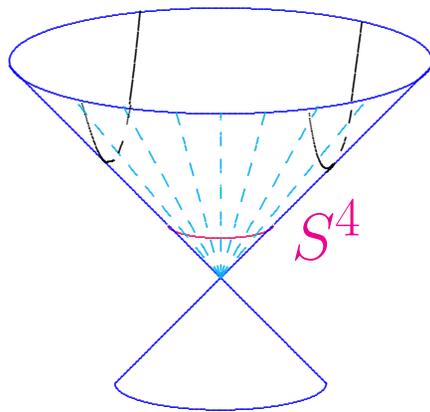


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Analogous to hyper-Kähler metrics, but different complex structures require different rescalings.

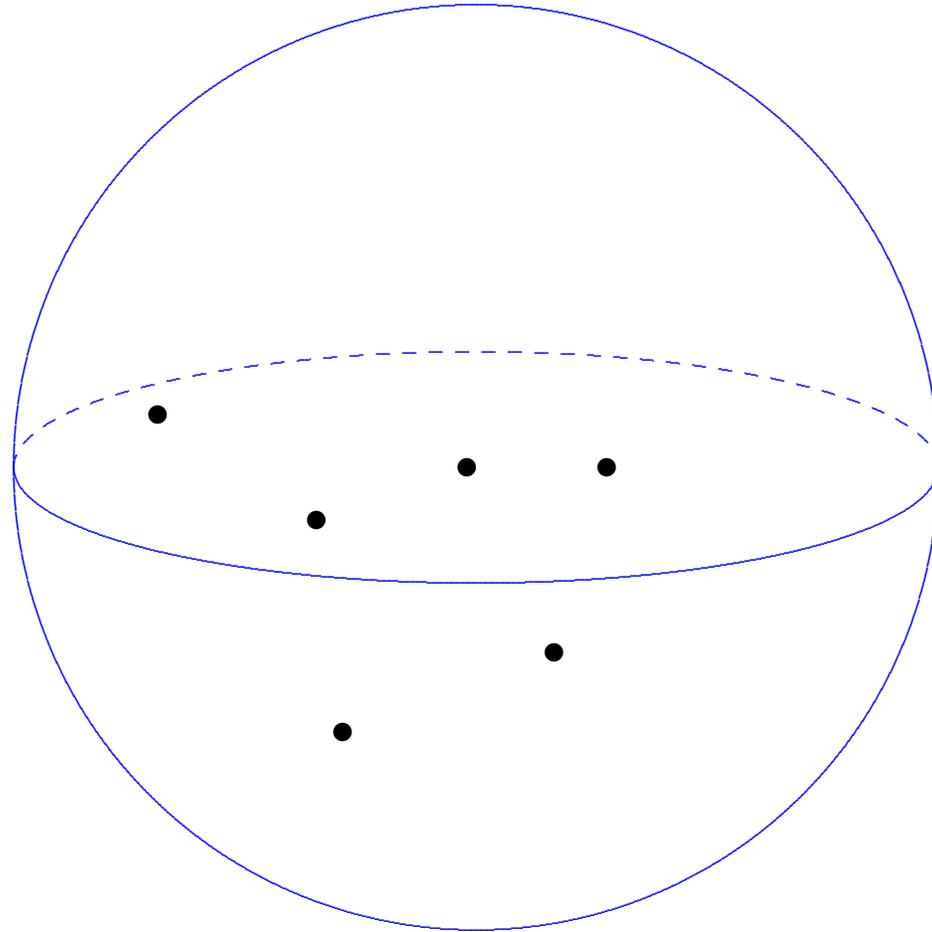
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(L '91)

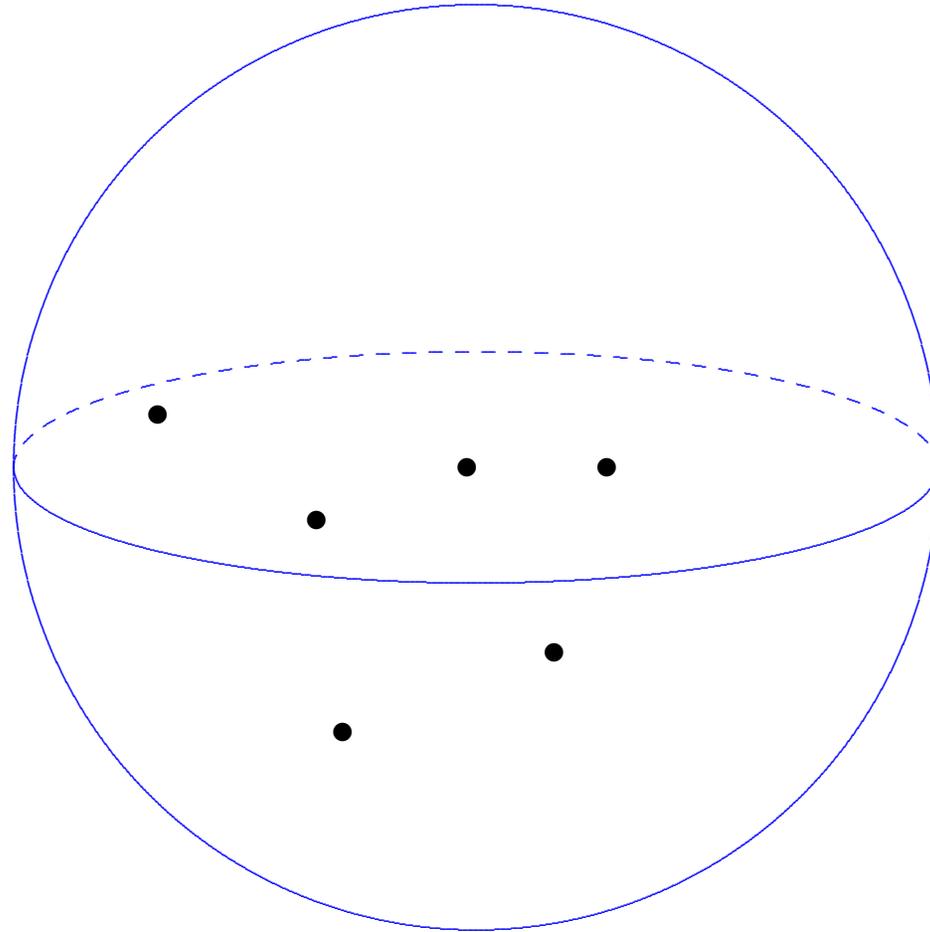
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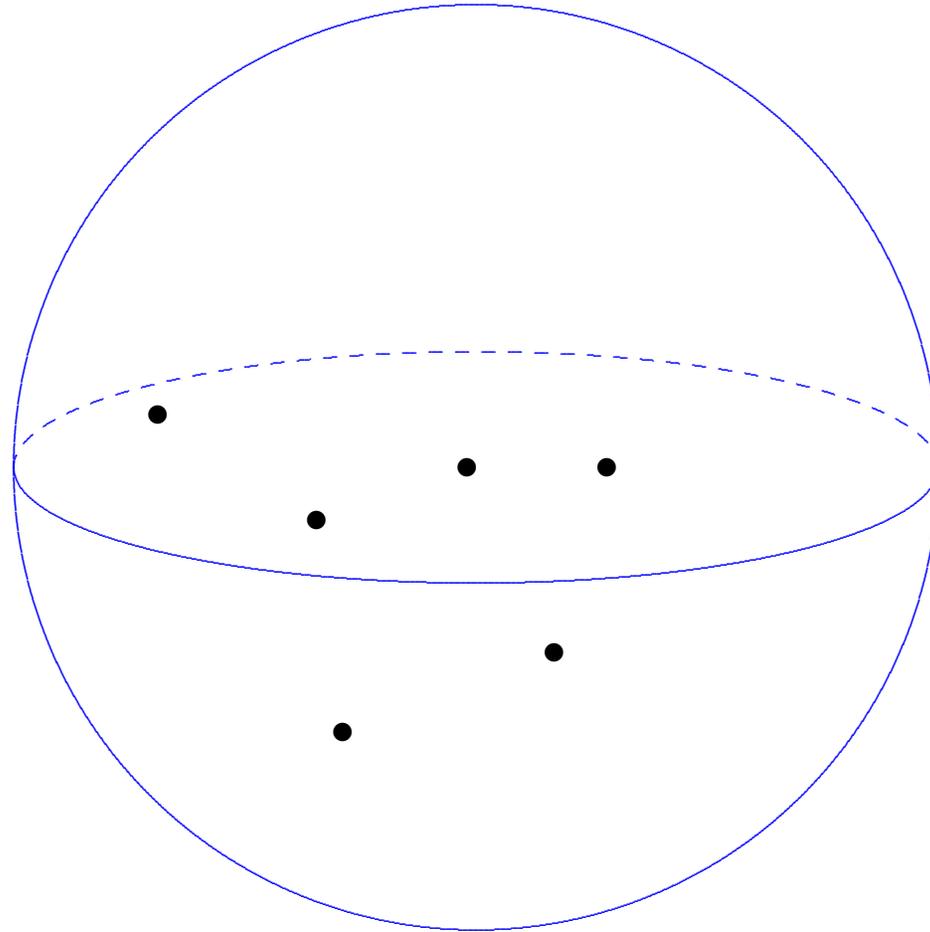
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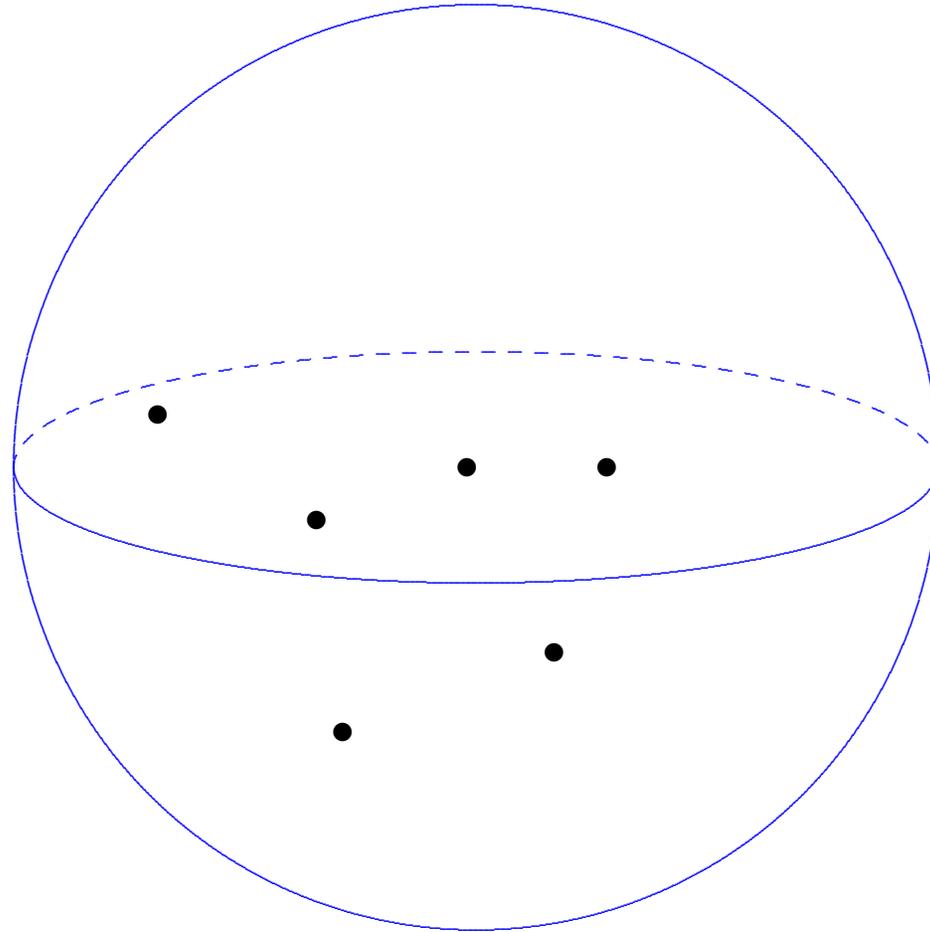
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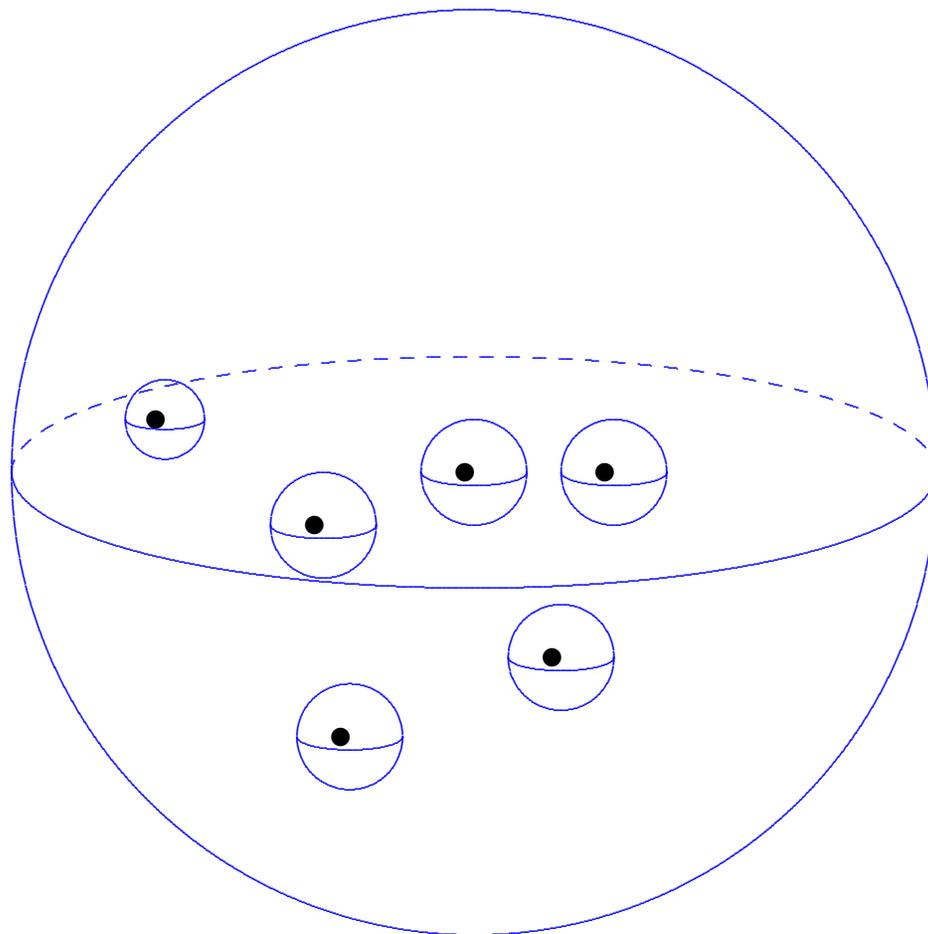
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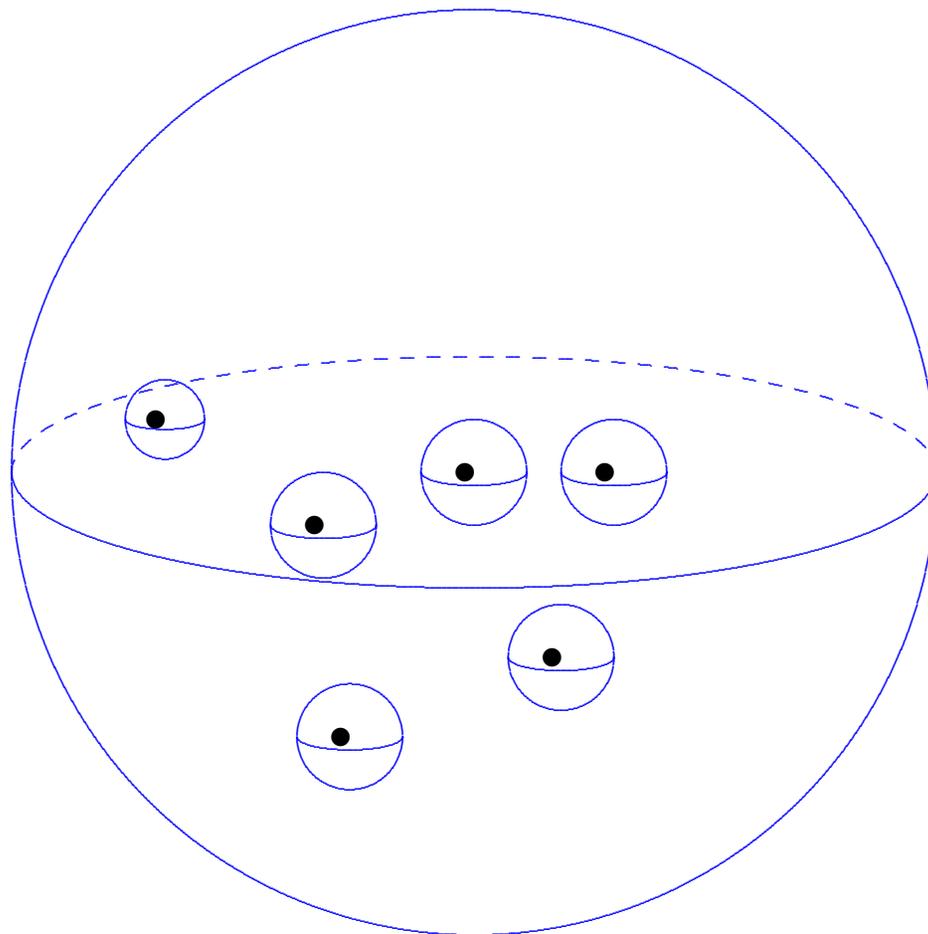
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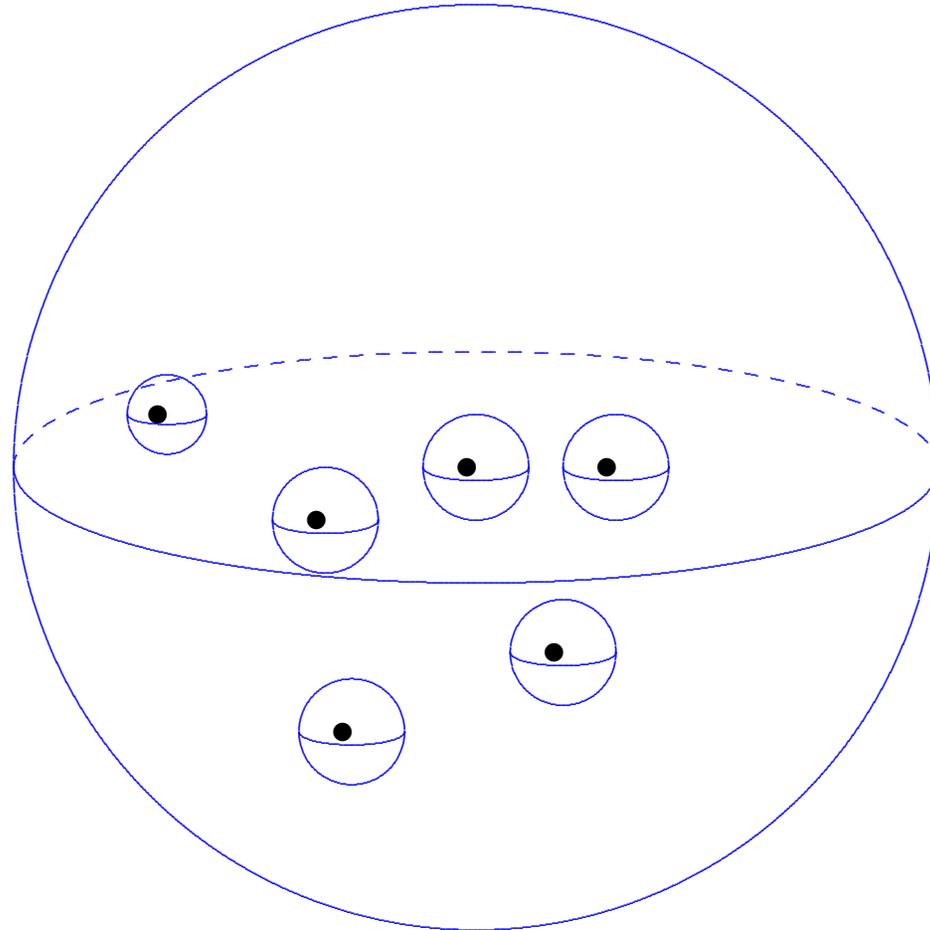
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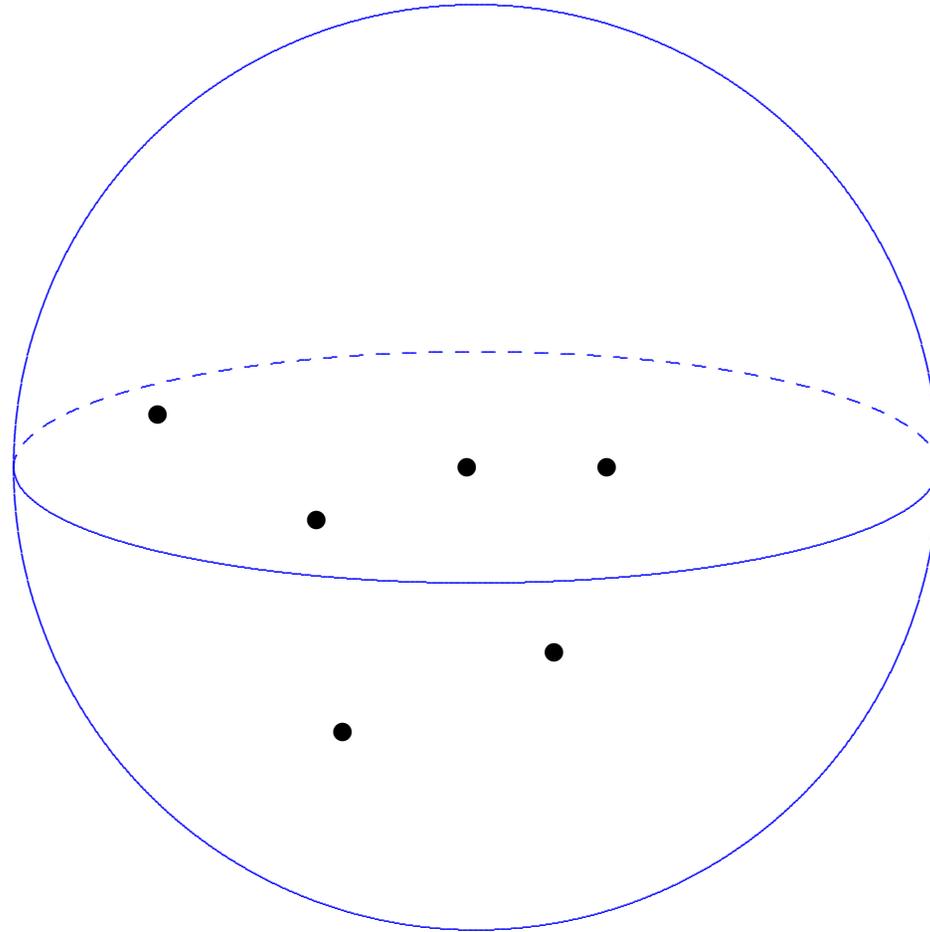
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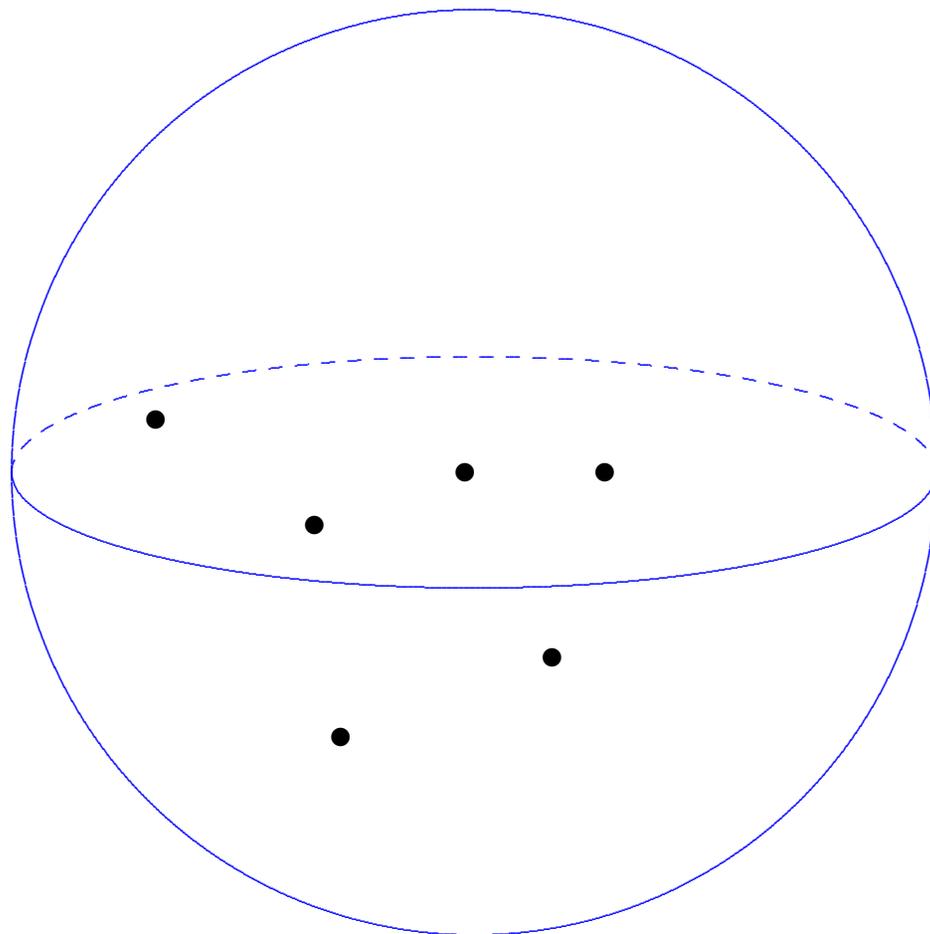
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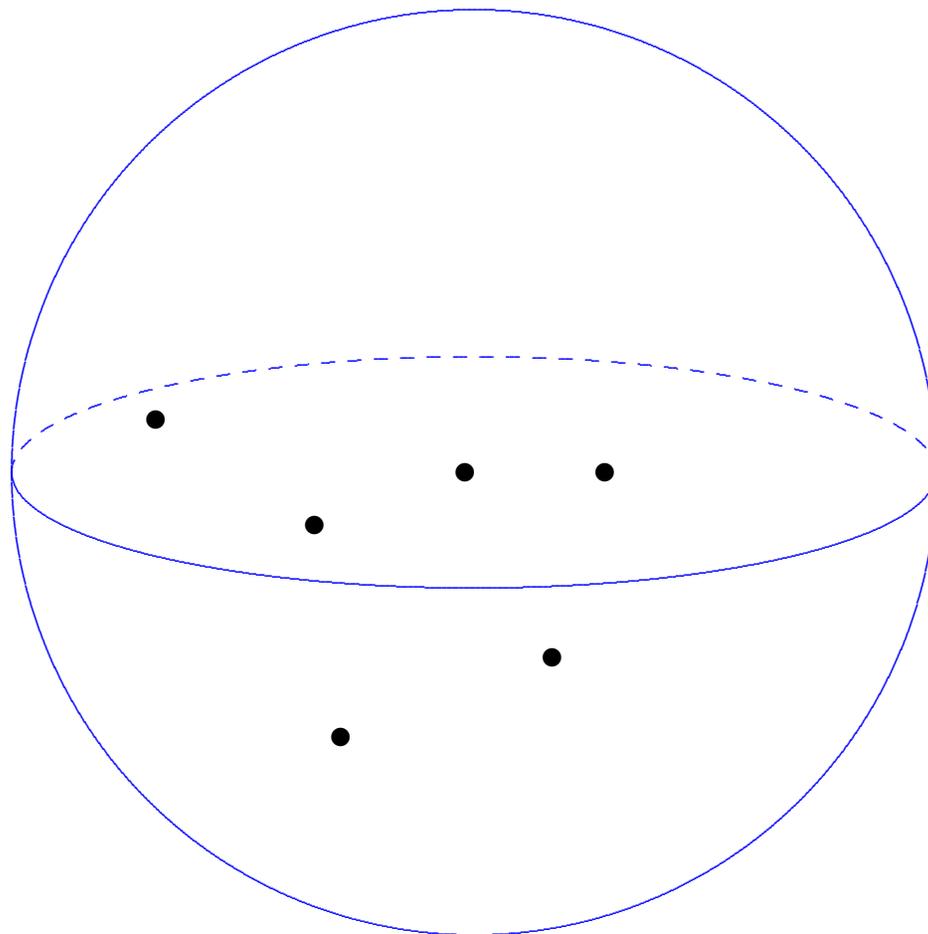
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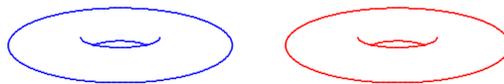
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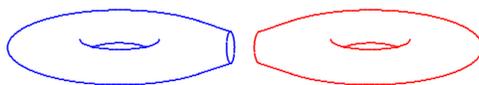


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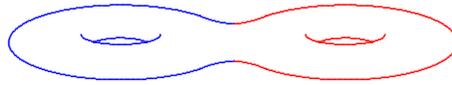


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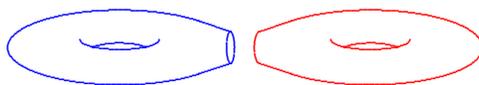


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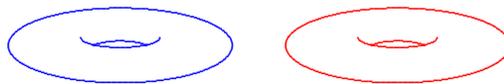


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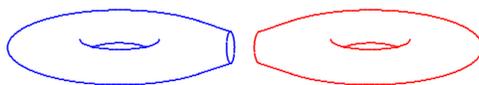


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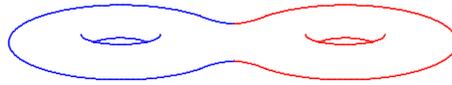


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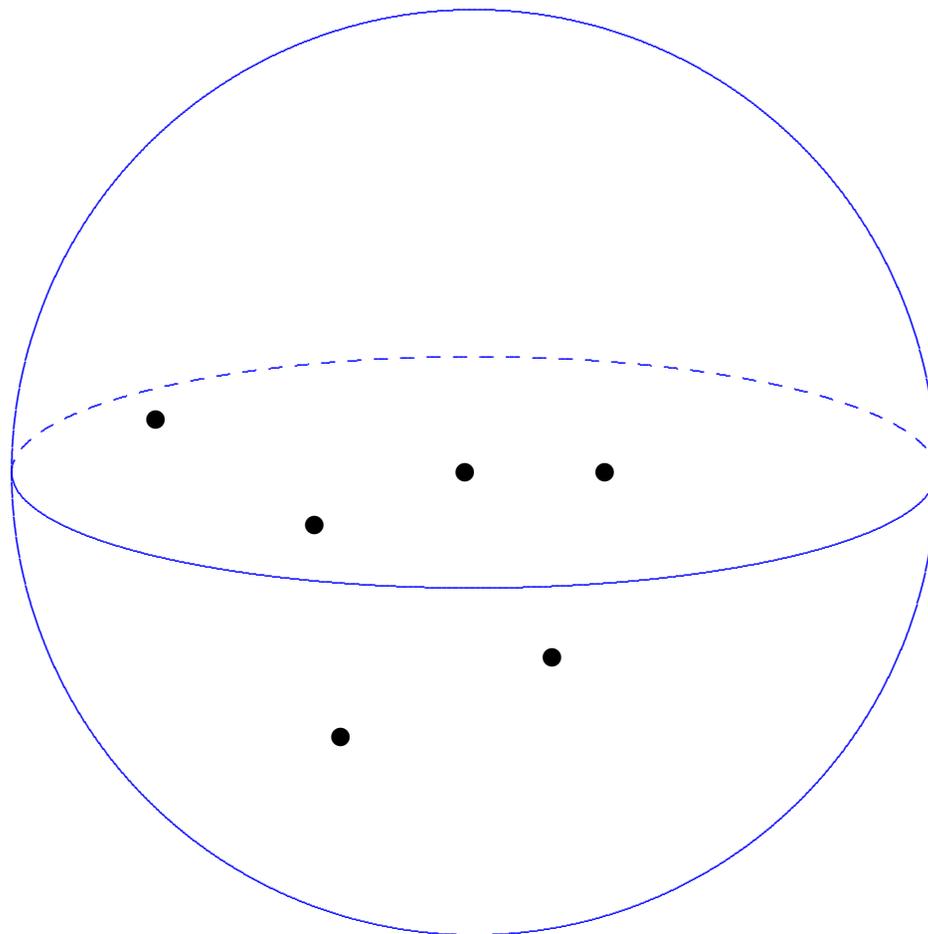
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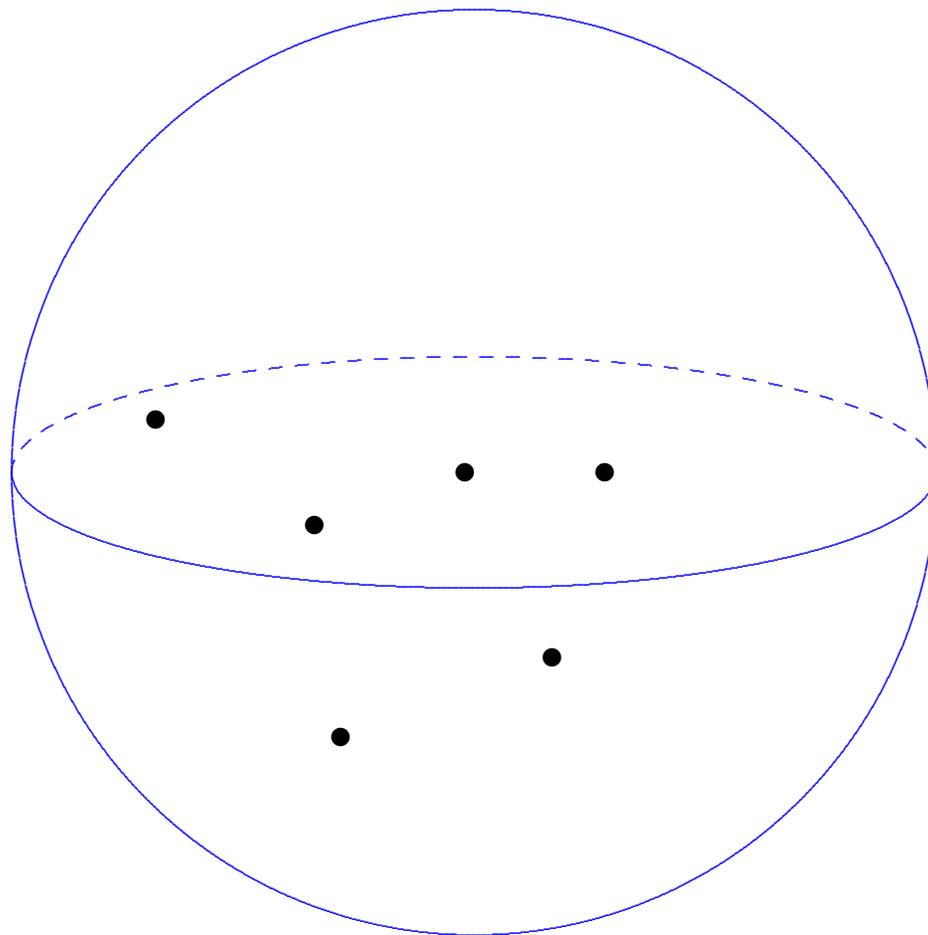
## Some Compact ASD 4-Manifolds



Riemannian completion is ASD metric on  $\underbrace{\overline{\mathbb{C}P}_2 \# \cdots \# \overline{\mathbb{C}P}_2}_k$ .

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This gives all possible homeotypes of compact,  $\pi_1 = 0$ , ASD 4-manifolds of positive scalar curvature.

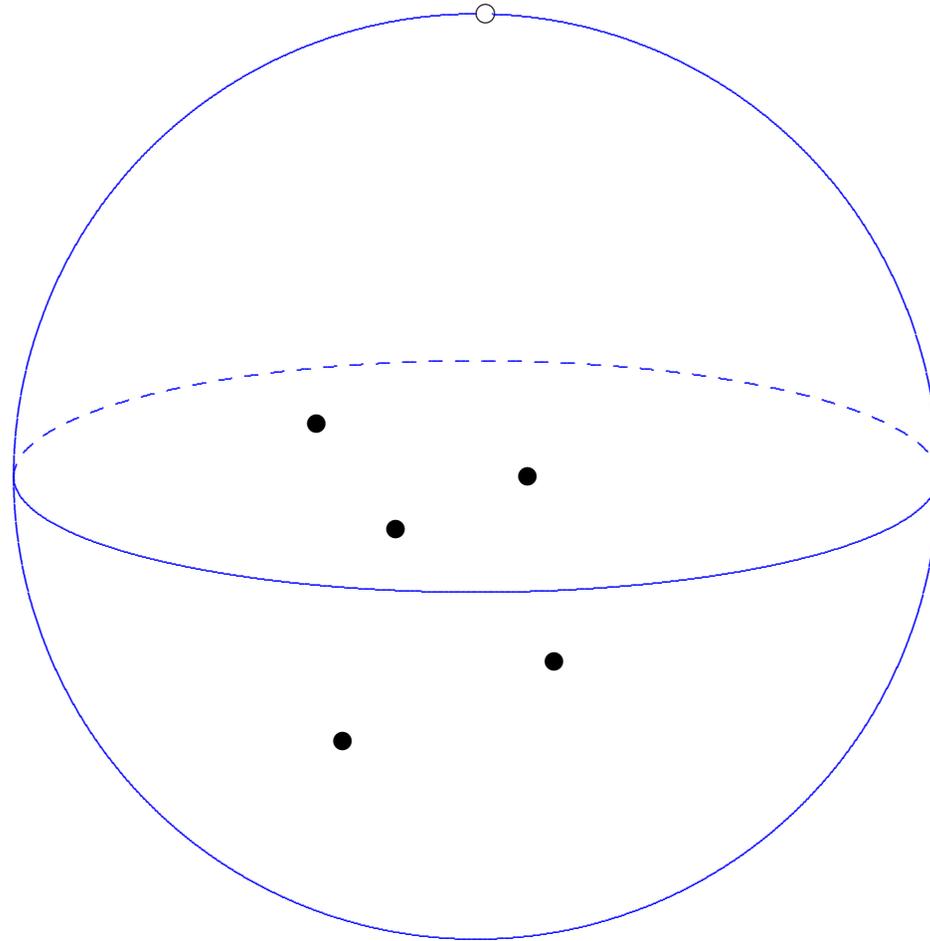
**Some AE Scalar-Flat Kähler Surfaces:**

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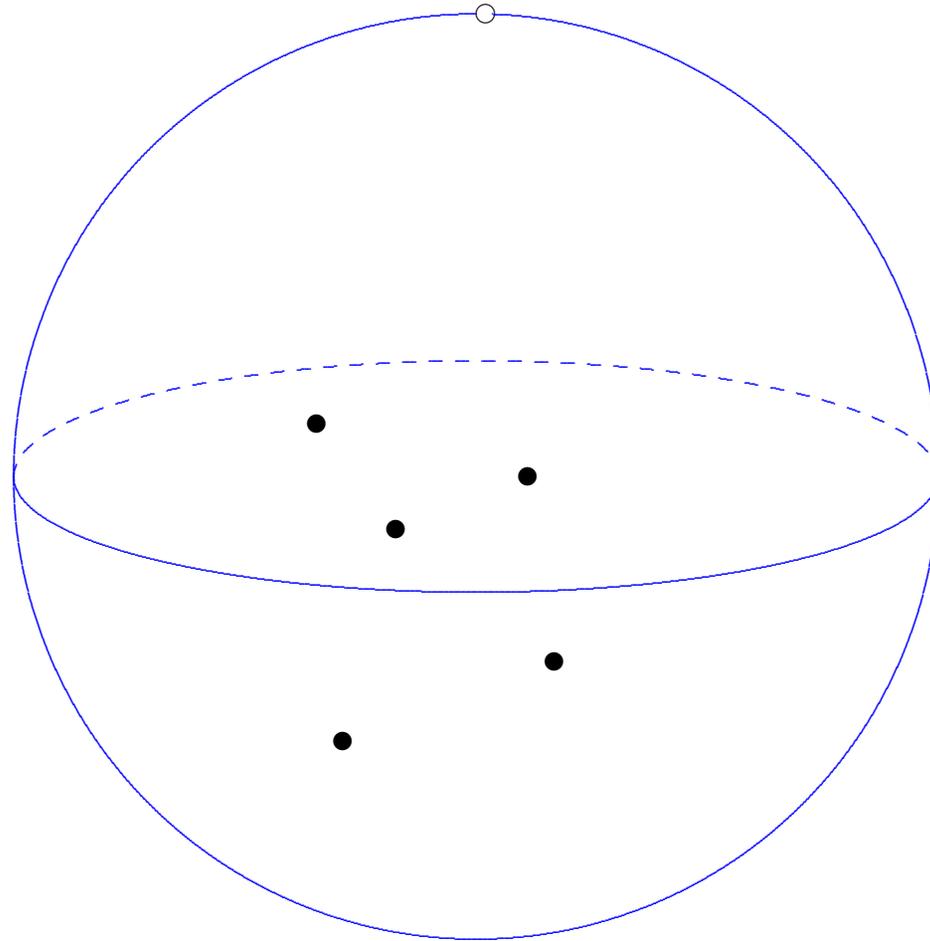
(AE = “asymptotically Euclidean”)

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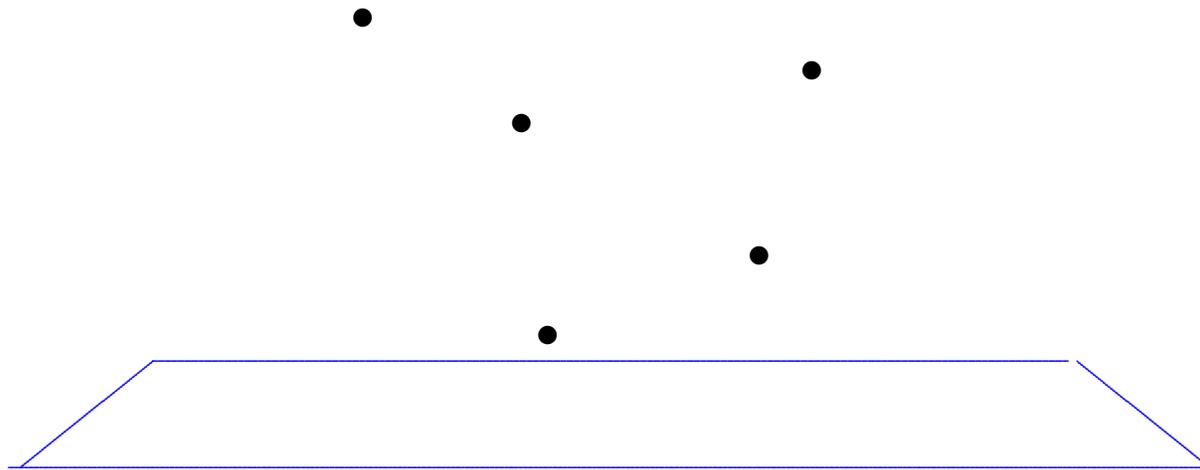


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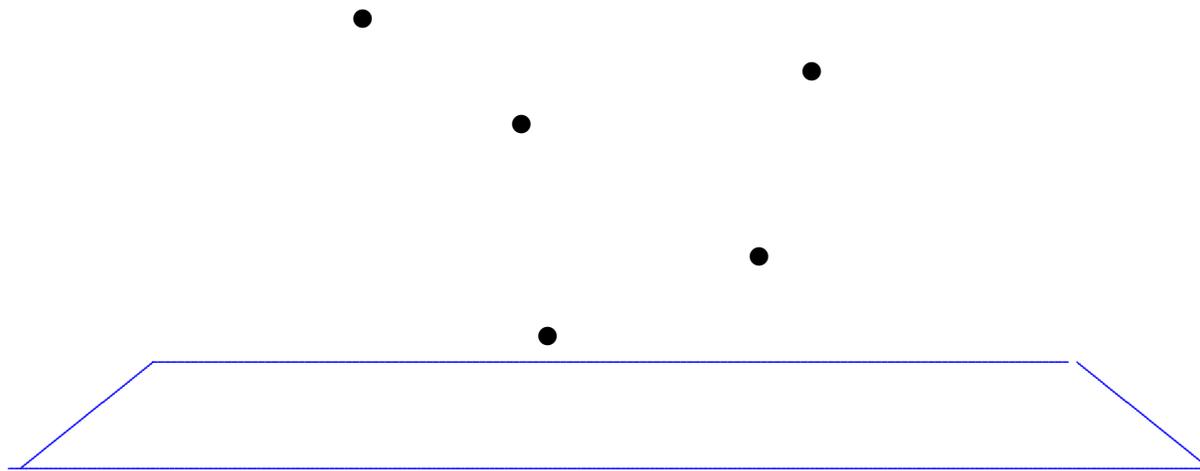
Data:  $k$  points in  $\mathcal{H}^3$  and one point at infinity.

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Data:  $k$  points in  $\mathcal{H}^3 =$  upper half-space model.

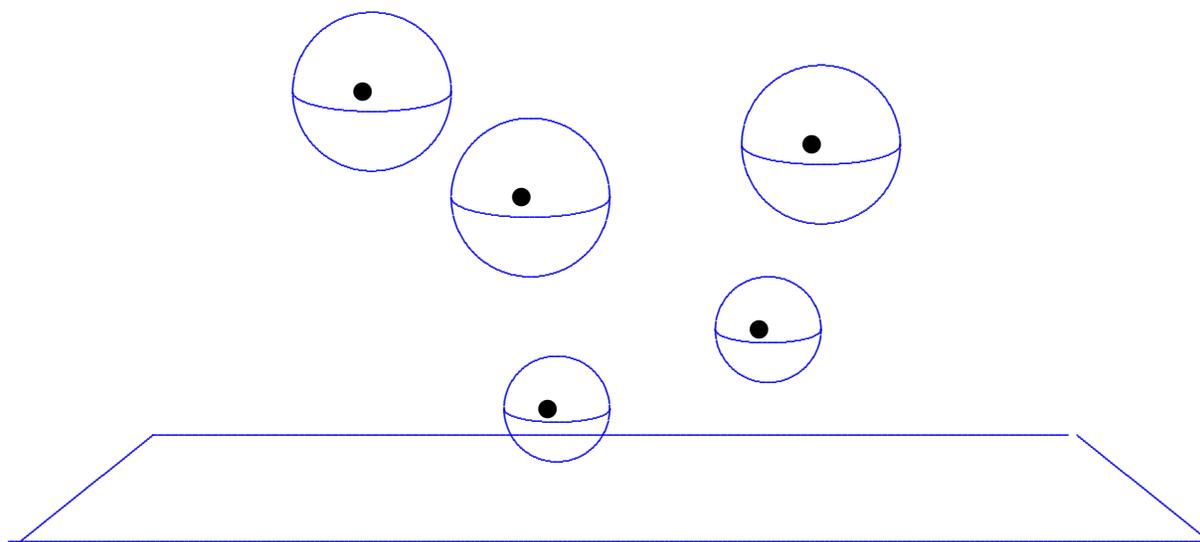
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Data:  $k$  points in  $\mathcal{H}^3$ .  $\implies V$  with  $\Delta V = 0$

$$V = 1 + \sum_{j=1}^k G_j$$

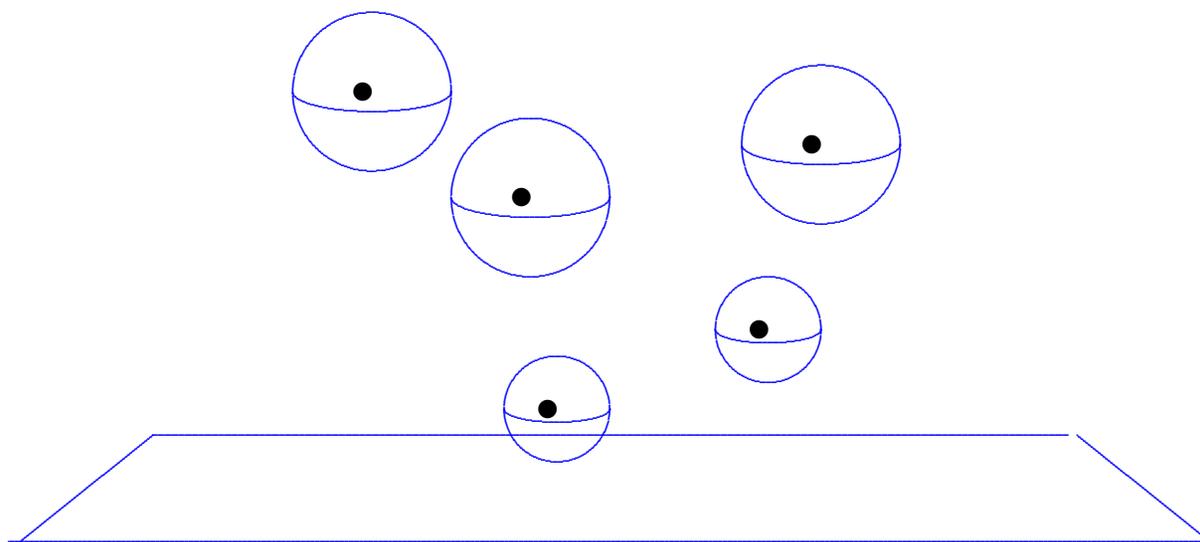
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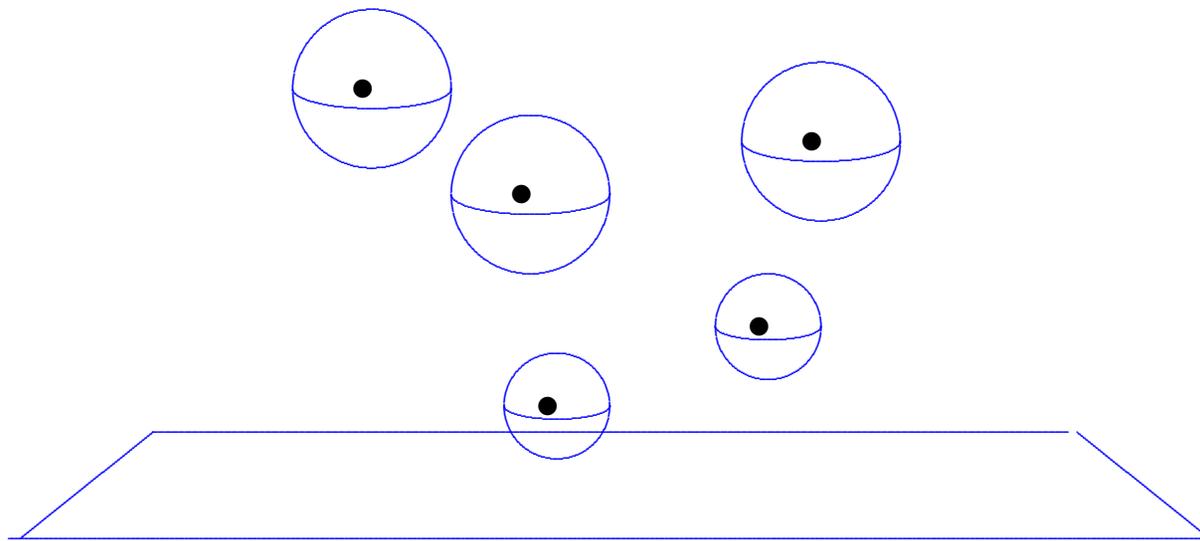
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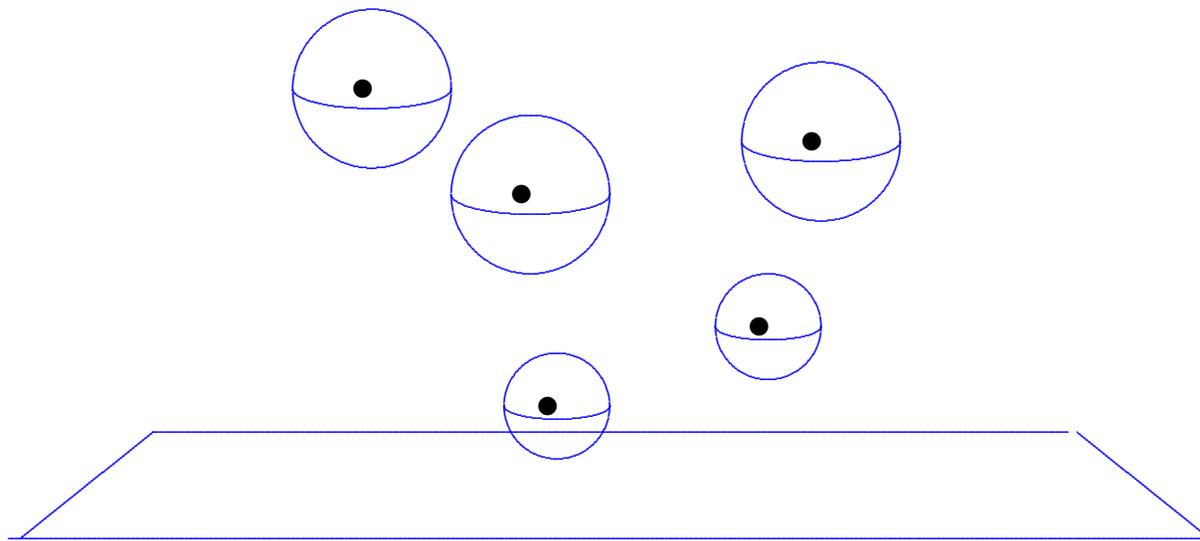
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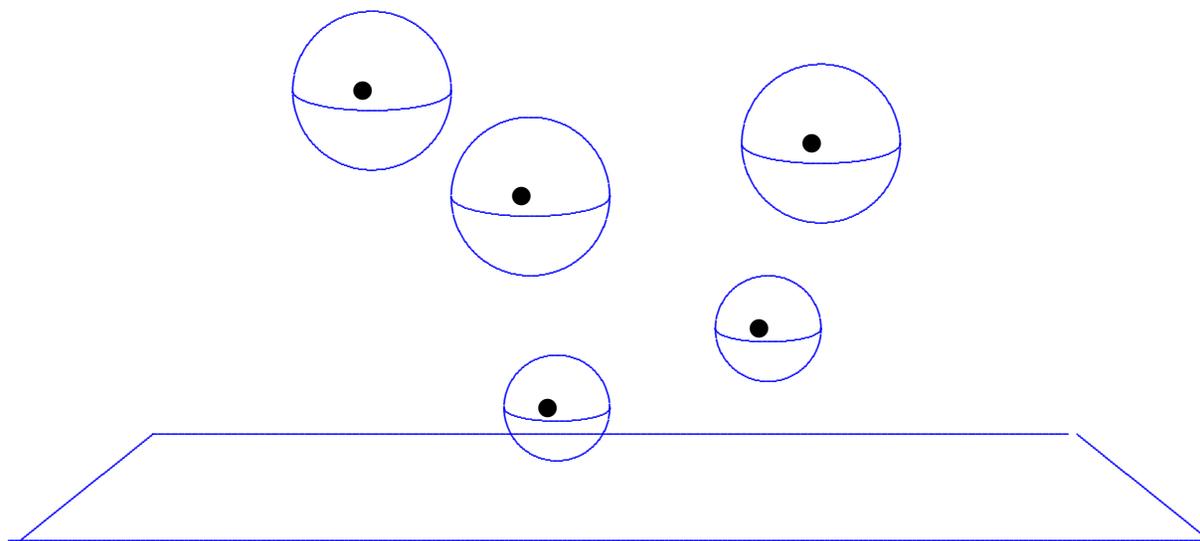
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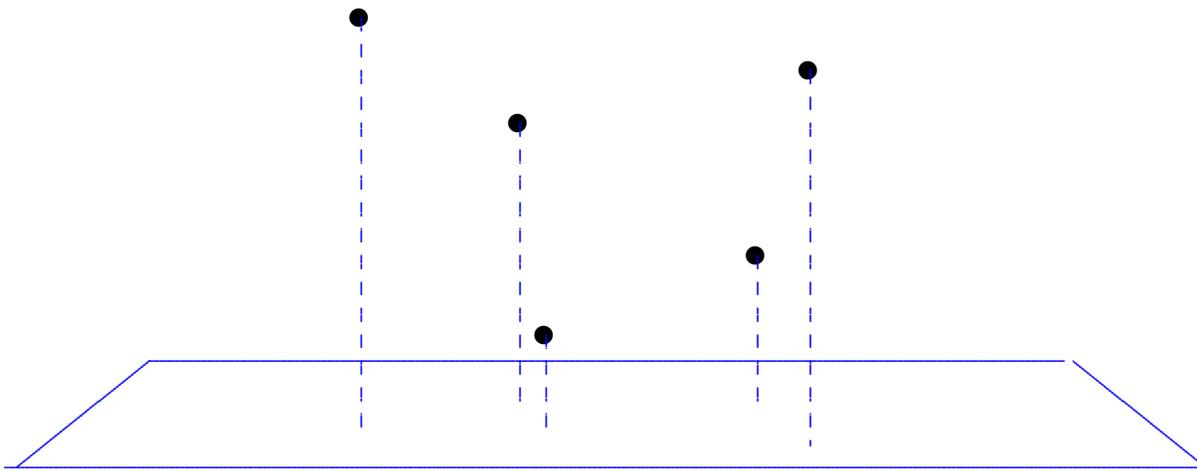
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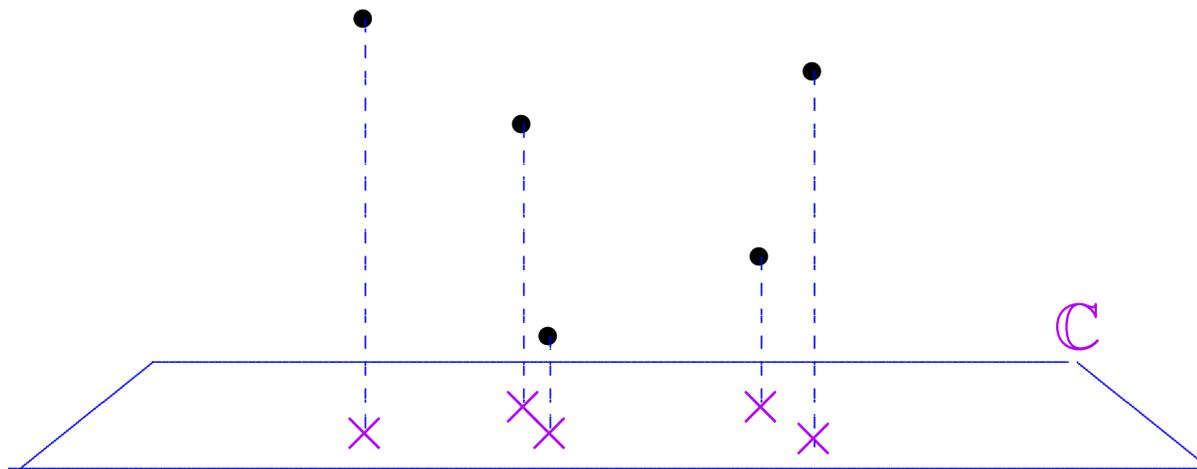
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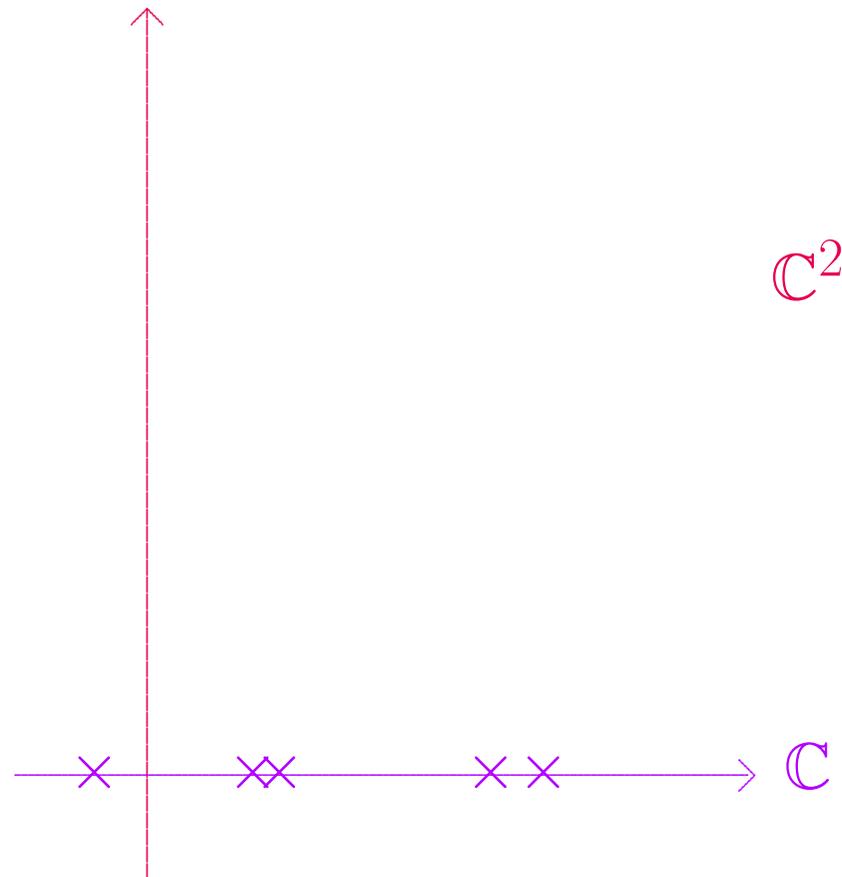
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Complex structure of scalar-flat Kähler surface...

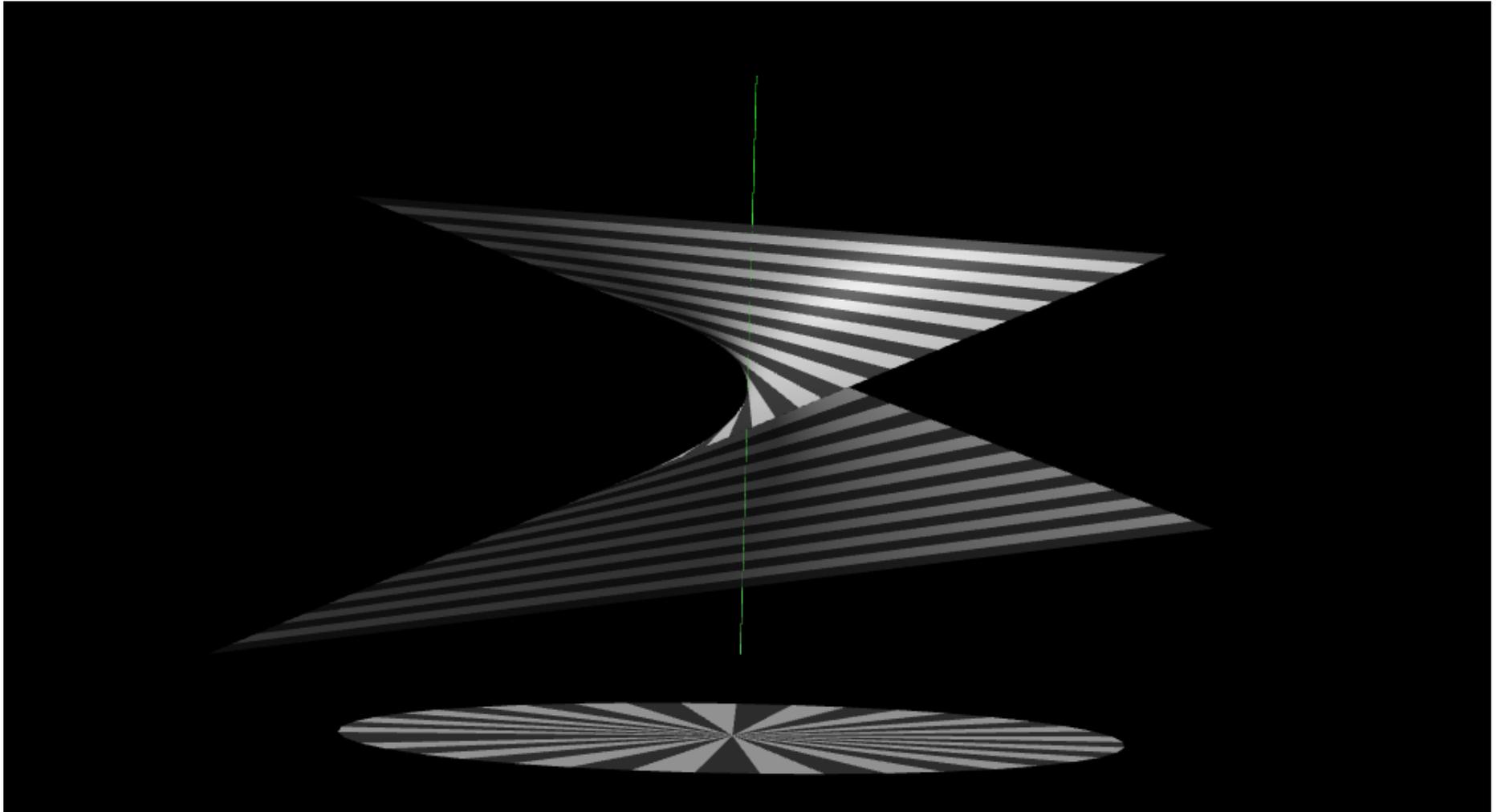
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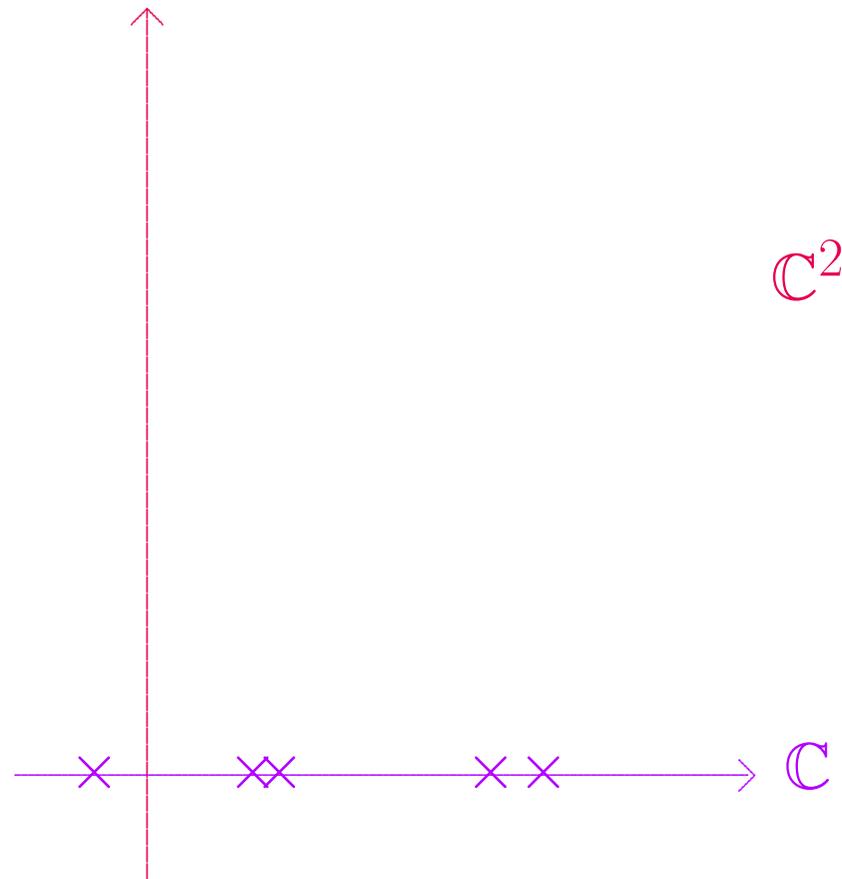


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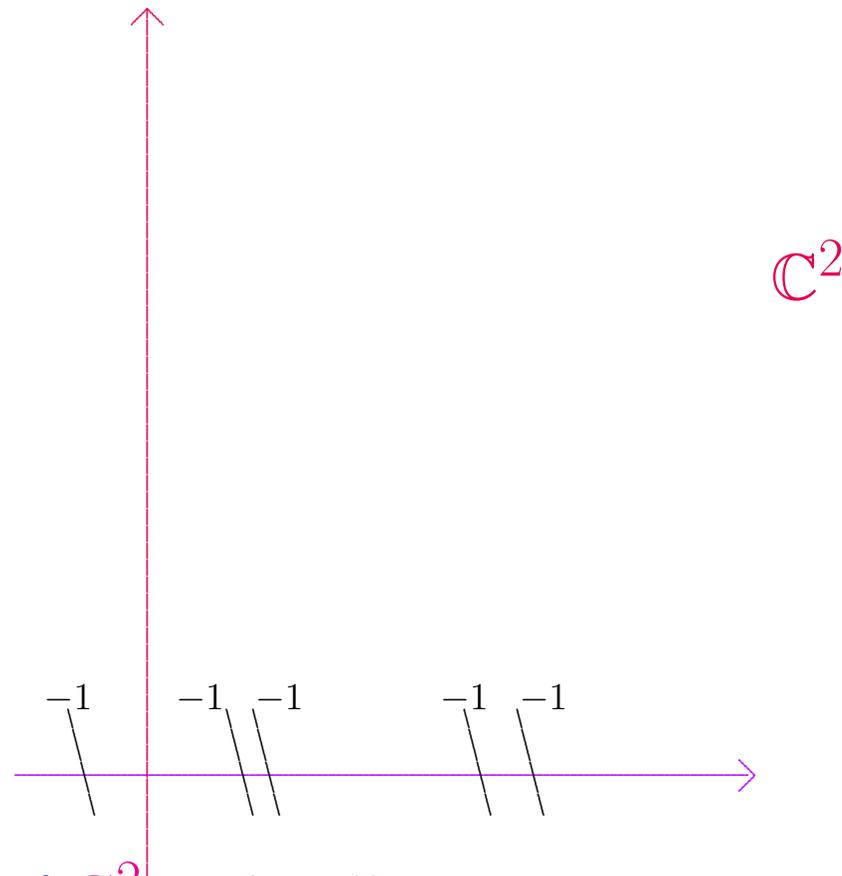
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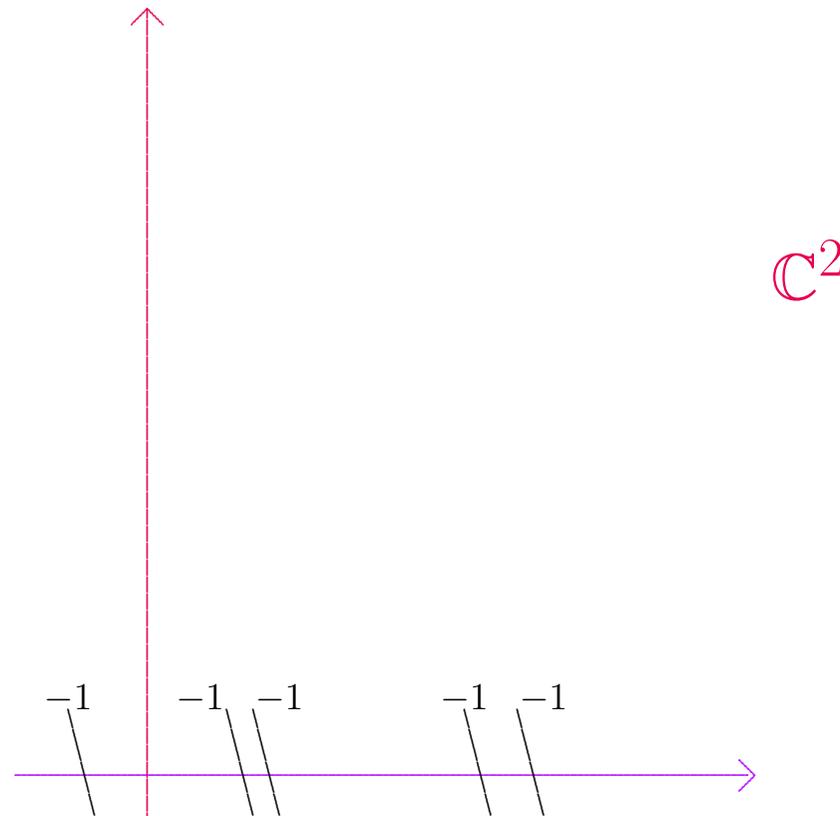
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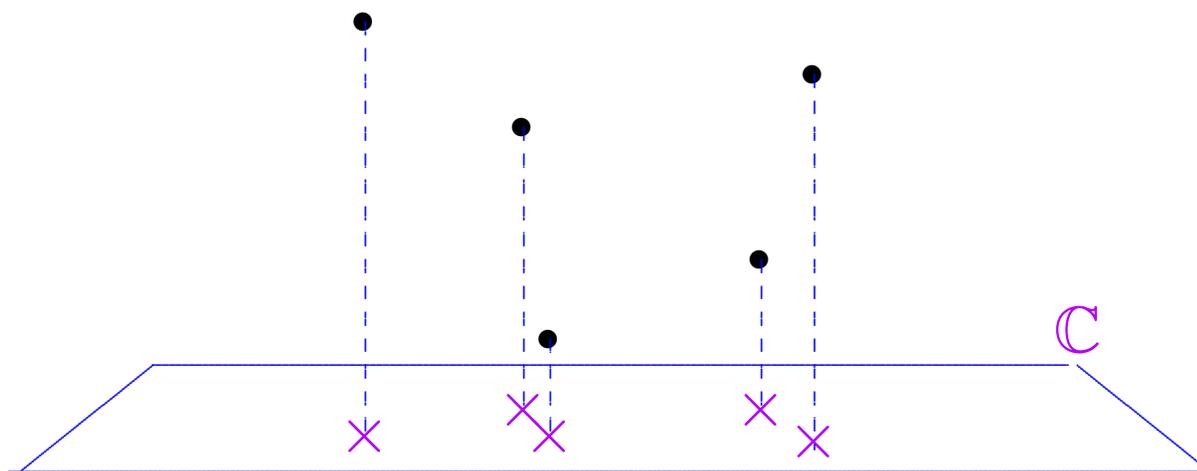
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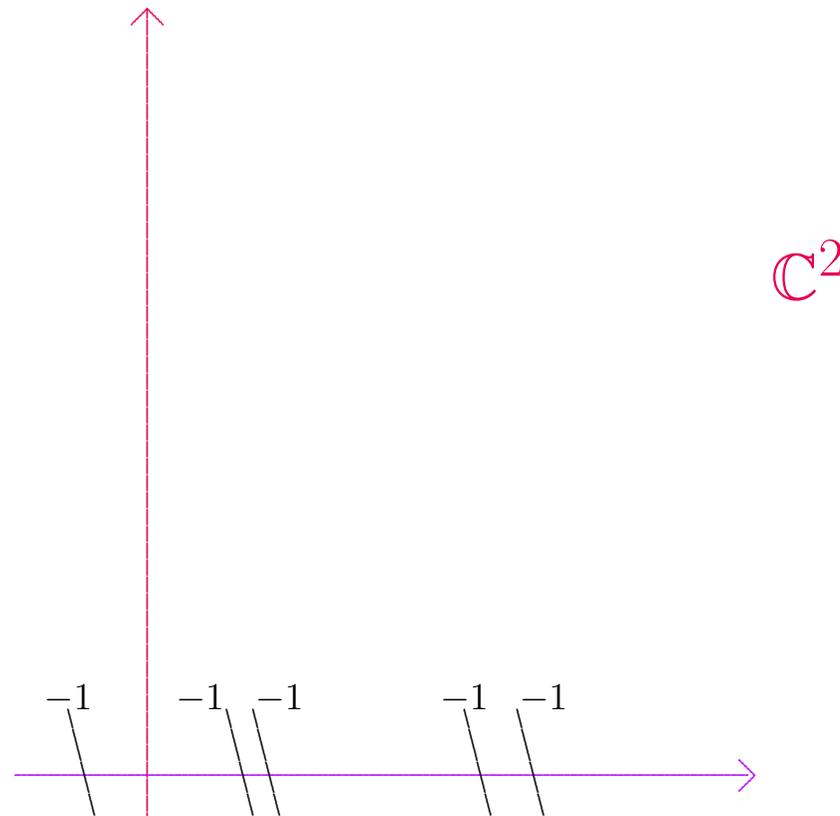
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Heights of centers  $\leftrightarrow$  areas exceptional divisors.

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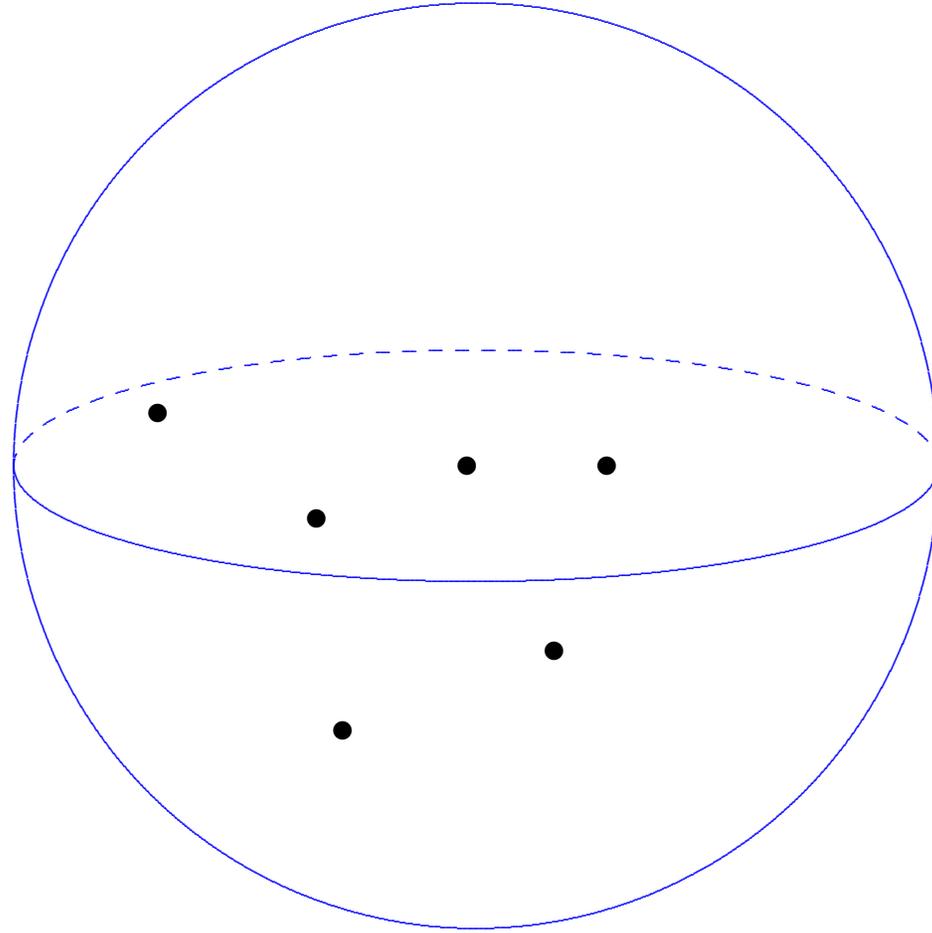
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Analogous to Hitchin's twistor spaces for Gibbons-Hawking gravitational instantons.

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Twistor spaces are Moishezon — i.e. they are bimeromorphic to algebraic varieties.

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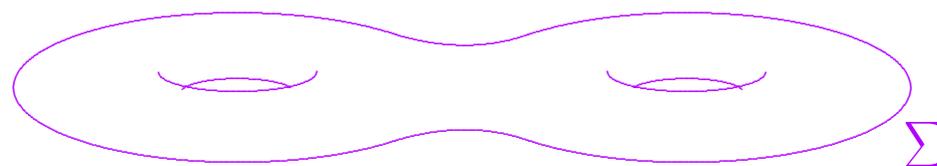
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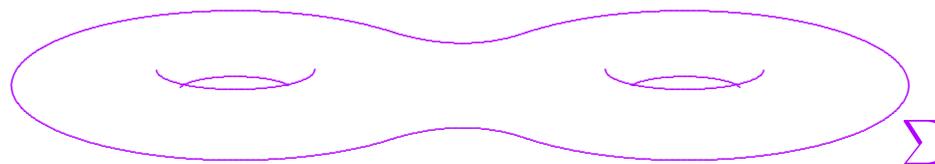
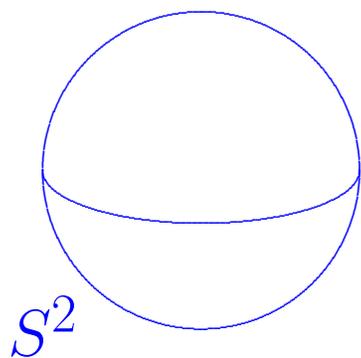
If compact, Kodaira-Enriques  $\implies$  either

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- ruled, as a complex surface.

Example.

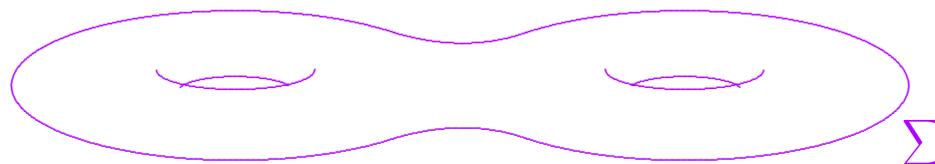
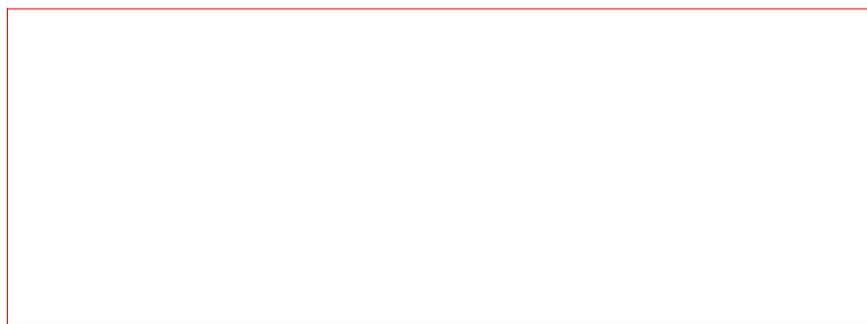
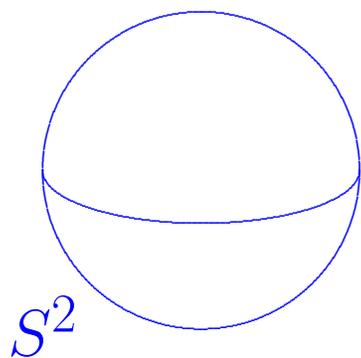


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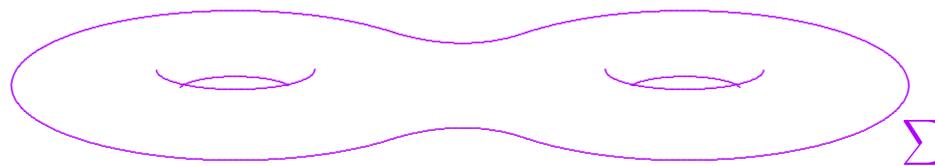
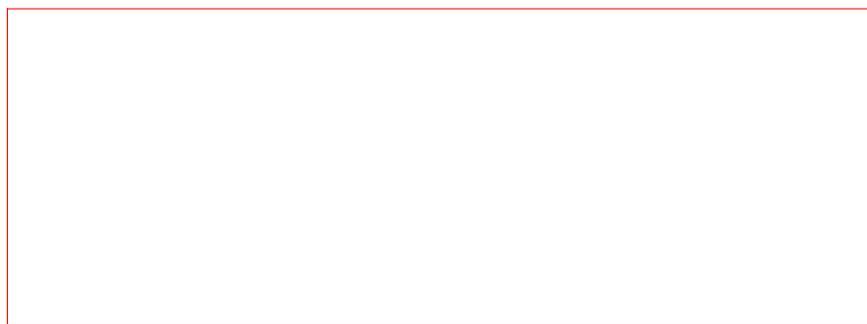
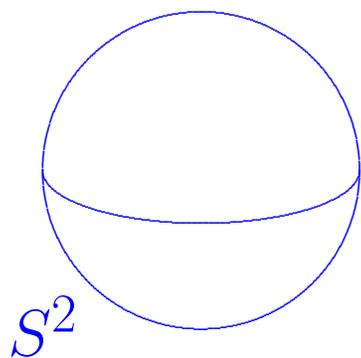
Example.

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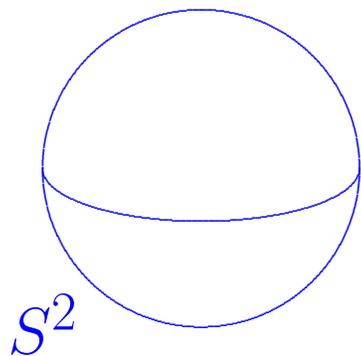
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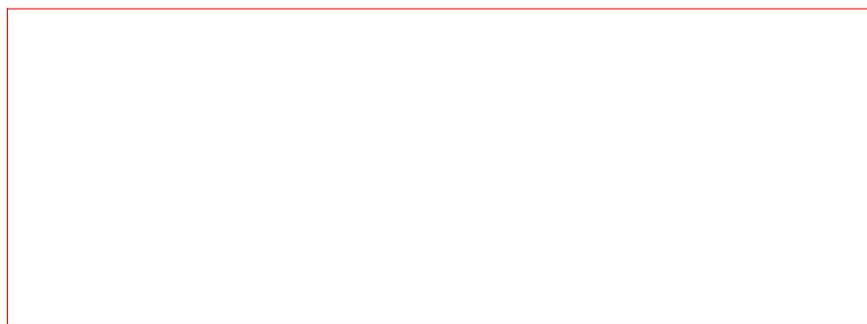
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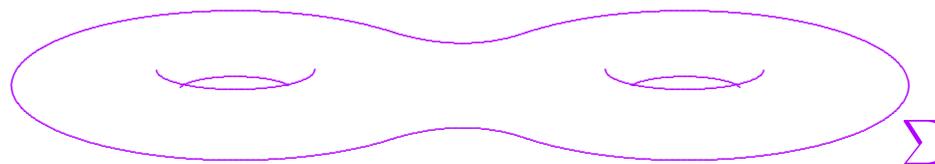
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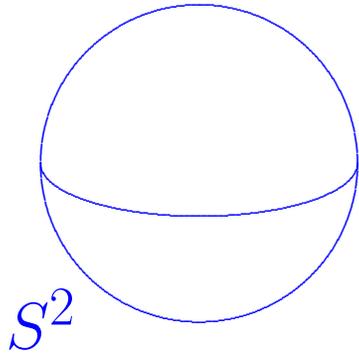


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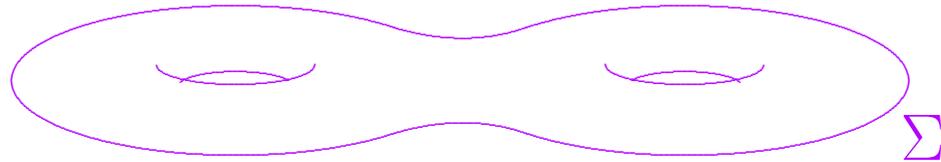
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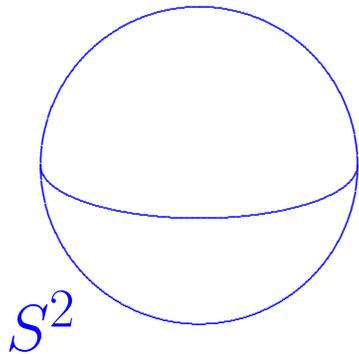
$$K = -1$$



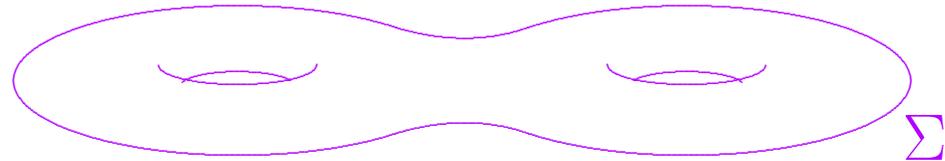
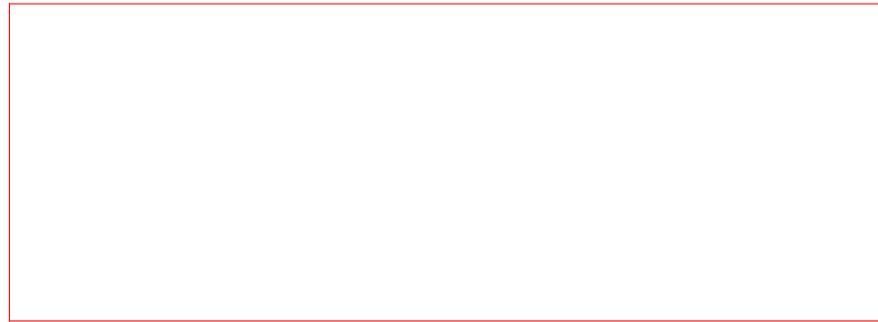
Product is scalar-flat

## Example.

$$K = +1$$



$$M = \Sigma \times S^2$$

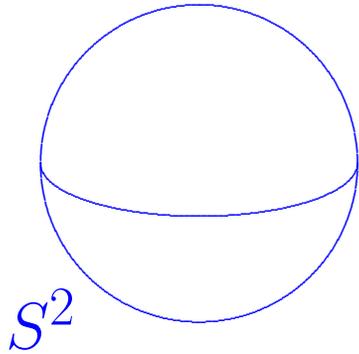


$$K = -1$$

Product is scalar-flat Kähler.

## Example.

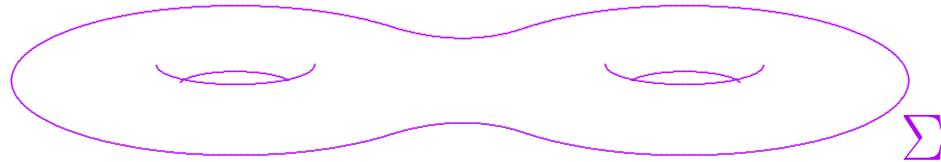
$$K = +1$$



$$M = \Sigma \times S^2$$



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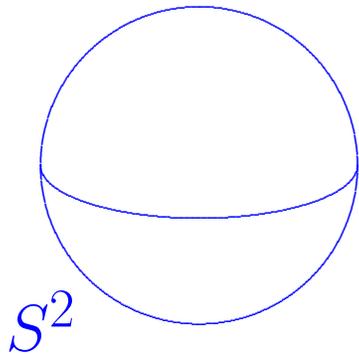


Product is scalar-flat Kähler.

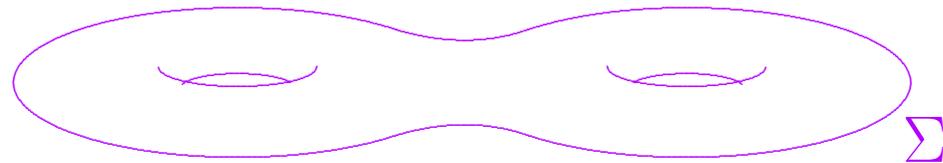
For both orientations!

## Example.

$$K = +1$$



$$M = \Sigma \times S^2$$



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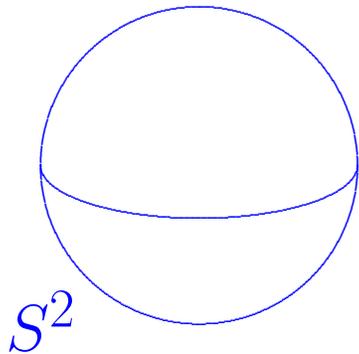
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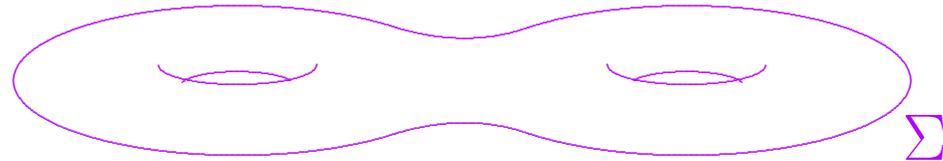
$$W_+ = 0.$$

## Example.

$$K = +1$$



$$M = \Sigma \times S^2$$



$$K = -1$$

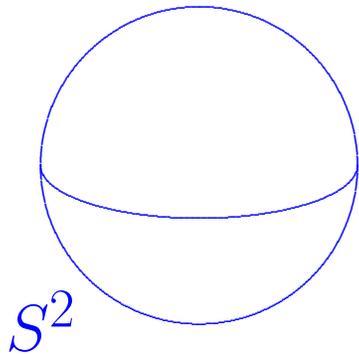
Product is scalar-flat Kähler.

For both orientations!

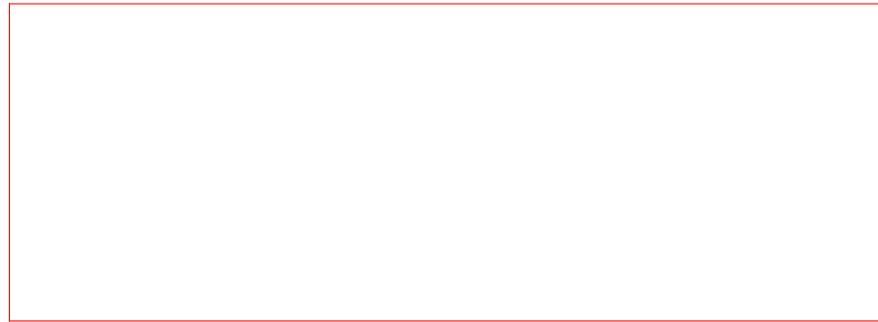
$$W_{\pm} = 0.$$

## Example.

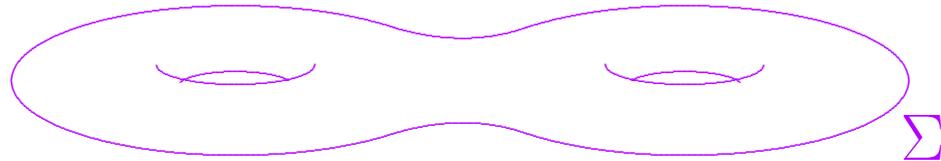
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Product is scalar-flat Kähler.

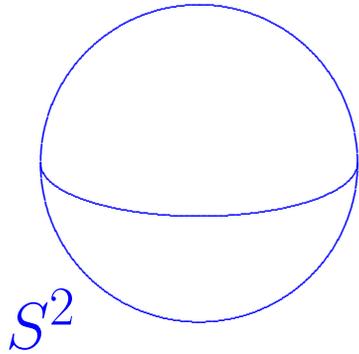
For both orientations!

$$W_{\pm} = 0.$$

Locally conformally flat!

# Example.

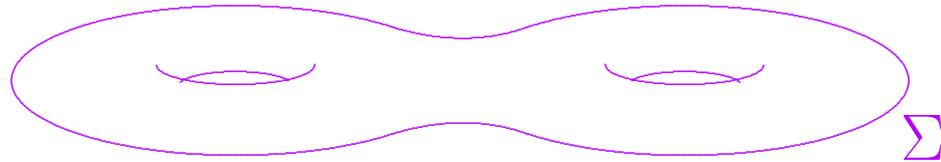
$$K = +1$$



$$M = \Sigma \times S^2$$



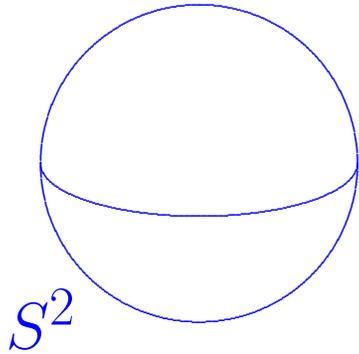
$$K = -1$$



$$\widetilde{M} = \mathcal{H}^2 \times S^2$$

## Example.

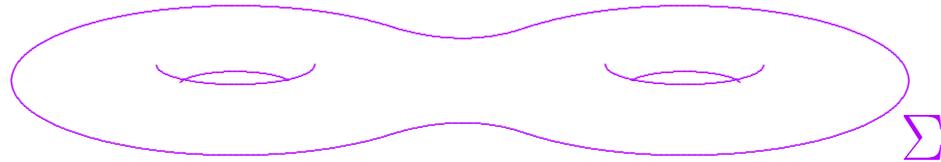
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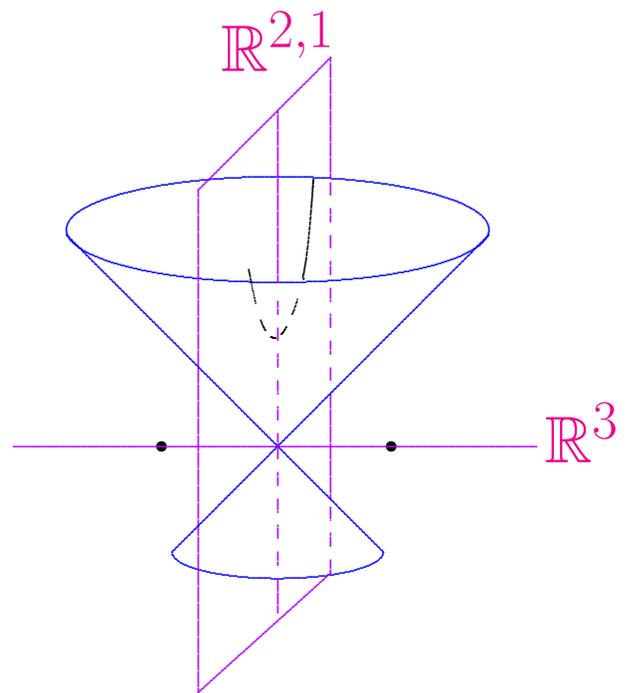
$$\widetilde{M} = \mathcal{H}^2 \times S^2 = S^4 - S^1$$

**Example:**

$\mathcal{H}^2 \times S^2$  conformal to  $S^4 - S^1$

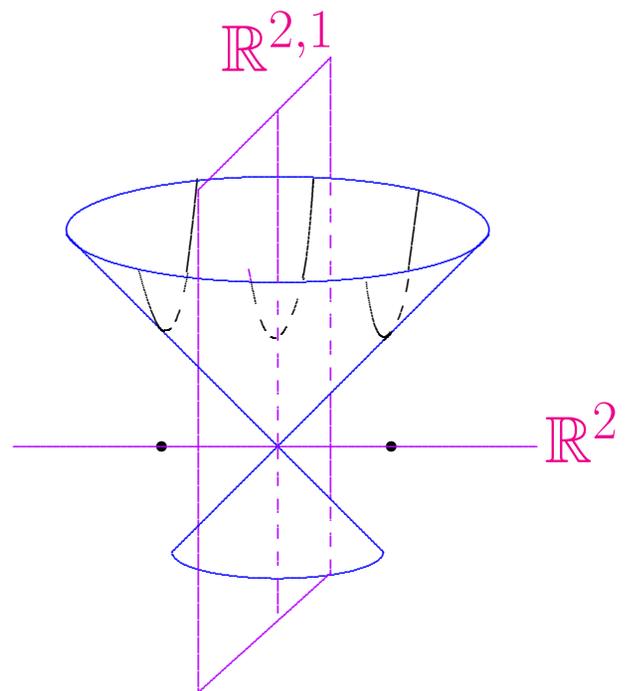
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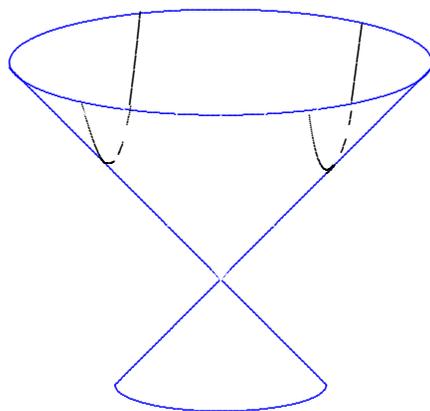
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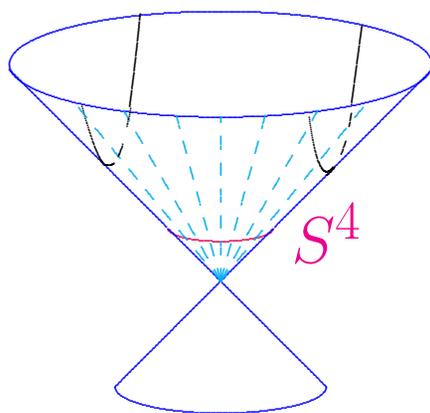
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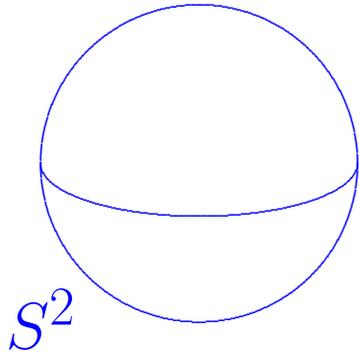
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## Example.

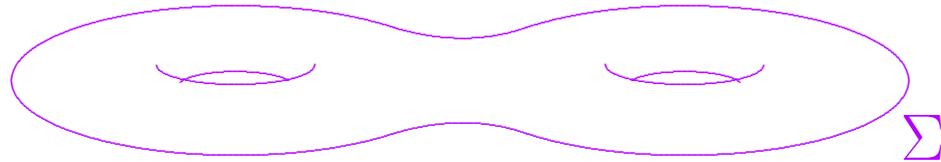
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$$\widetilde{M} = \mathcal{H}^2 \times S^2 = S^4 - S^1$$

What about non-minimal ruled surfaces?

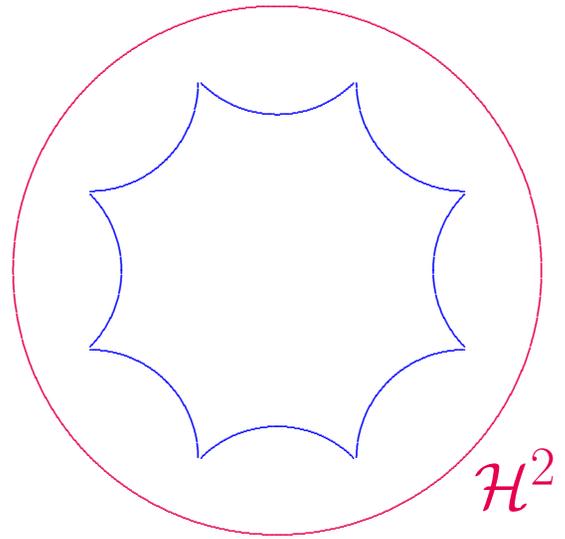
**What about non-minimal ruled surfaces?**

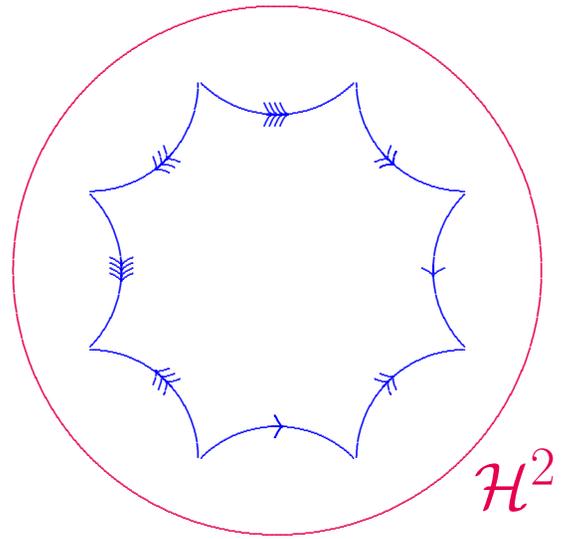
Hyperbolic ansatz also provides such examples!

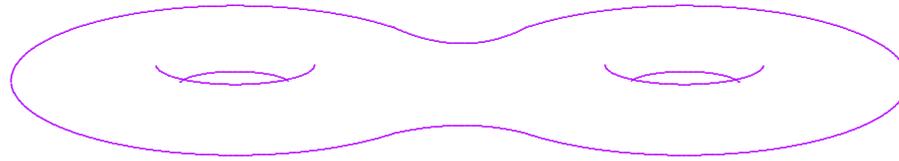
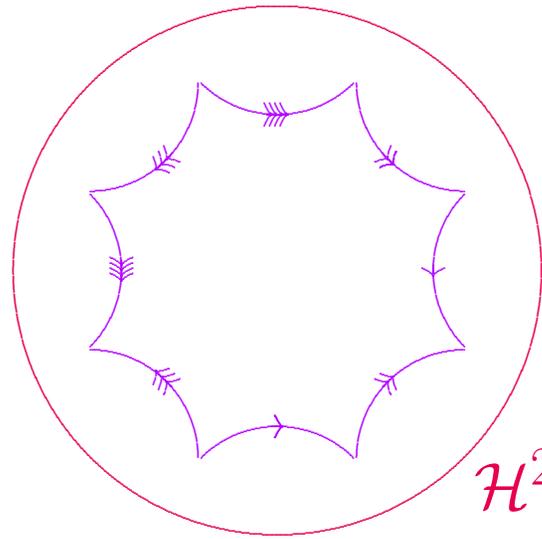
## What about non-minimal ruled surfaces?

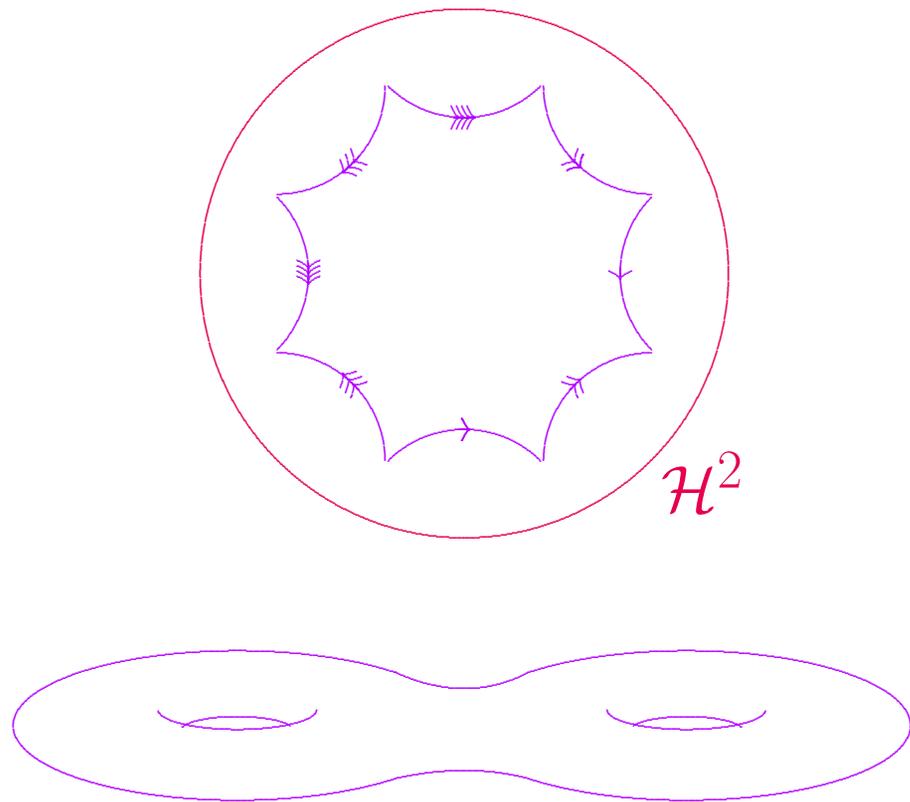
Hyperbolic ansatz also provides such examples!

Construction based on Fuchsian groups. L '91

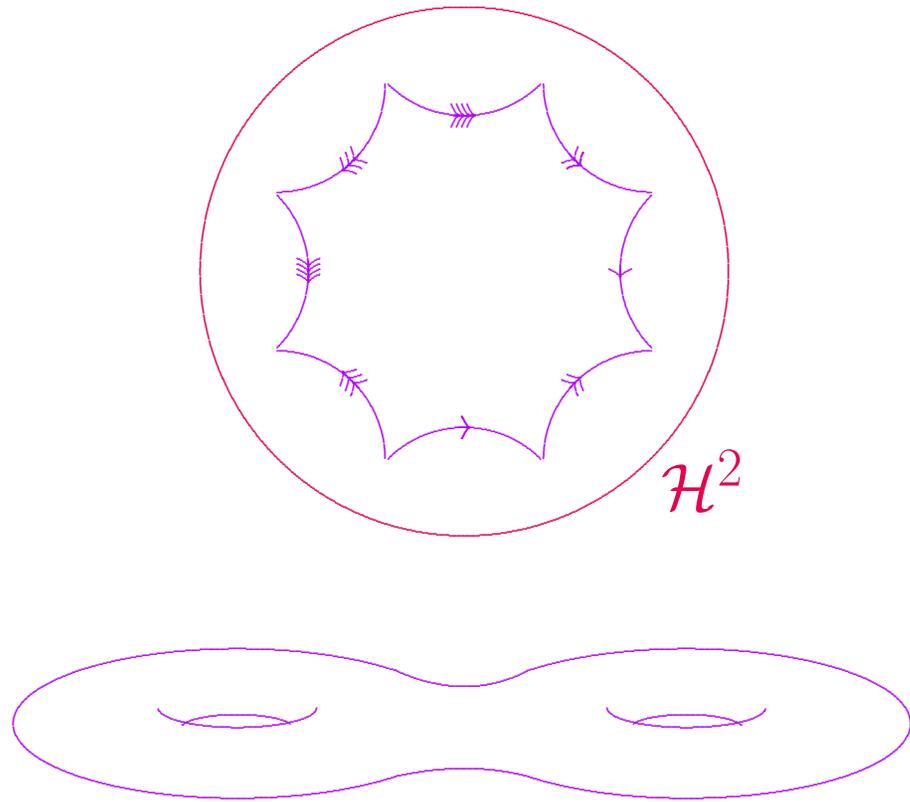




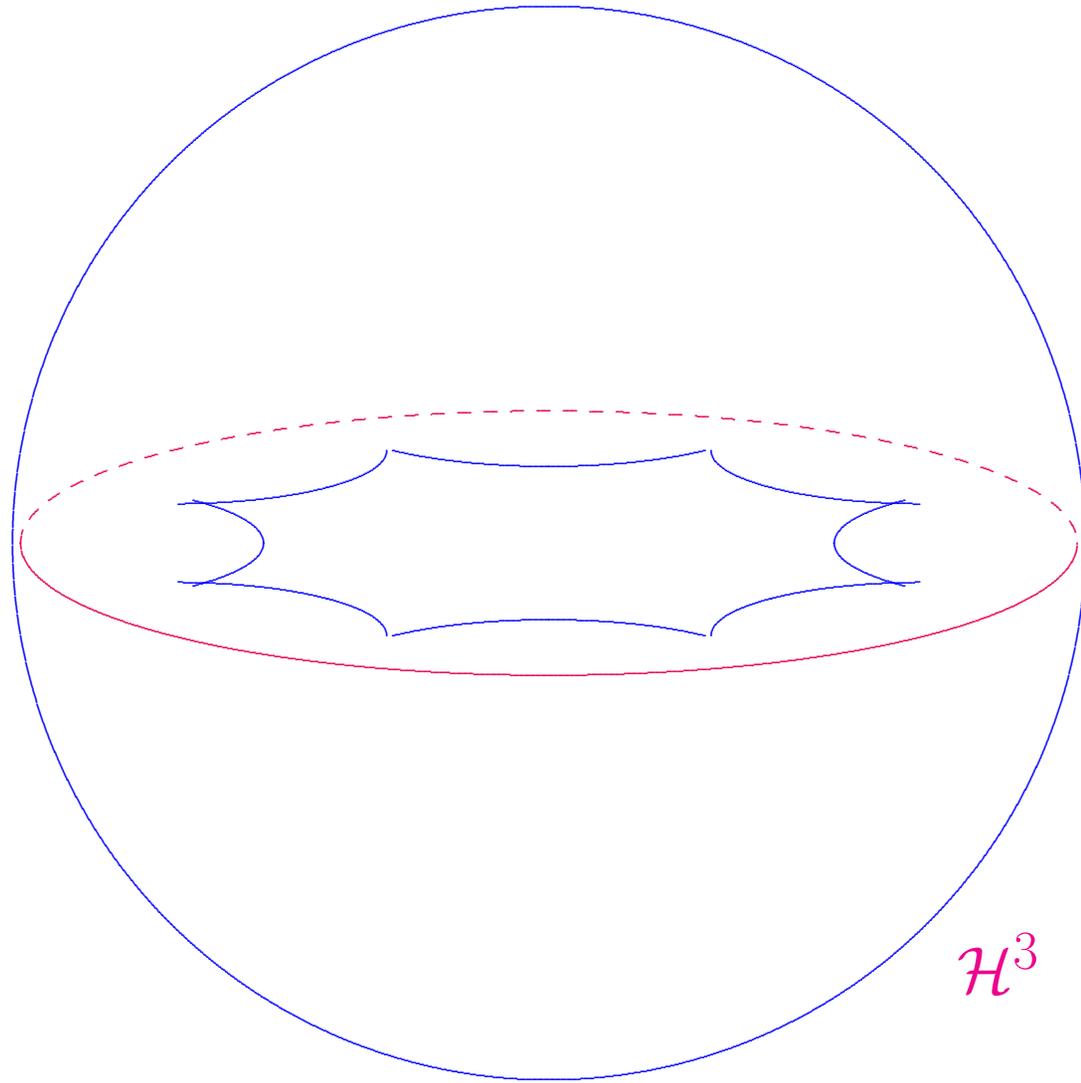




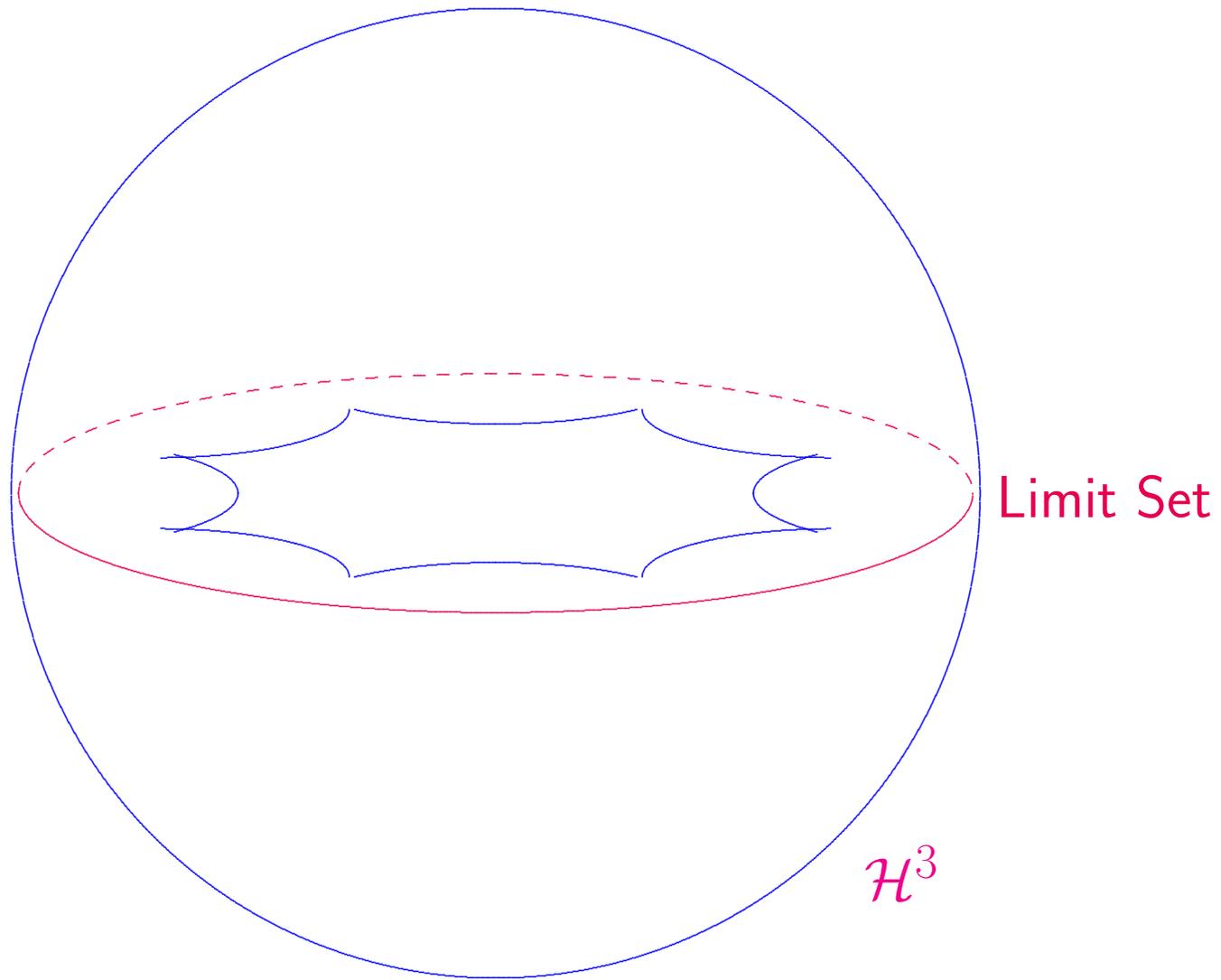
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R})$$

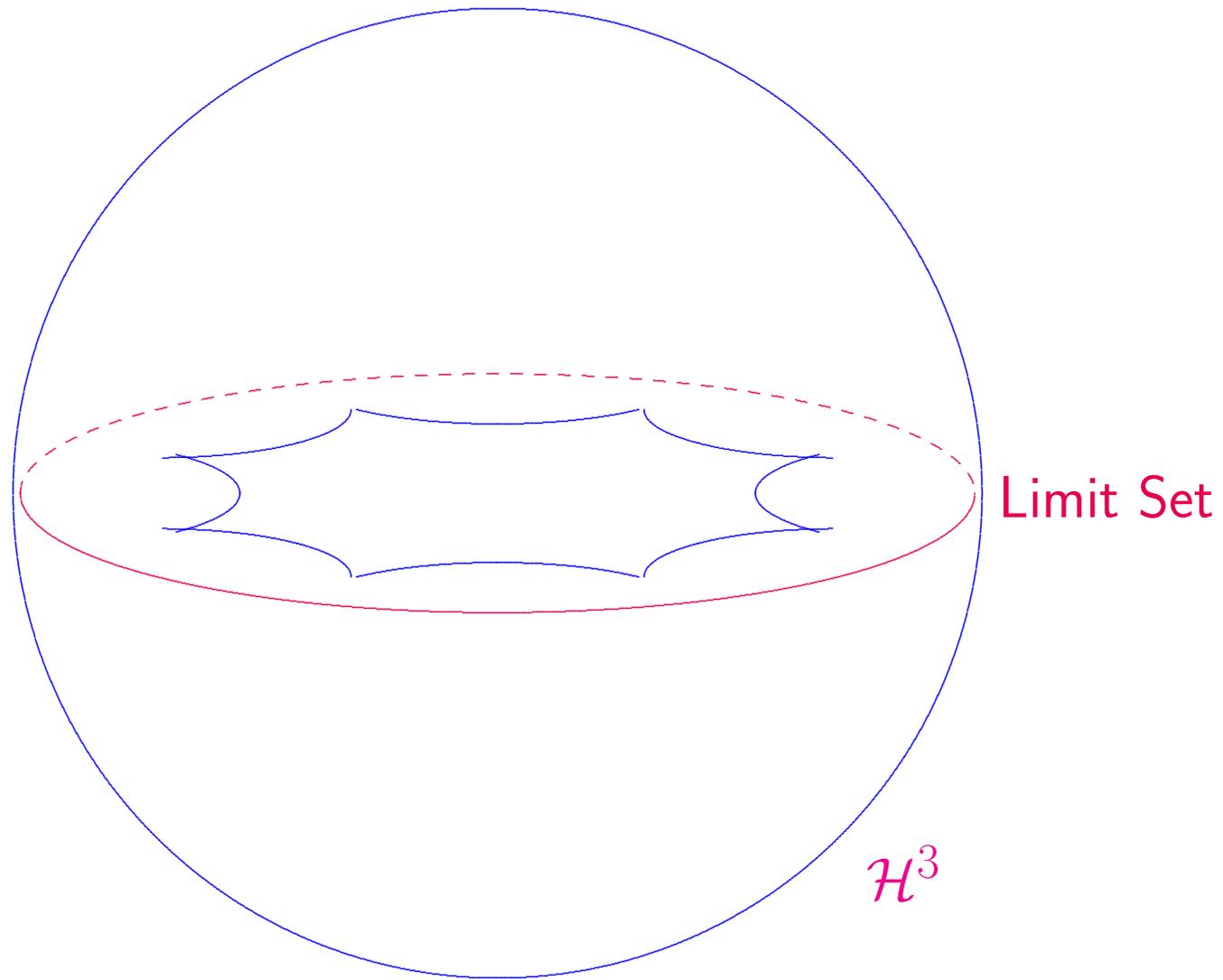


$$\begin{array}{ccc}
 \pi_1(\Sigma) & \hookrightarrow & \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R}) \\
 & & \cap \\
 & & \mathbf{SO}_+(1, 3) = \mathbf{PSL}(2, \mathbb{C})
 \end{array}$$

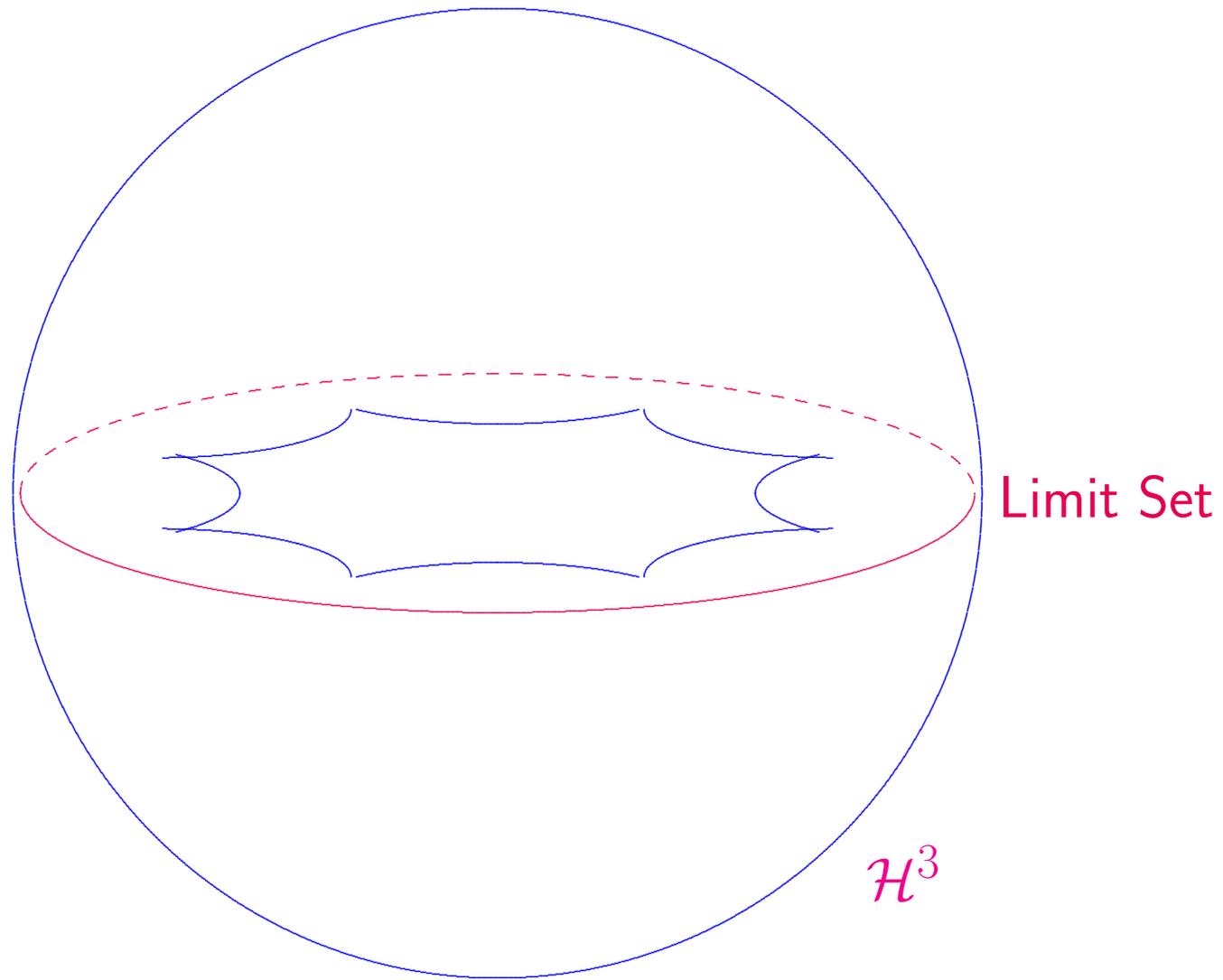


$\mathcal{H}^3$

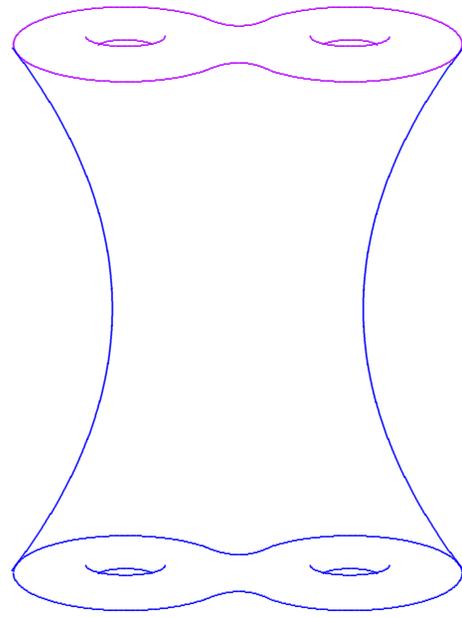


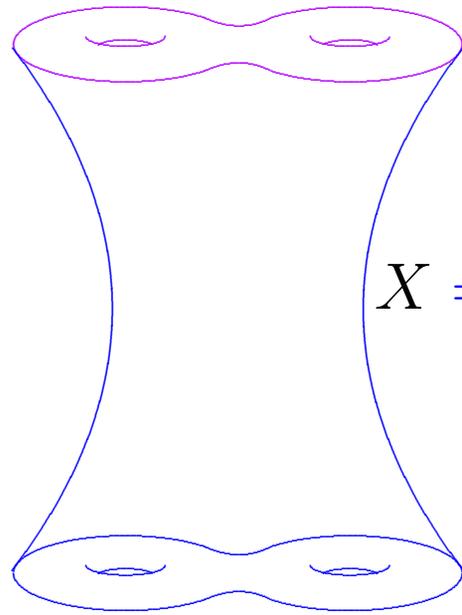


$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{R}) \text{ Fuchsian group}$$

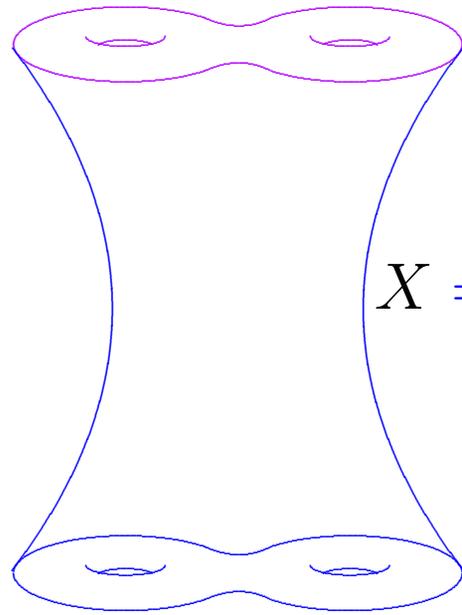


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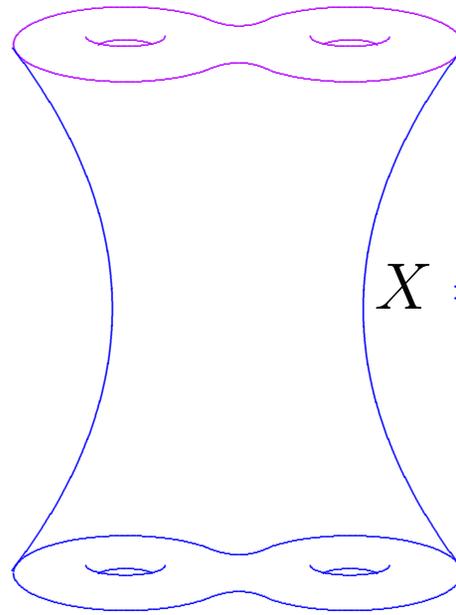


$$X = \mathcal{H}^3 / \Gamma$$



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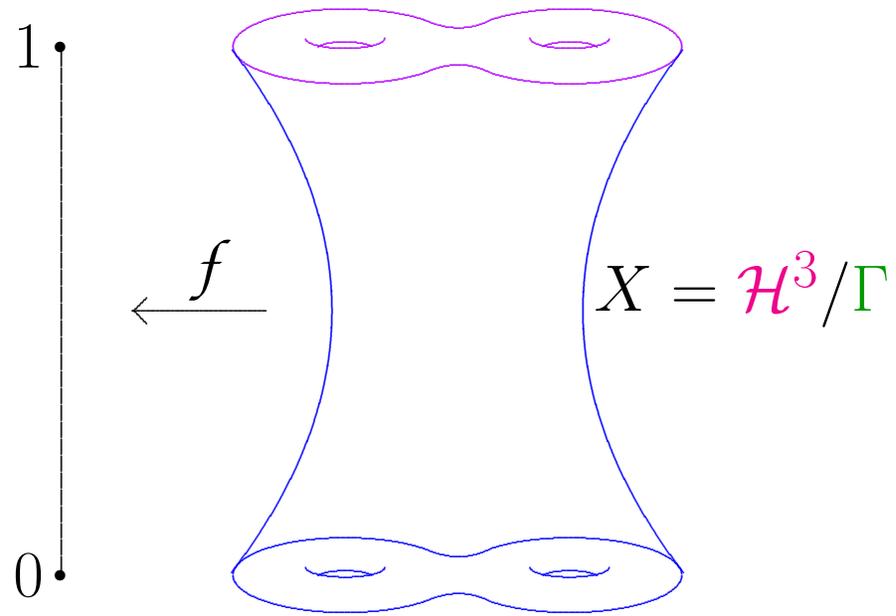
$\Gamma$  Fuchsian



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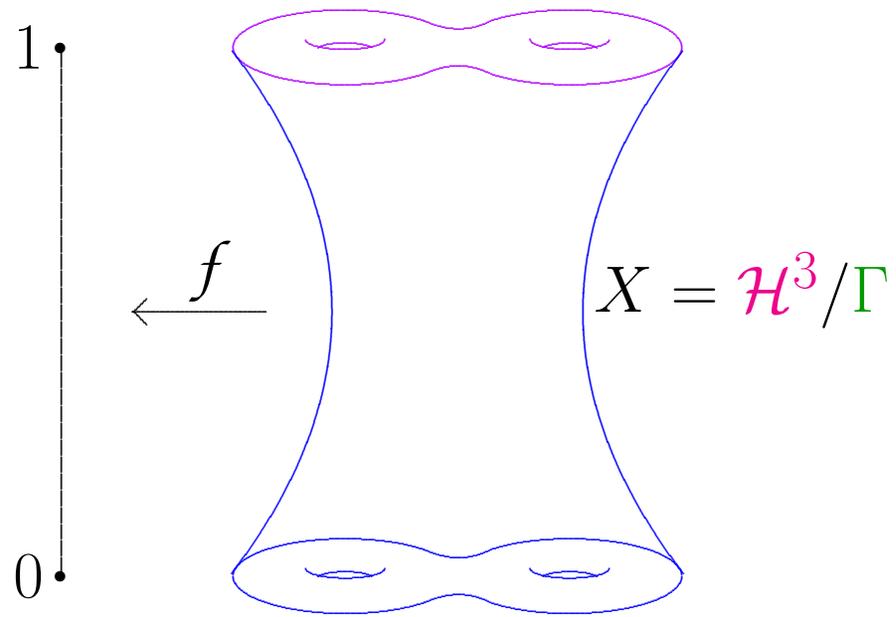
$\Gamma$  Fuchsian

$$X \approx \Sigma \times \mathbb{R}$$



$\Gamma$  Fuchsian

$$\overline{X} \approx \Sigma \times [0, 1]$$

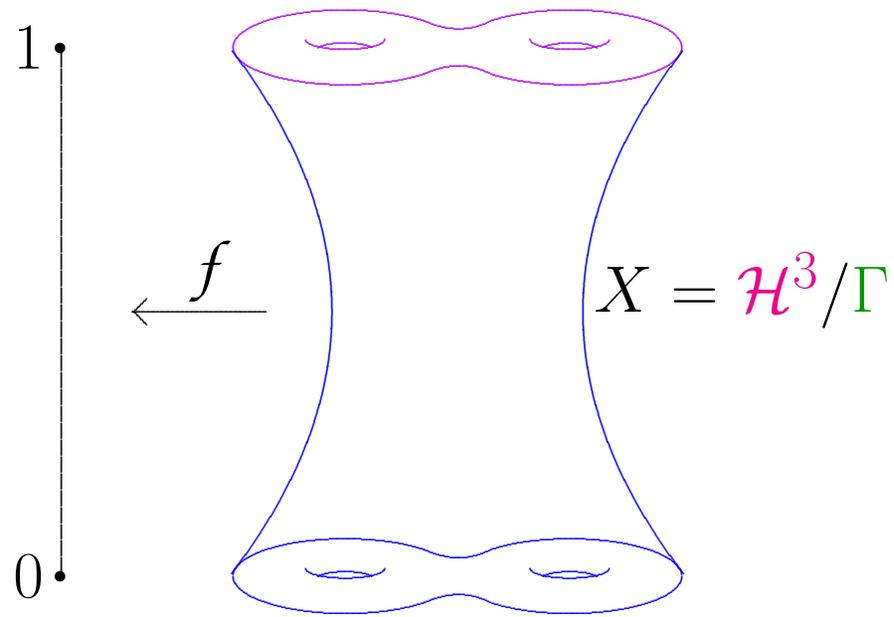


$\Gamma$  Fuchsian

$$\overline{X} \approx \Sigma \times [0, 1]$$

Tunnel-Vision function:

$$f : \overline{X} \rightarrow [0, 1]$$



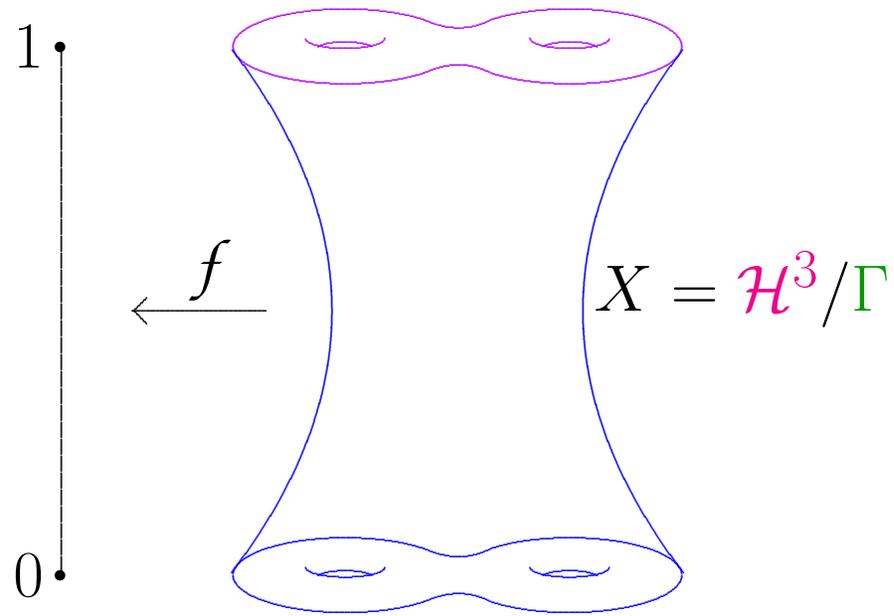
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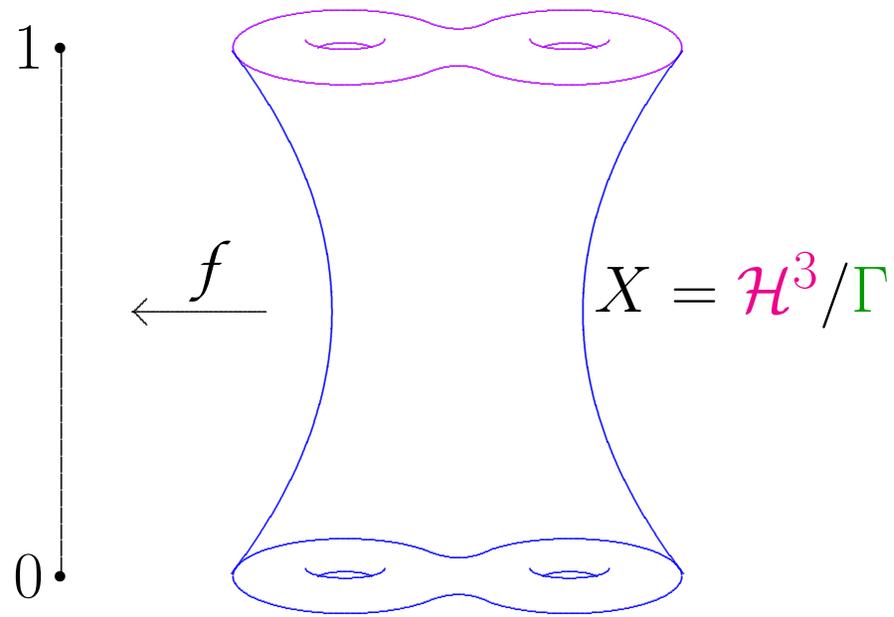
$\Gamma$  Fuchsian

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Tunnel-Vision function:

$$f = \frac{1}{1 + e^{-2t}}$$

$t =$  signed distance from  $\mathcal{H}^2/\Gamma$



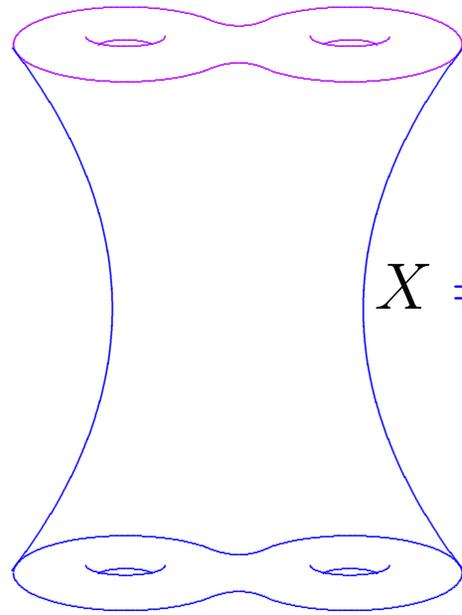
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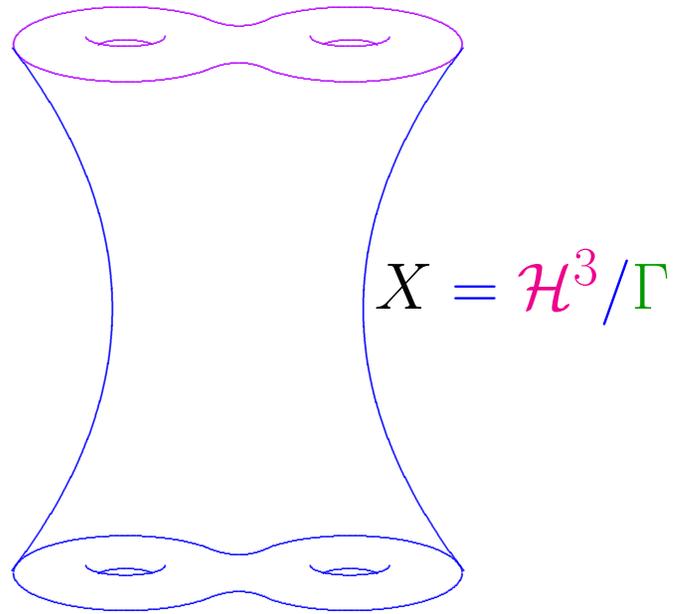
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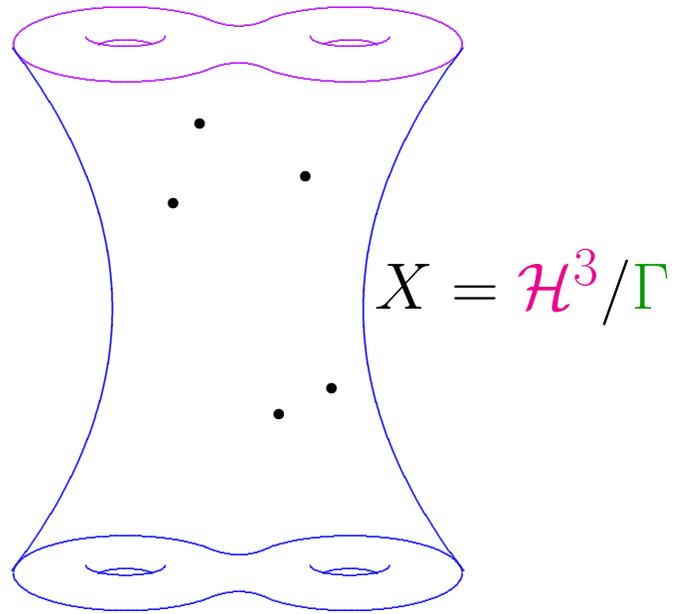
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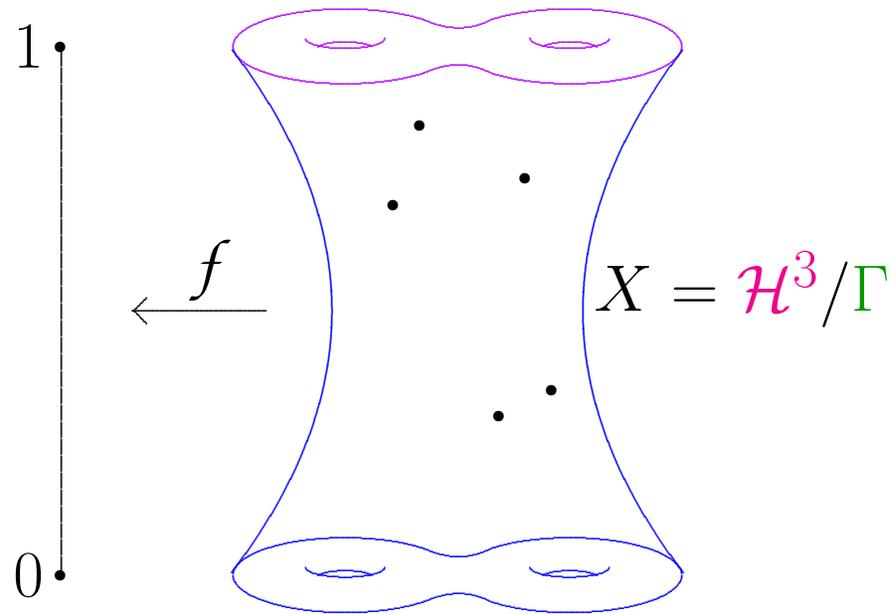


Construction of scalar-flat Kähler manifolds:



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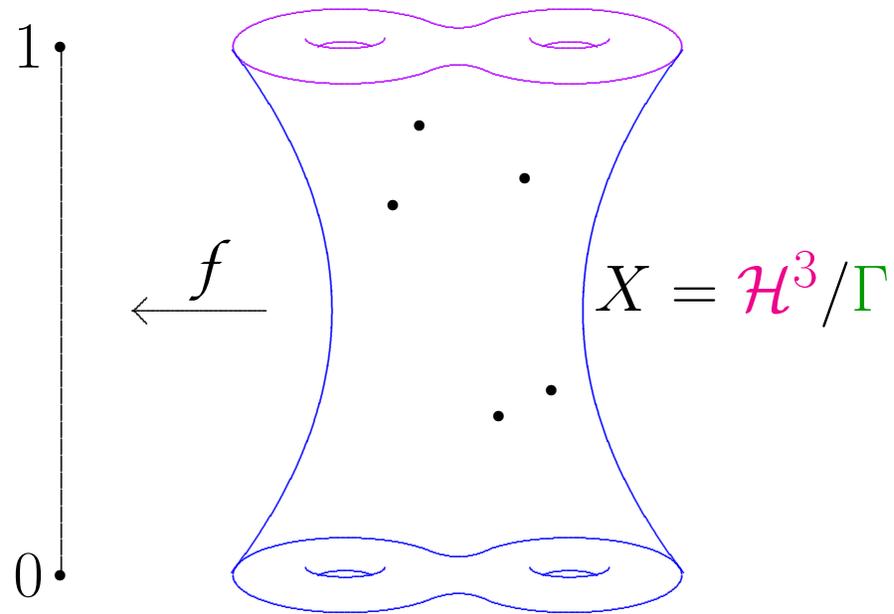
Choose  $k$  points  $p_1, \dots, p_k \in X$



Construction of scalar-flat Kähler manifolds:

Choose  $k$  points  $p_1, \dots, p_k \in X$

satisfying  $\sum_{j=1}^k f(p_j) \in \mathbb{Z}$ .

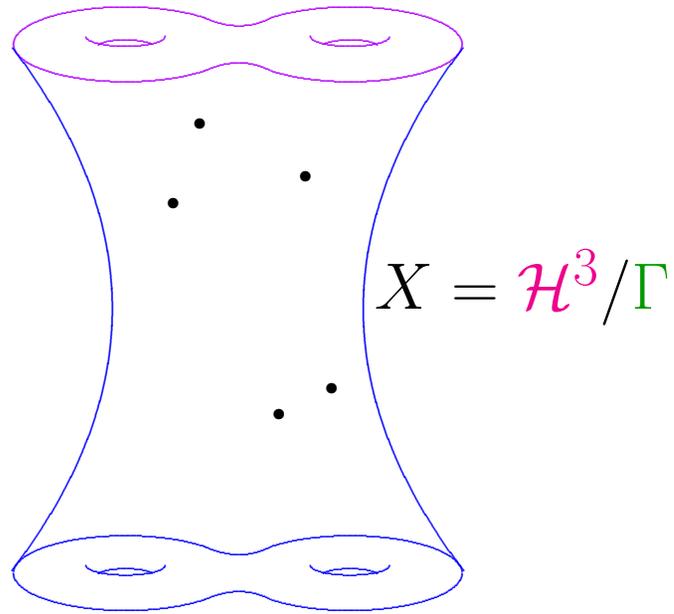


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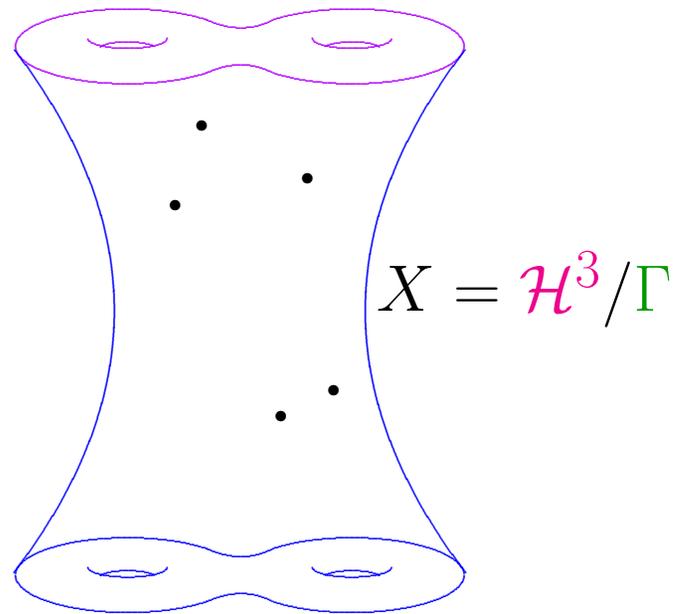
satisfying  $\sum_{j=1}^k f(p_j) \in \mathbb{Z}$ .

Can do if  $k \neq 1$ .



Construction of scalar-flat Kähler manifolds:

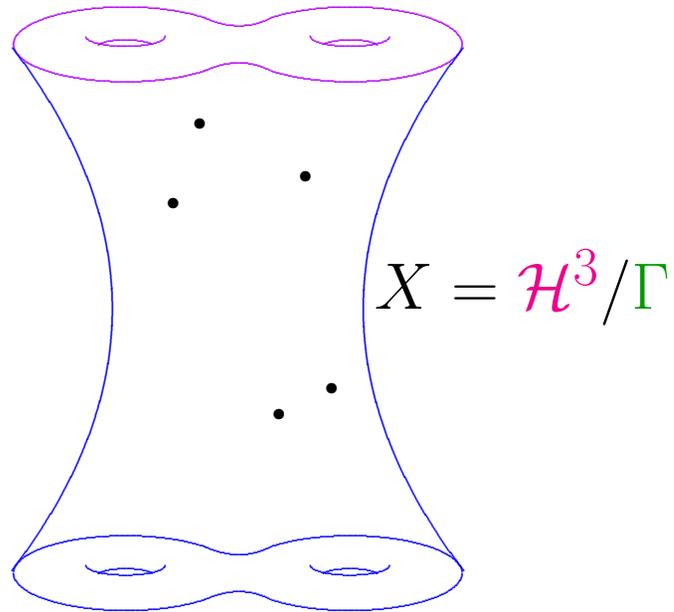
Let  $G_j$  be the Green's function of  $p_j$ :



Construction of scalar-flat Kähler manifolds:

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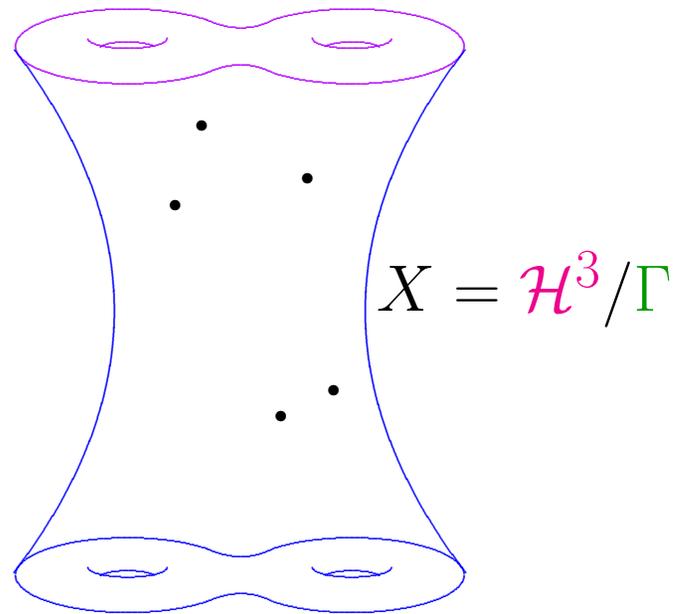
$$\Delta G_j = 2\pi\delta_{p_j}, \quad G_j \rightarrow 0 \text{ at } \partial\bar{X}$$



Construction of scalar-flat Kähler manifolds:

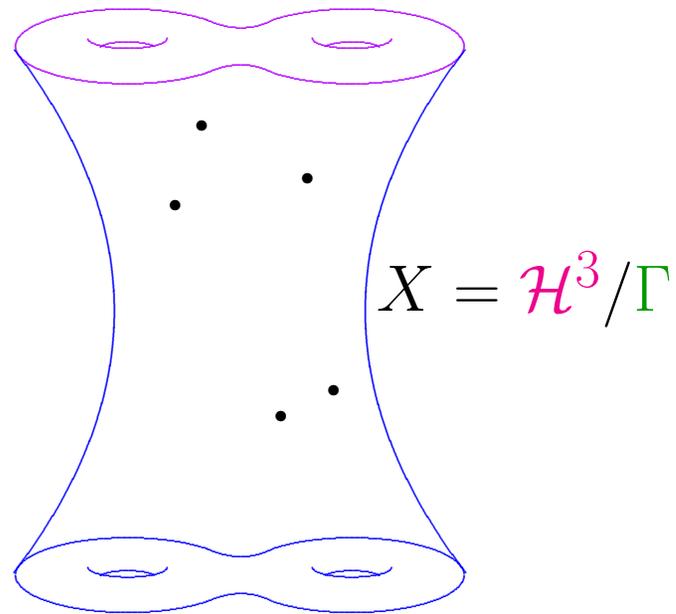
Let  $G_j$  be the Green's function of  $p_j$ , and set

$$V = 1 + \sum_{j=1}^k G_j.$$



Construction of scalar-flat Kähler manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

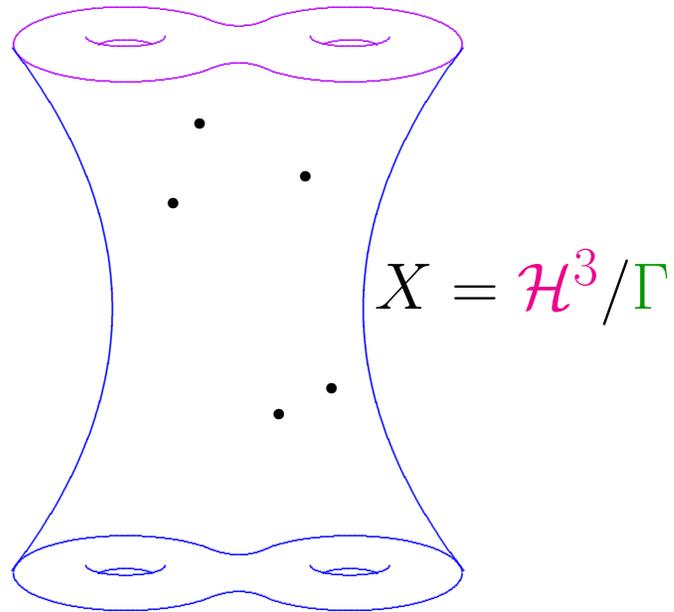


Construction of scalar-flat Kähler manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

Choose  $P \rightarrow (X - \{p_1, \dots, p_k\})$  circle bundle with connection form  $\theta$  such that

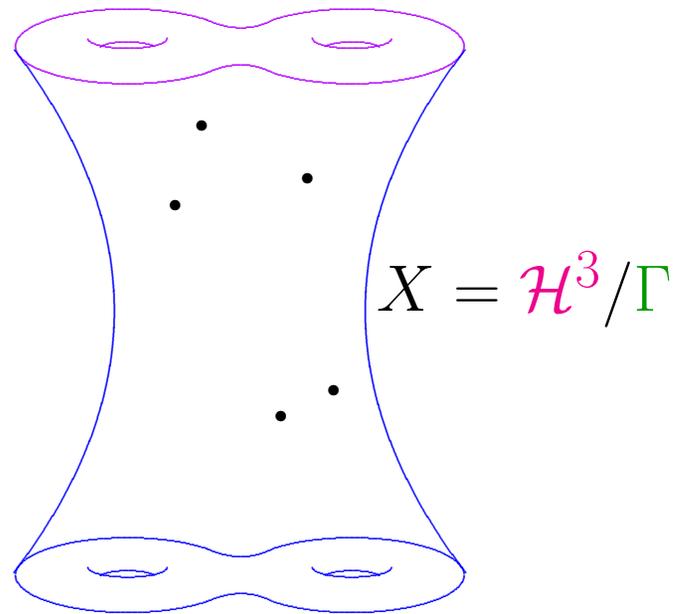
$$d\theta = \star dV.$$



Construction of scalar-flat Kähler manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

$$\sum_{j=1}^k f(p_j) \in \mathbb{Z} \iff \frac{1}{2\pi} [\star dV] \in H^2(X - \{p_j\}, \mathbb{Z})$$

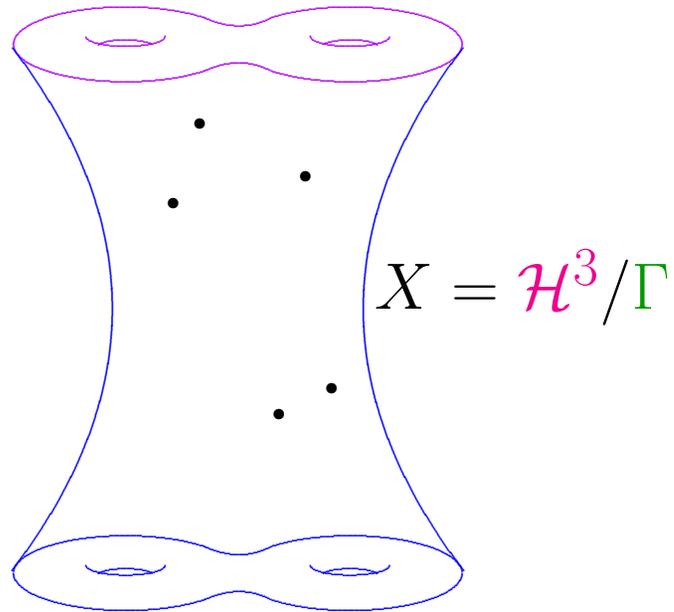


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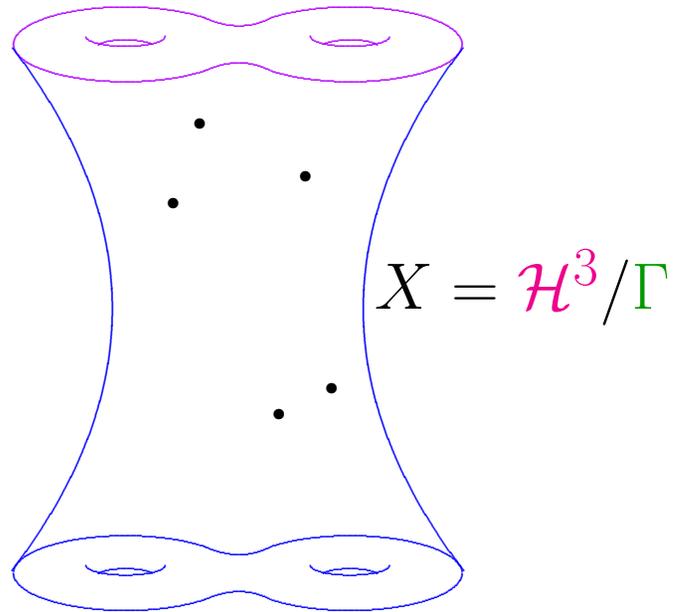


Construction of scalar-flat Kähler manifolds:

$$g = Vh + V^{-1}\theta^2$$

$$V = 1 + \sum_{j=1}^k G_j$$

$$d\theta = \star dV$$

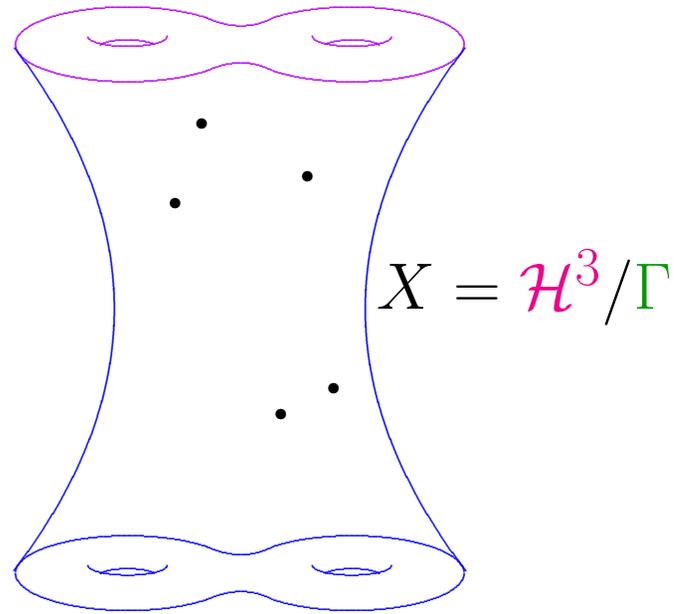


Construction of scalar-flat Kähler manifolds:

$$g = f(1 - f)[Vh + V^{-1}\theta^2]$$

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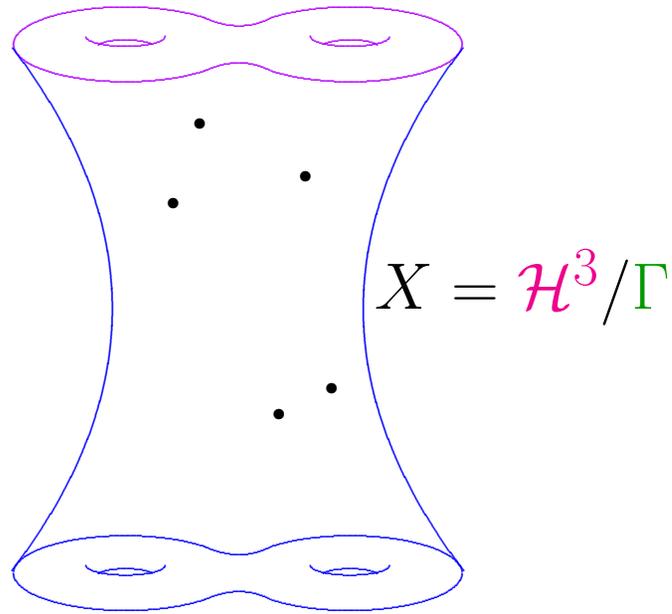
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Construction of scalar-flat Kähler manifolds:

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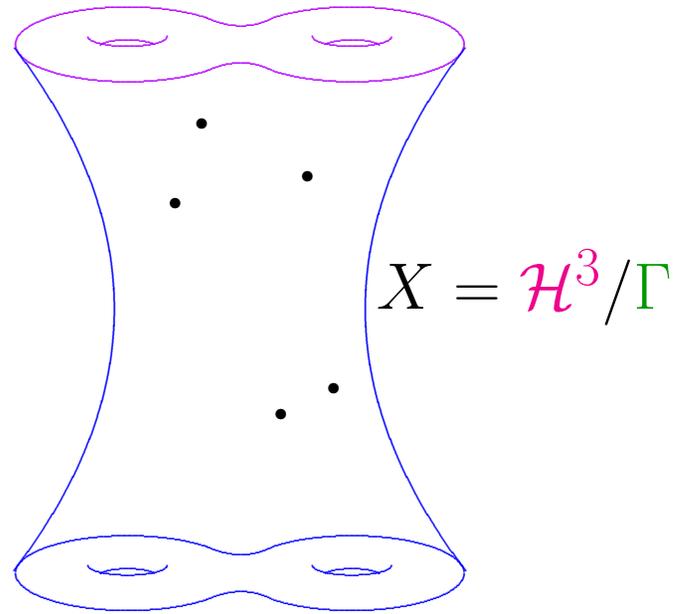
$$M = P \cup \{\hat{p}_1, \dots, \hat{p}_k\} \cup \partial\bar{X}$$



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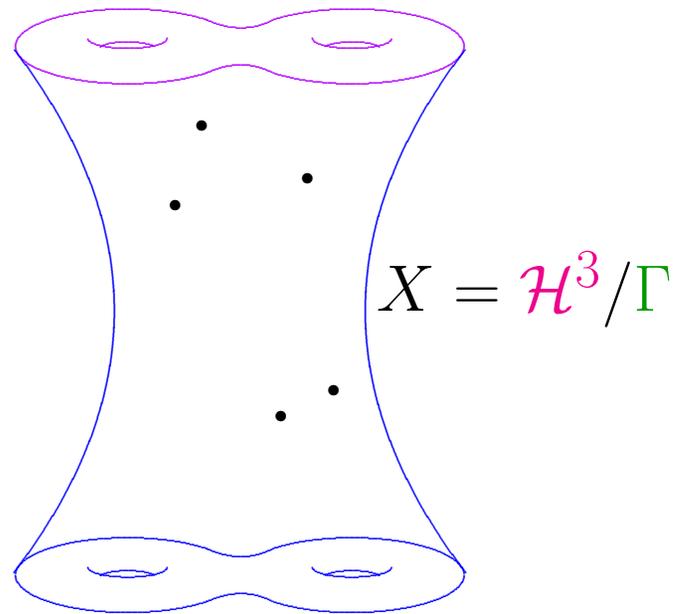
$$\begin{array}{ccccccc}
 M & = & P & \cup & \{\hat{p}_1, \dots, \hat{p}_k\} & \cup & \partial \bar{X} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \bar{X} & = & X - \{p_1, \dots, p_k\} & \cup & \{p_1, \dots, p_k\} & \cup & \partial \bar{X}
 \end{array}$$



Construction of scalar-flat Kähler manifolds:

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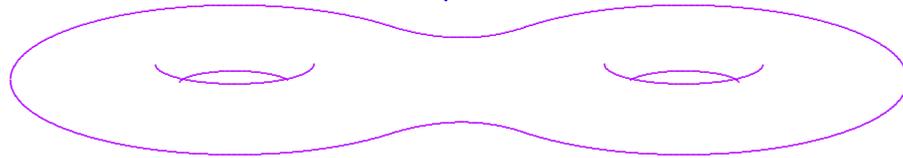
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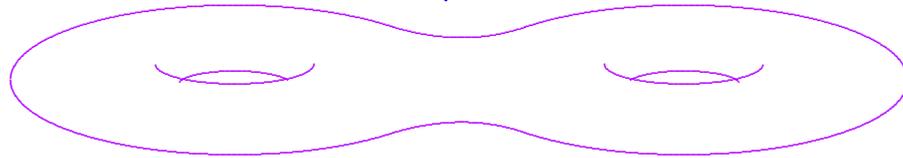
$$\approx (\Sigma \times S^2) \# k\bar{\mathbb{C}P}_2$$

$$\overline{X} = \Sigma \times [0, 1]$$



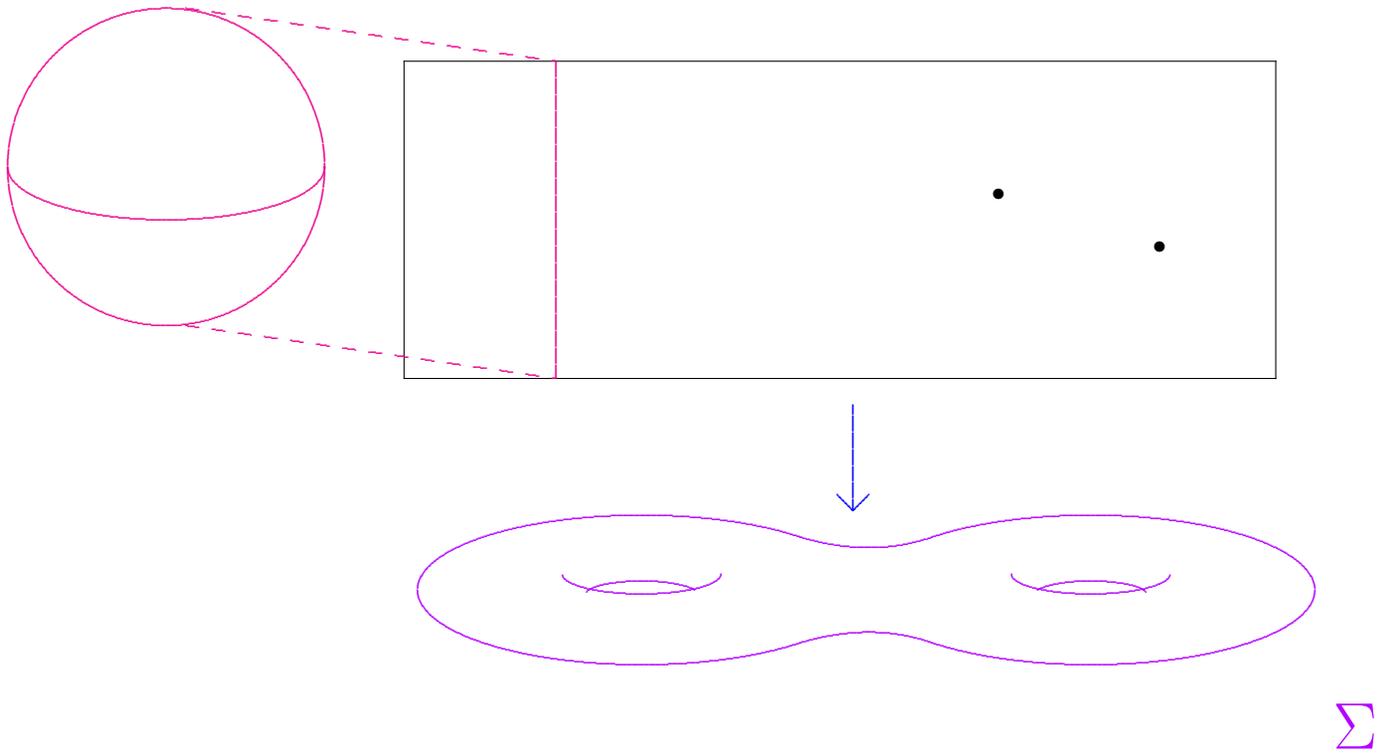
$\Sigma$

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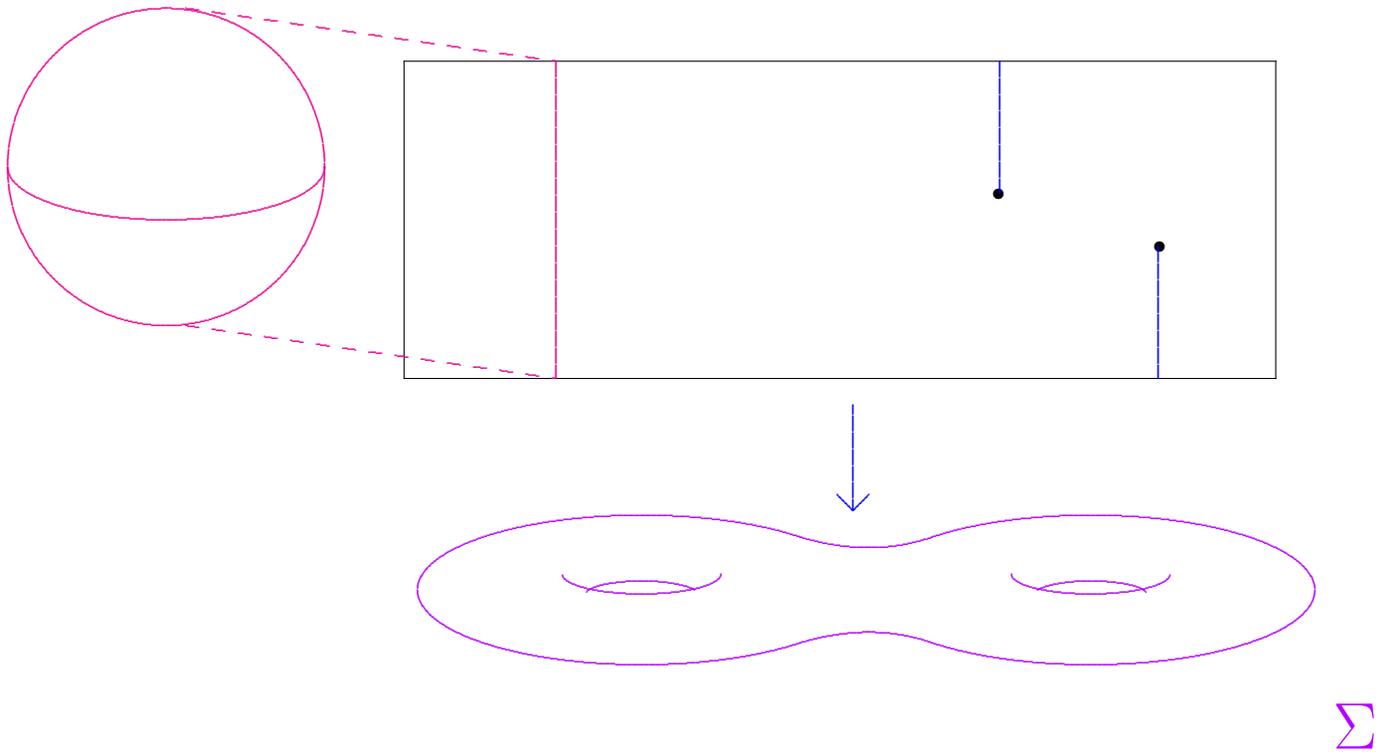


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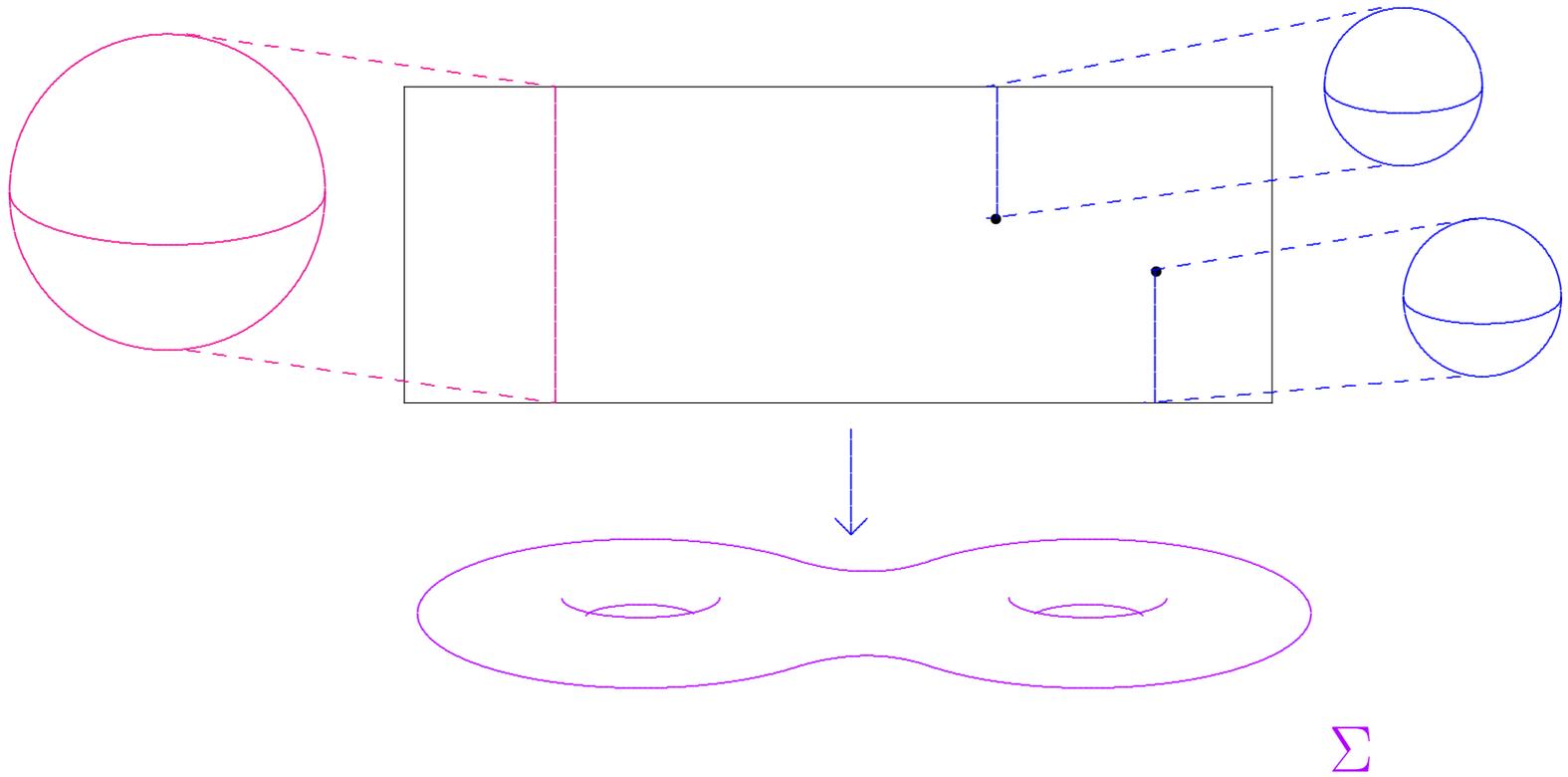
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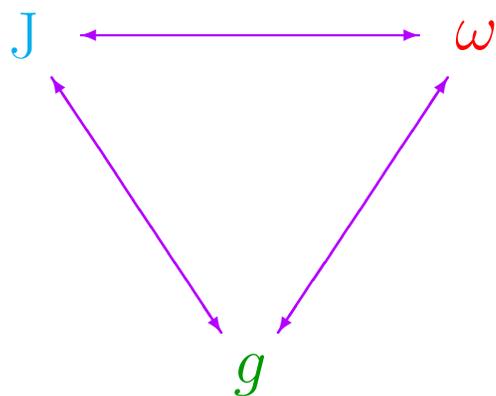
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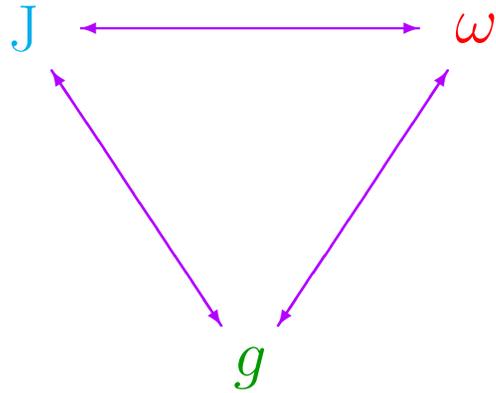
Recently explored in Bishop-L '20.

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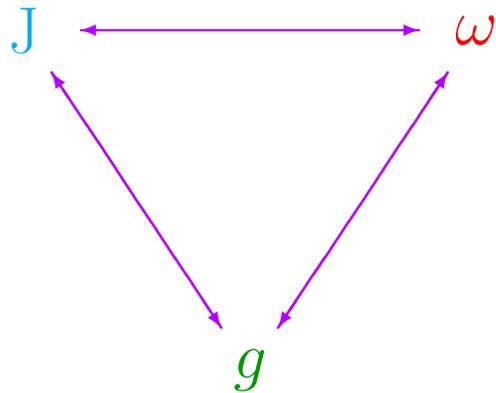


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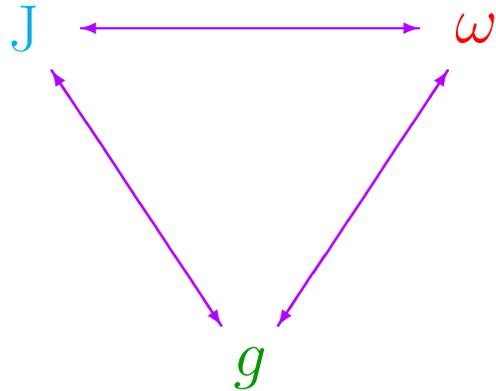


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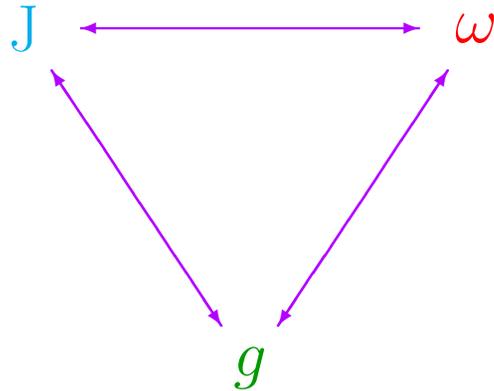
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Open condition on  $[g]$ , in  $C^2$  topology.

# Simultaneous Uniformization (Bers '60)

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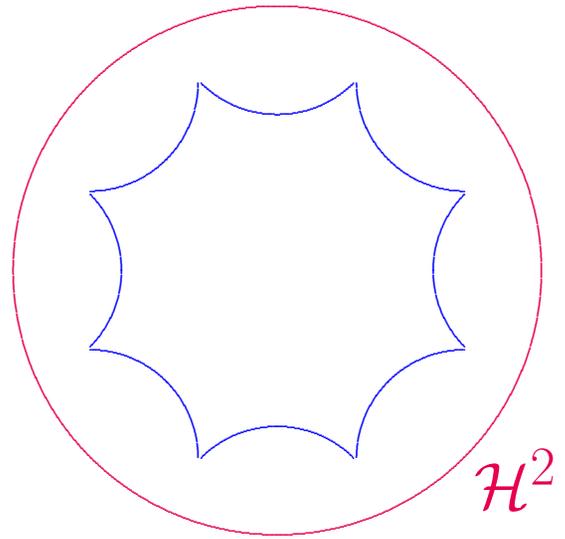
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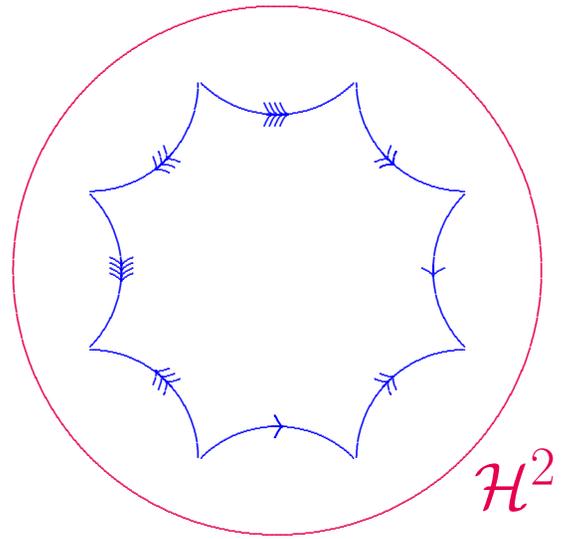
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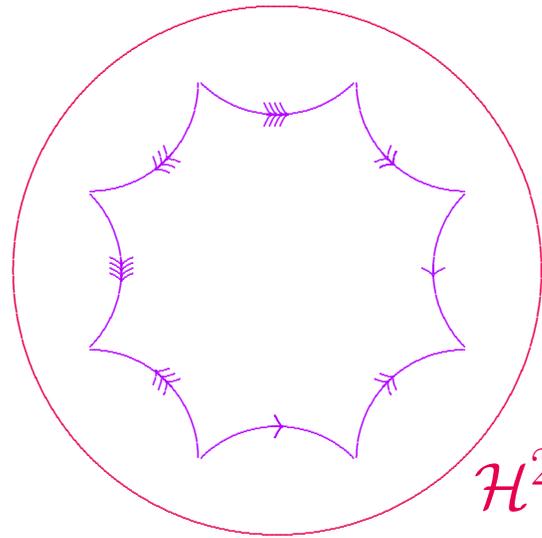
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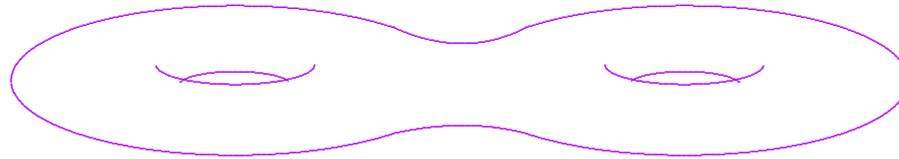
$\pi_1(\Sigma) \subset \mathbf{PSL}(2, \mathbb{C})$  called a quasi-Fuchsian group.

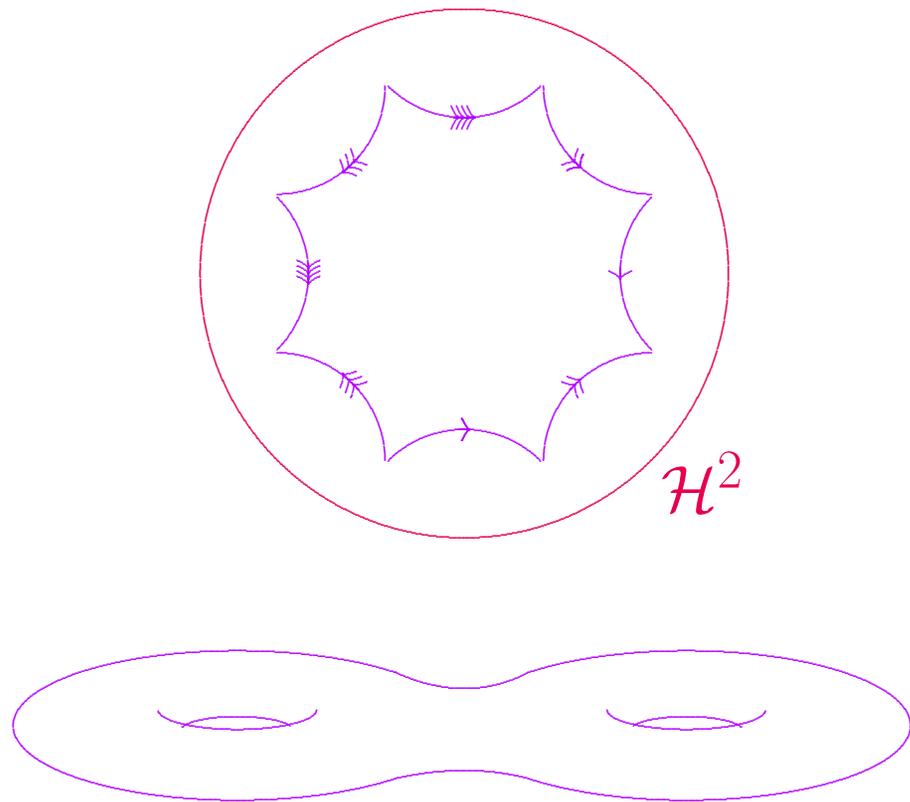




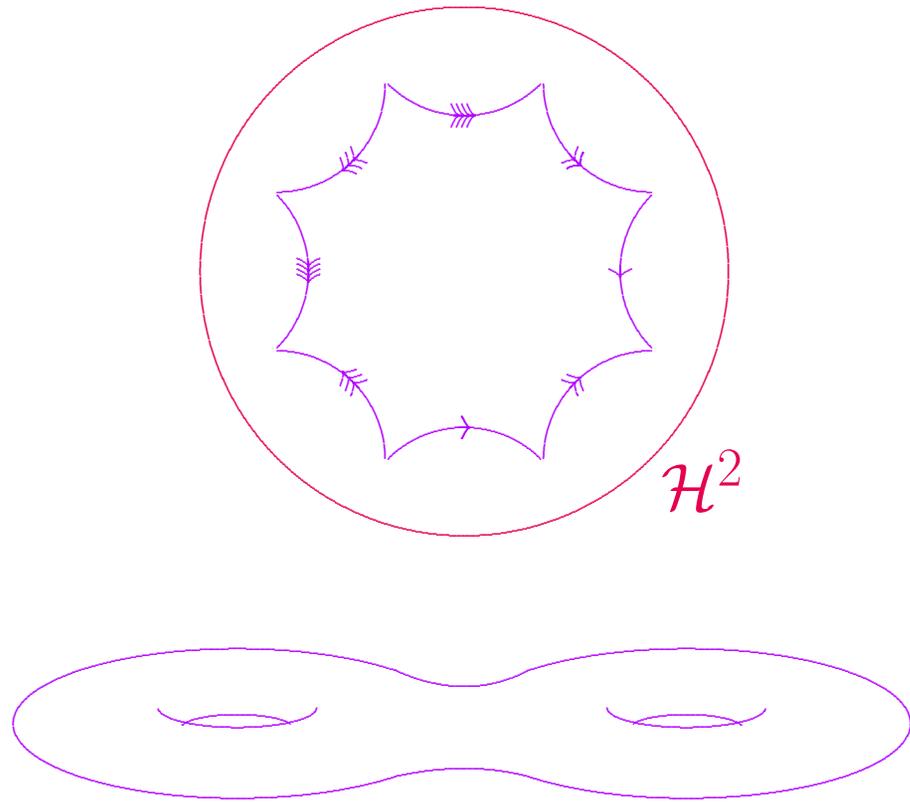


$\mathcal{H}^2$

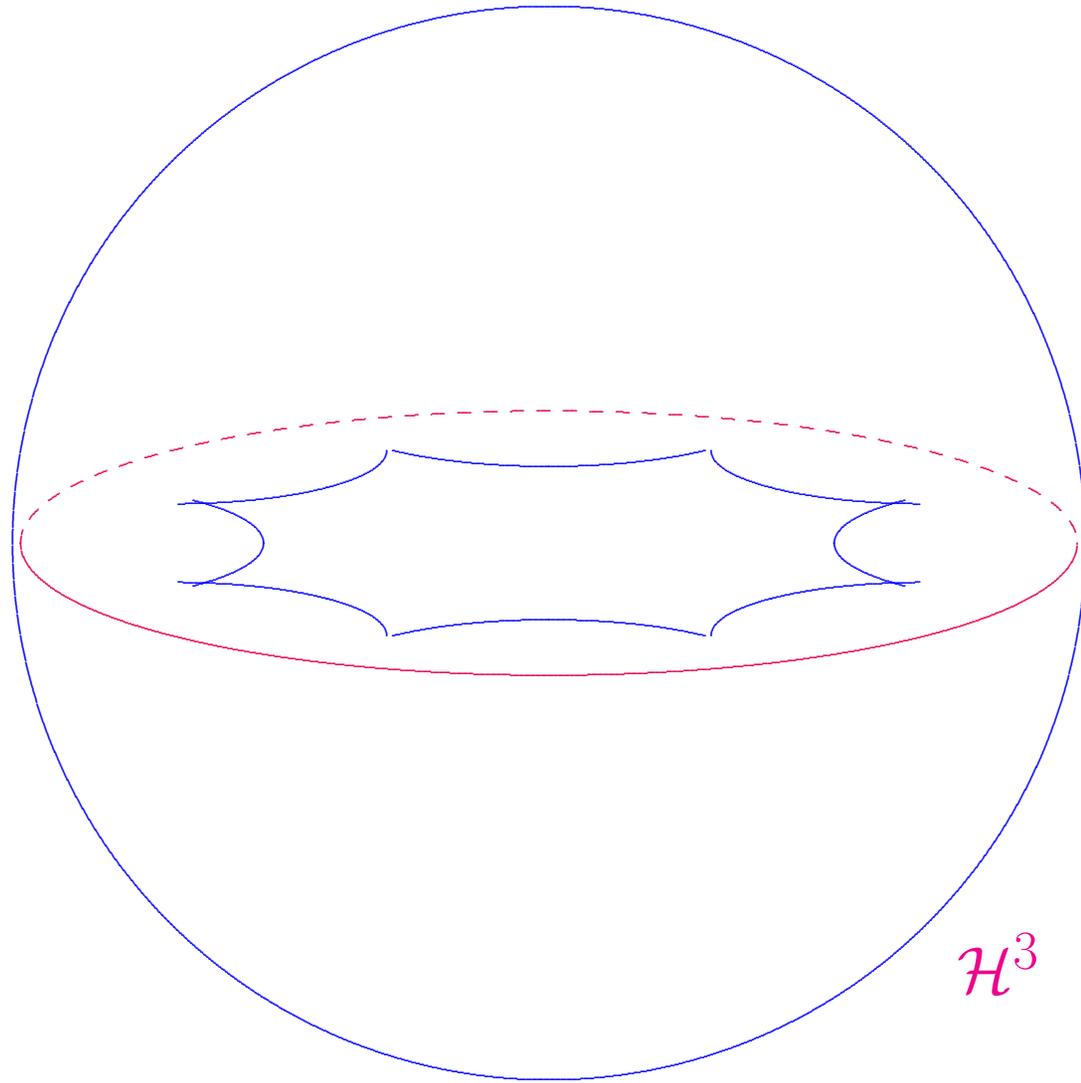




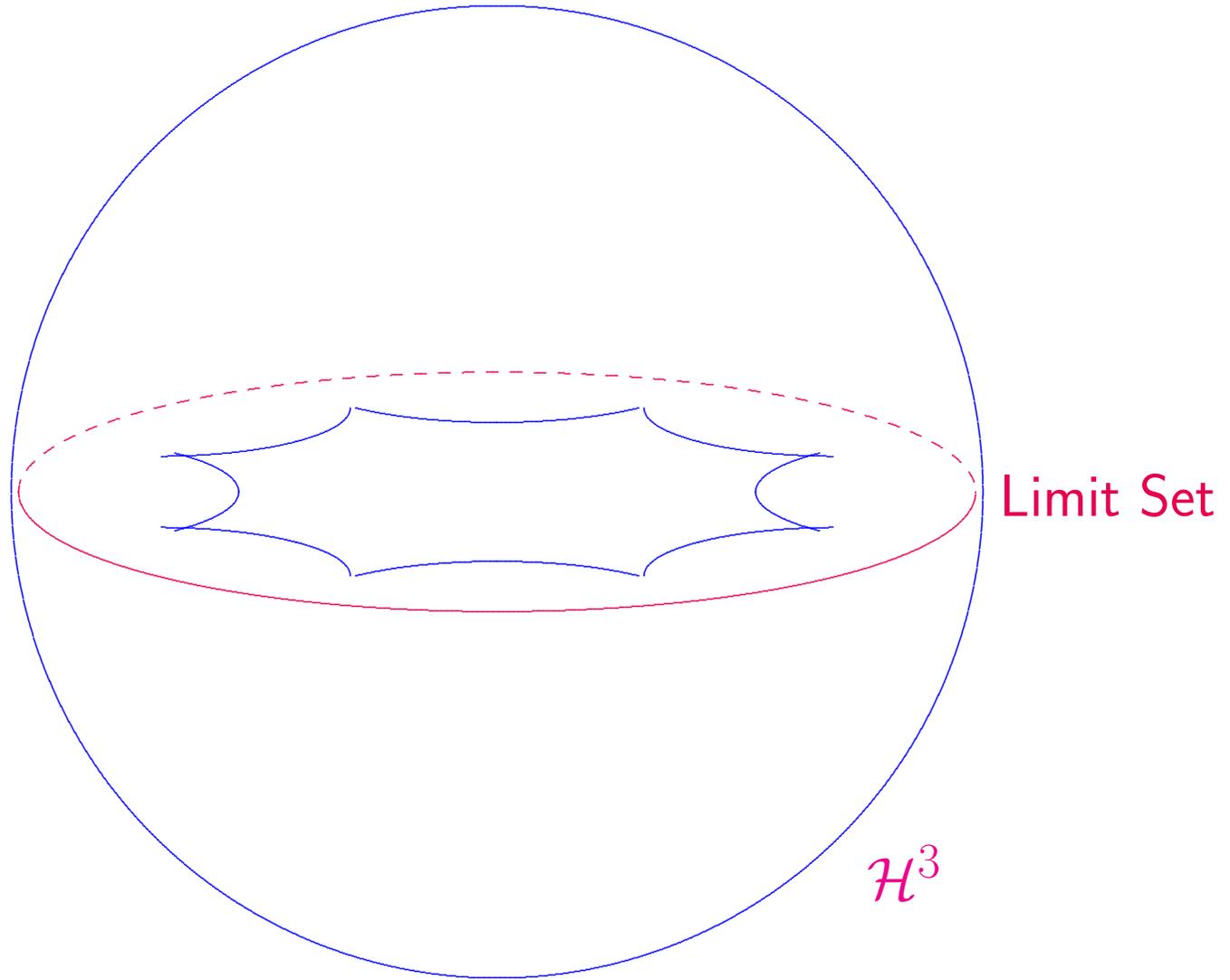
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R})$$

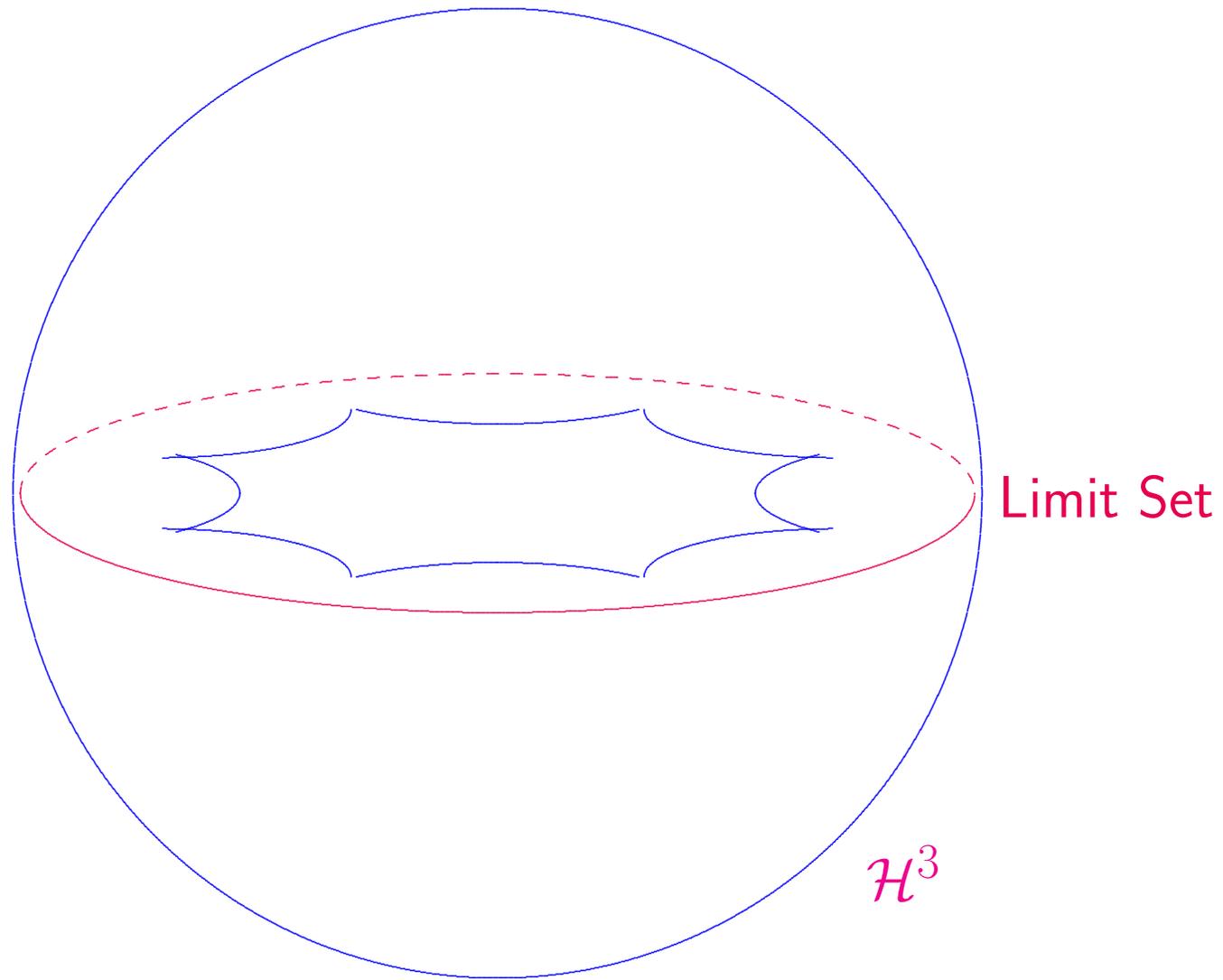


$$\begin{array}{ccc}
 \pi_1(\Sigma) & \hookrightarrow & \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R}) \\
 & & \cap \qquad \qquad \cap \\
 & & \mathbf{SO}_+(1, 3) = \mathbf{PSL}(2, \mathbb{C})
 \end{array}$$

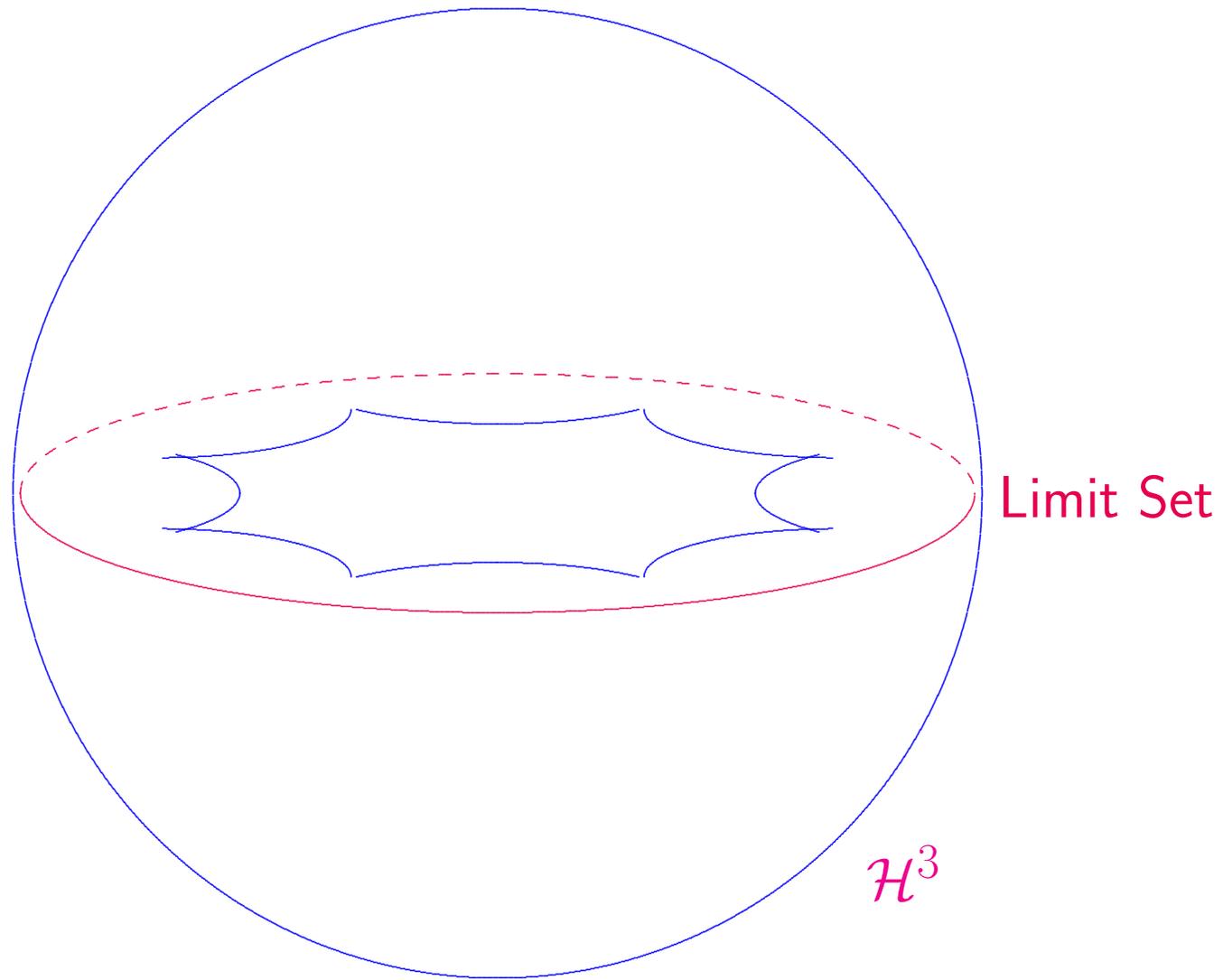


$\mathcal{H}^3$

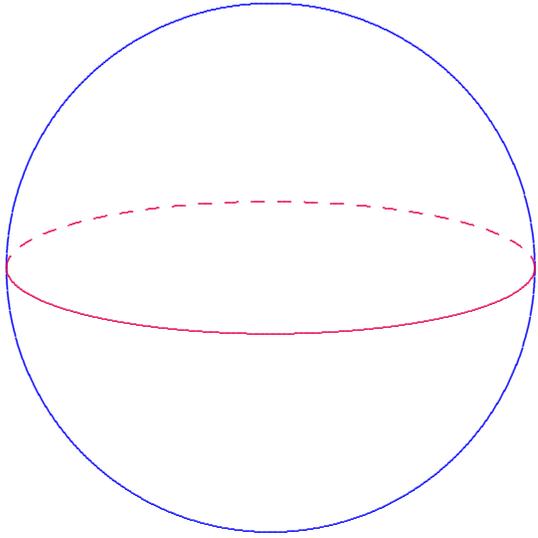


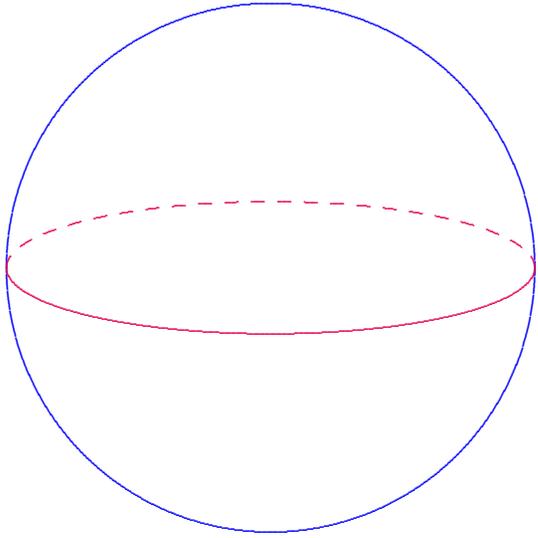


$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{R}) \text{ Fuchsian group}$$

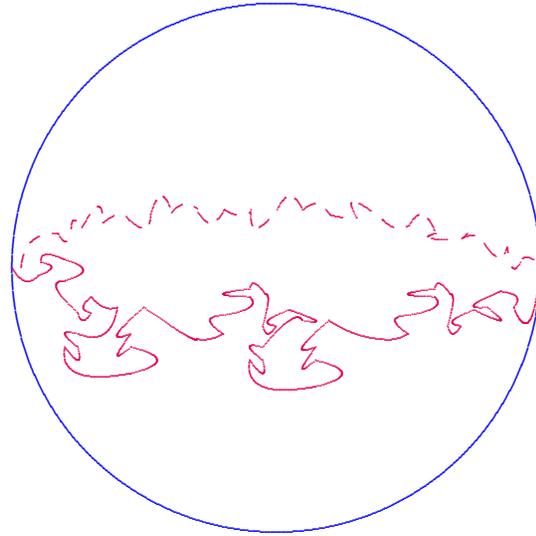
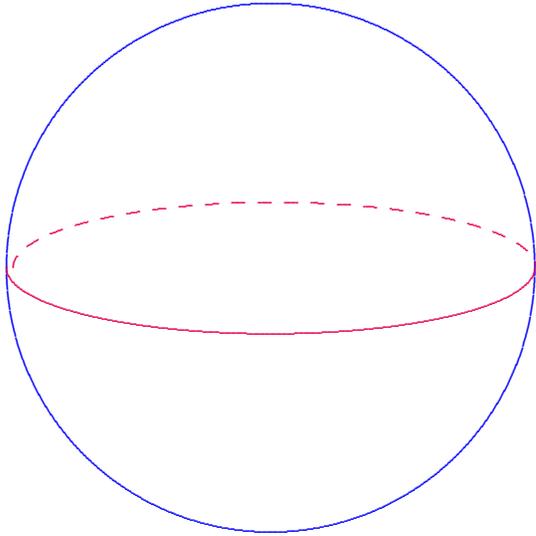


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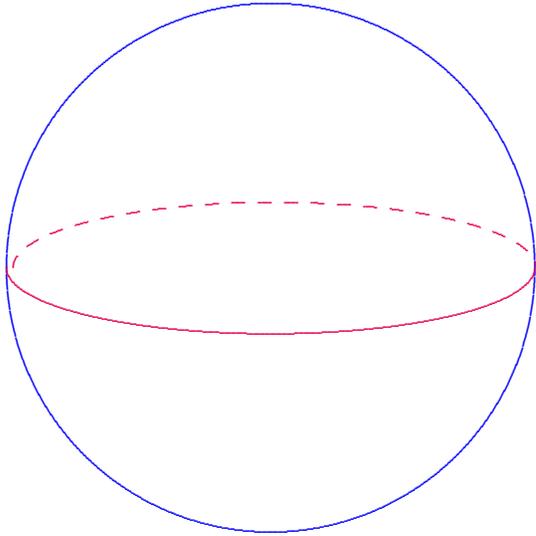




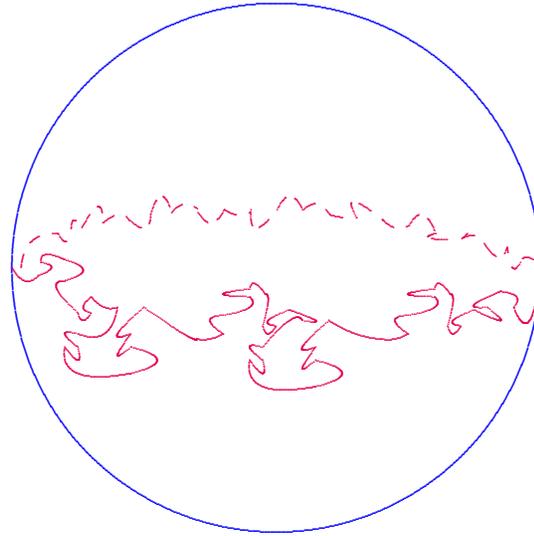
Fuchsian



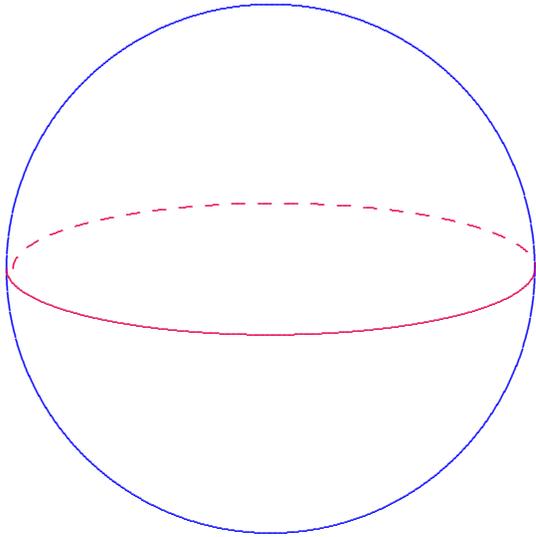
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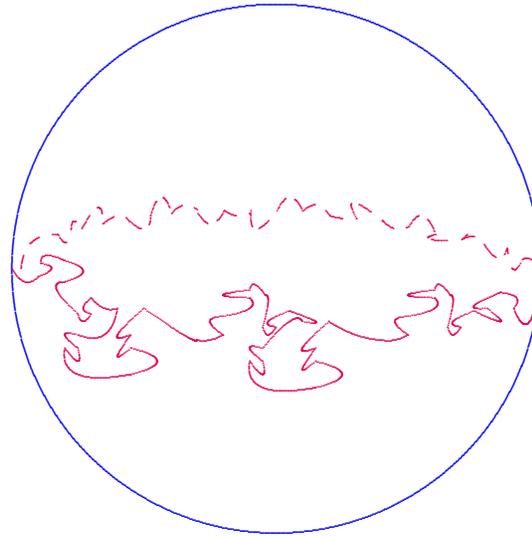
Fuchsian



quasi-Fuchsian

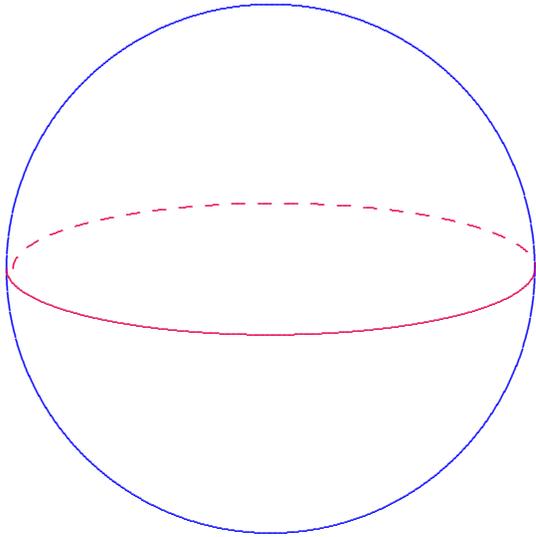


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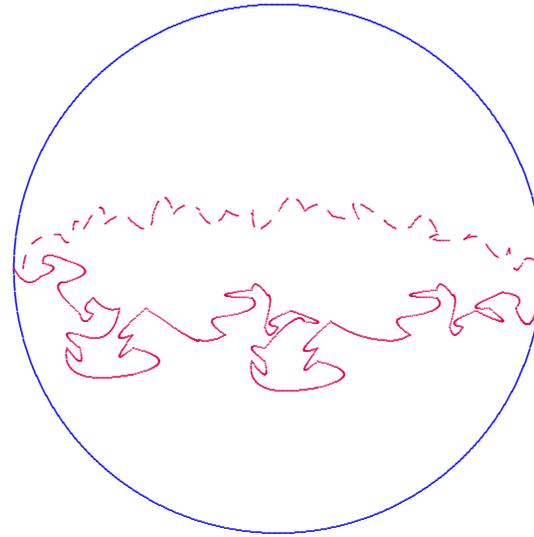


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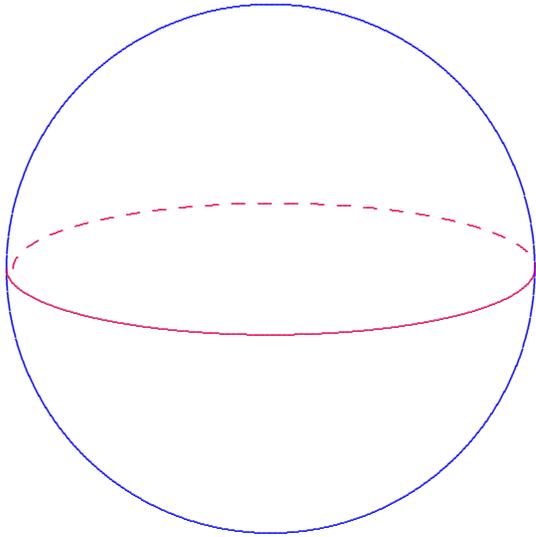


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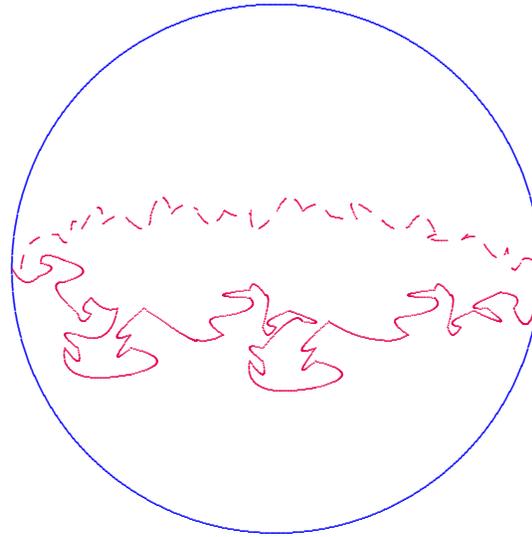


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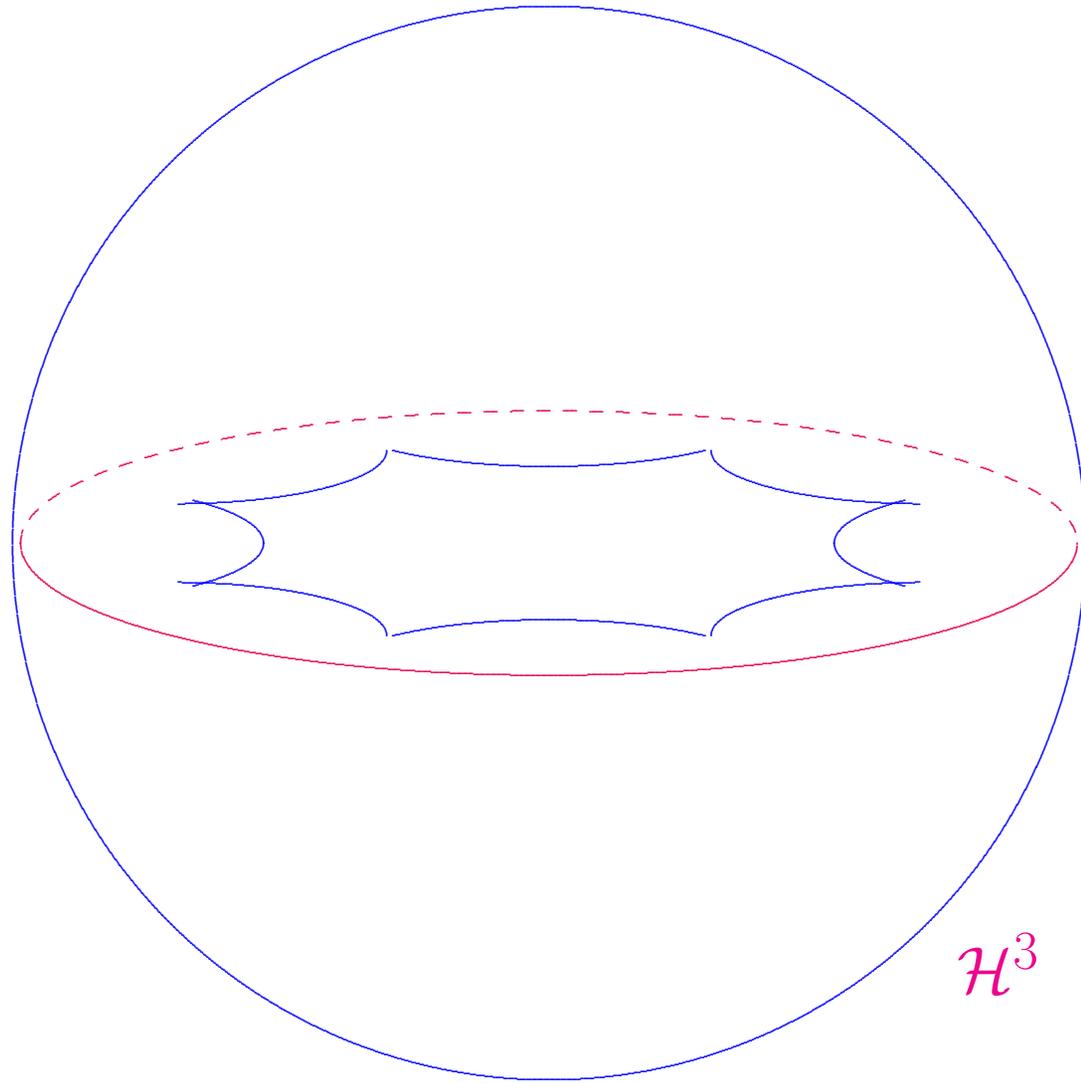
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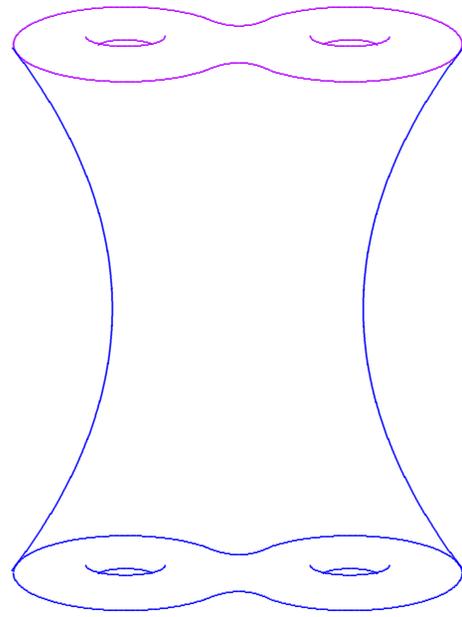
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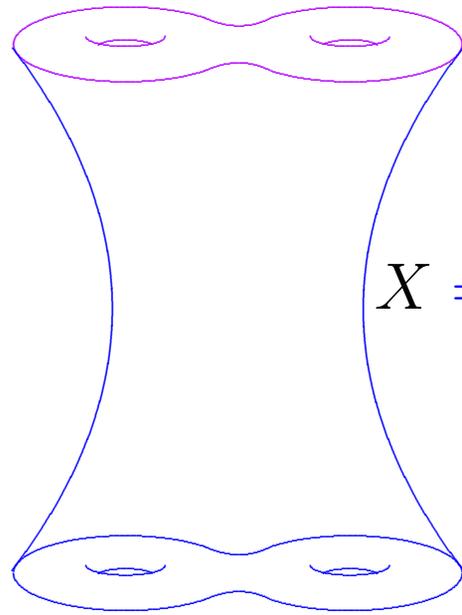
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Quasi-conformally conjugate to Fuchsian.

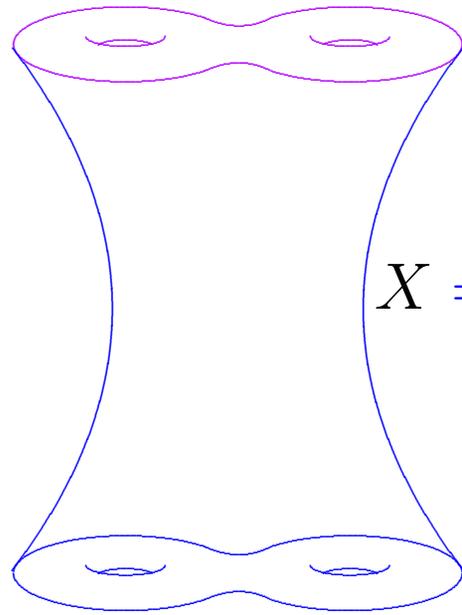


$\mathcal{H}^3$



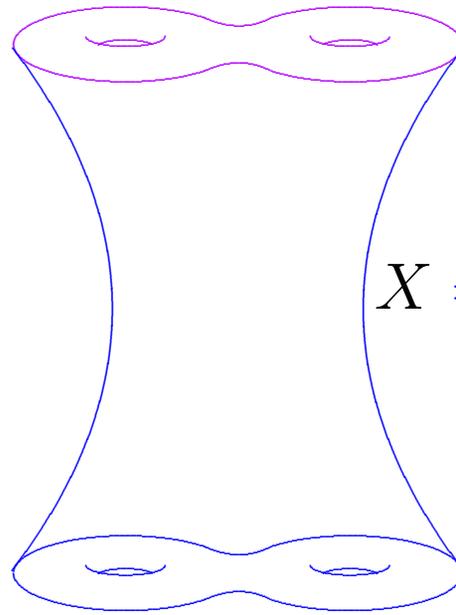


$$X = \mathcal{H}^3 / \Gamma$$



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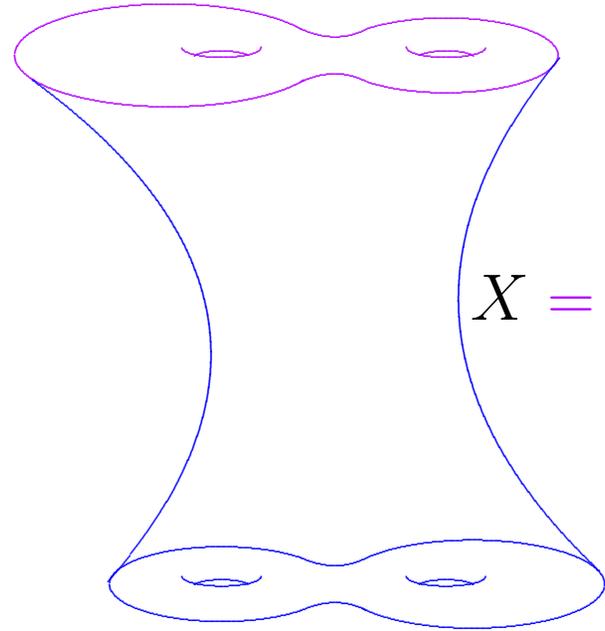
$\Gamma$  Fuchsian



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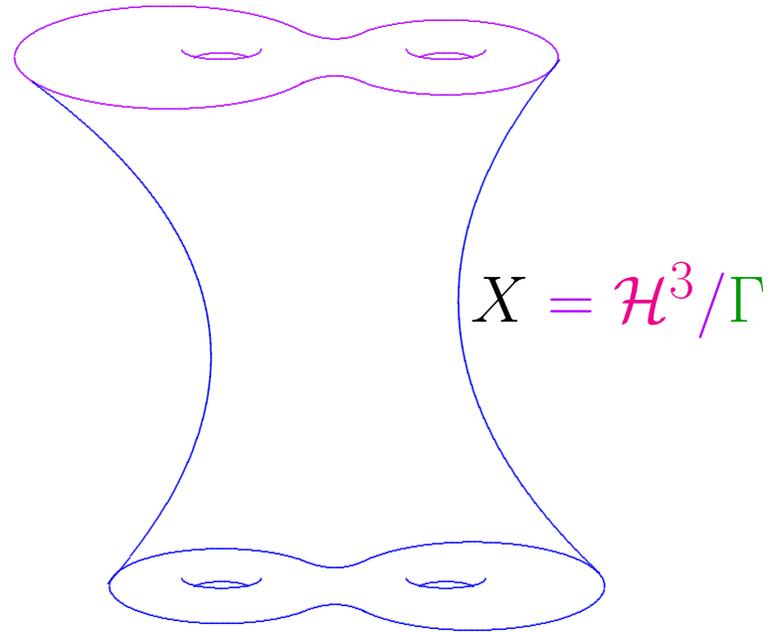
$$X \approx \Sigma \times \mathbb{R}$$



$$X = \mathcal{H}^3 / \Gamma$$

$\Gamma$  quasi-Fuchsian

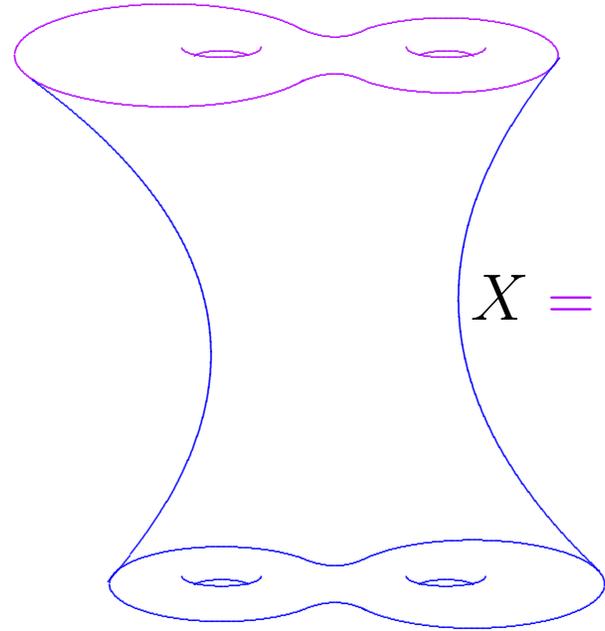
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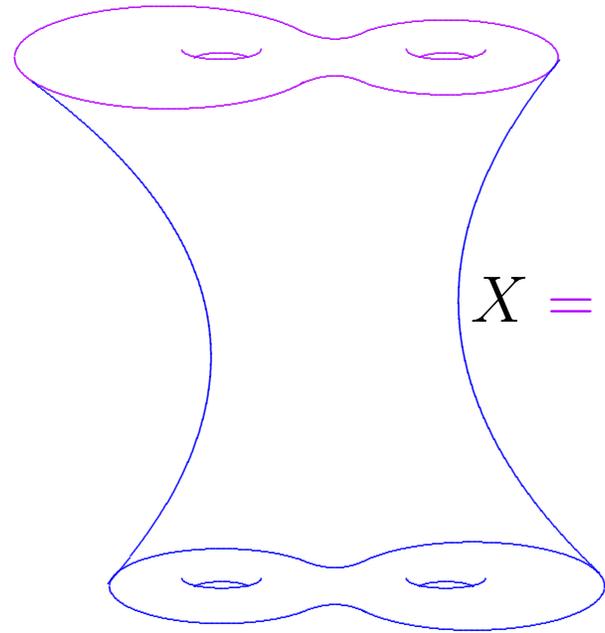
Freedom: two points in Teichmüller space.



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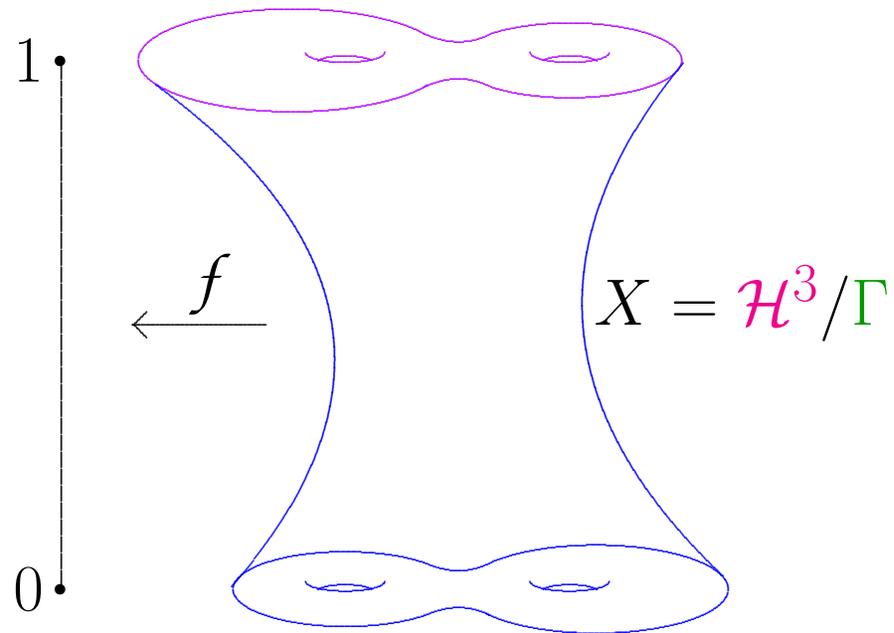
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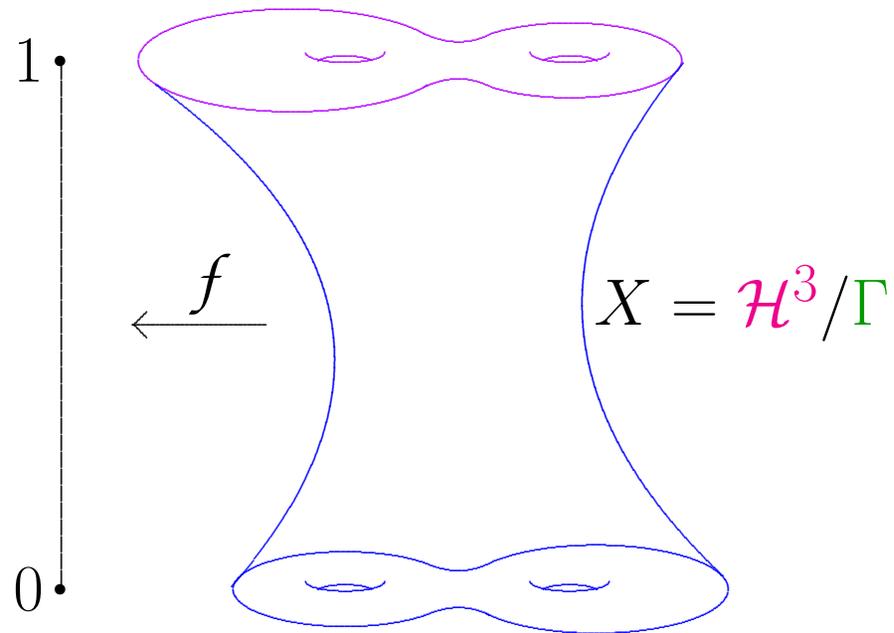


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Tunnel-Vision function:

$$f : \overline{X} \rightarrow [0, 1]$$



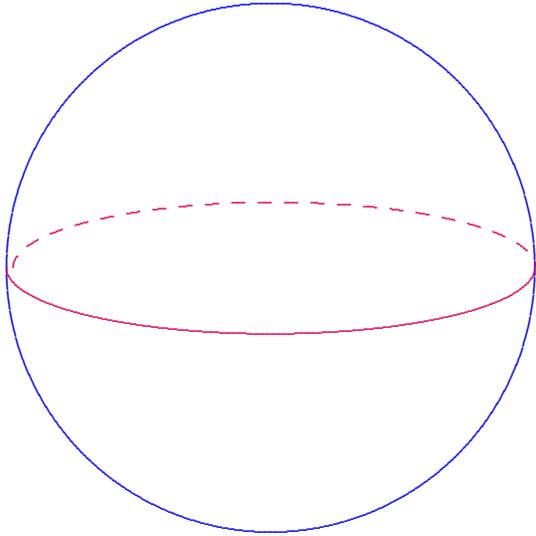
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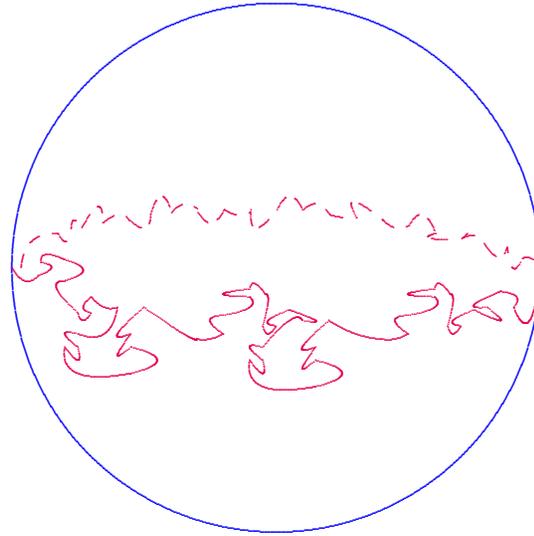
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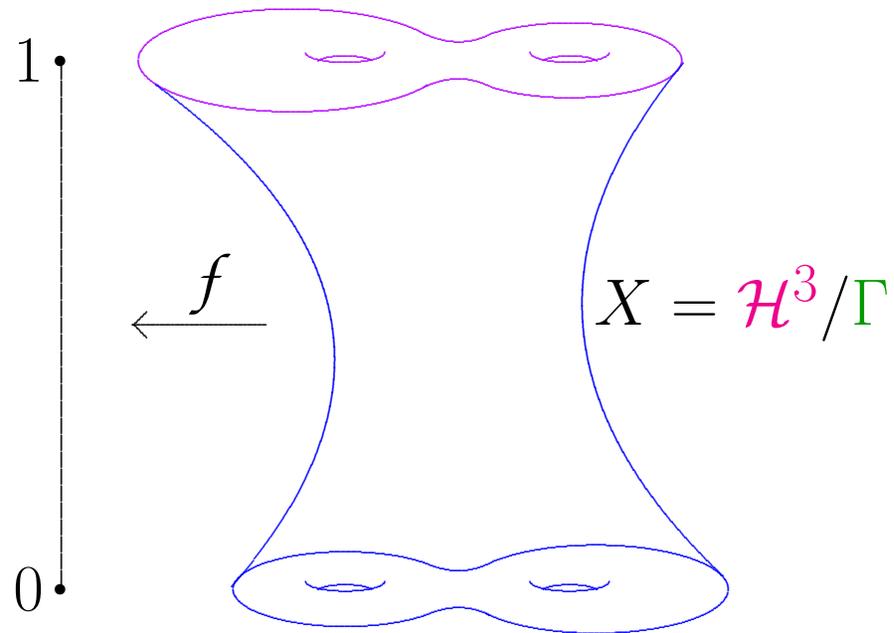
$$\Delta f = 0$$



Fuchsian



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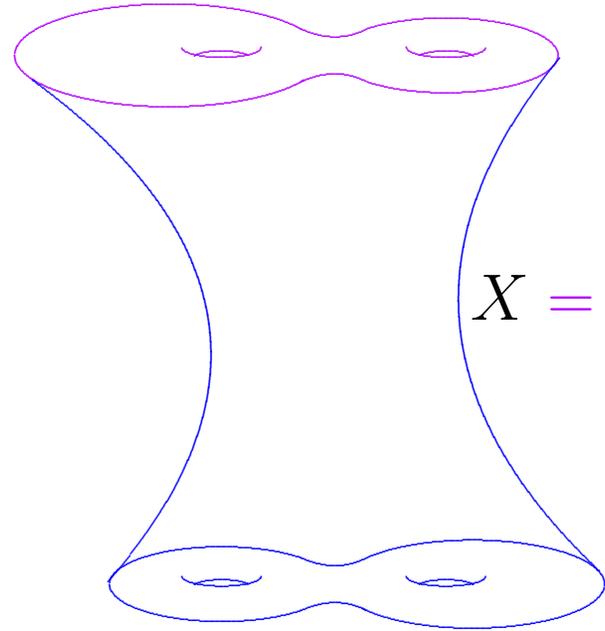
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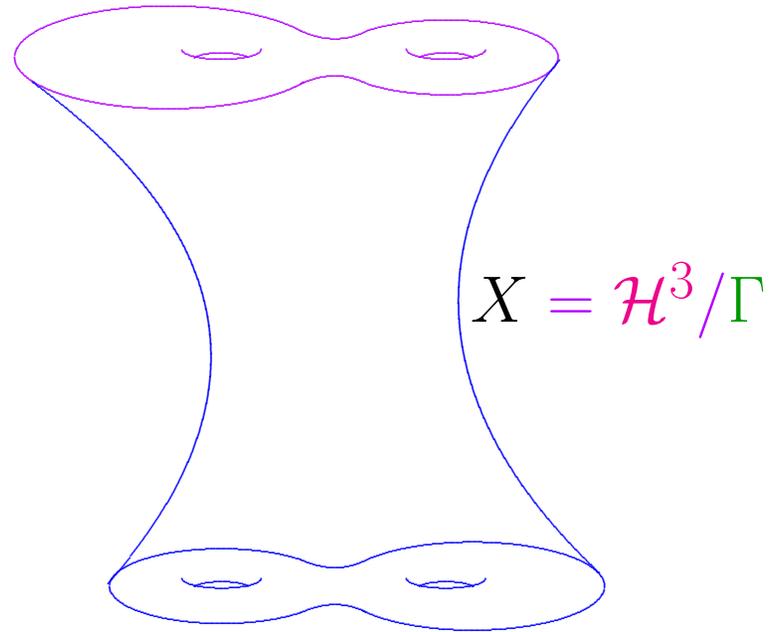
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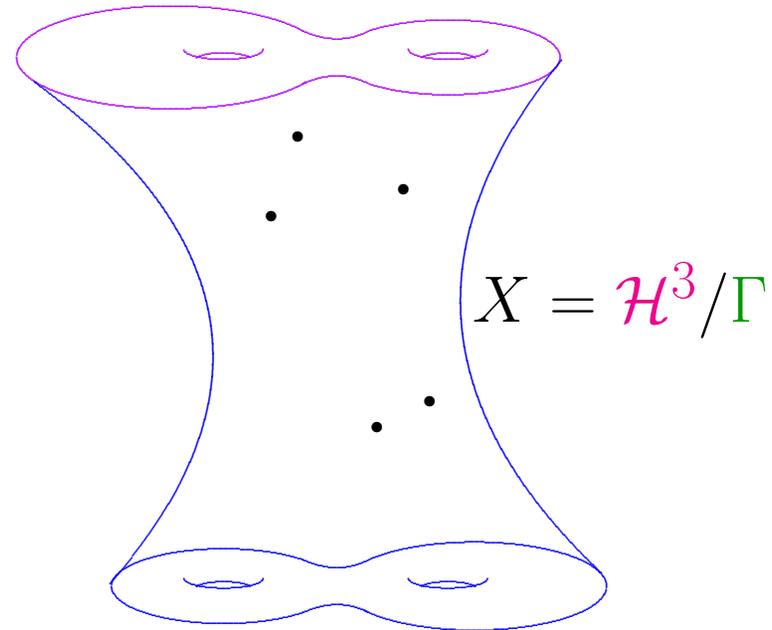
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$$X = \mathcal{H}^3 / \Gamma$$

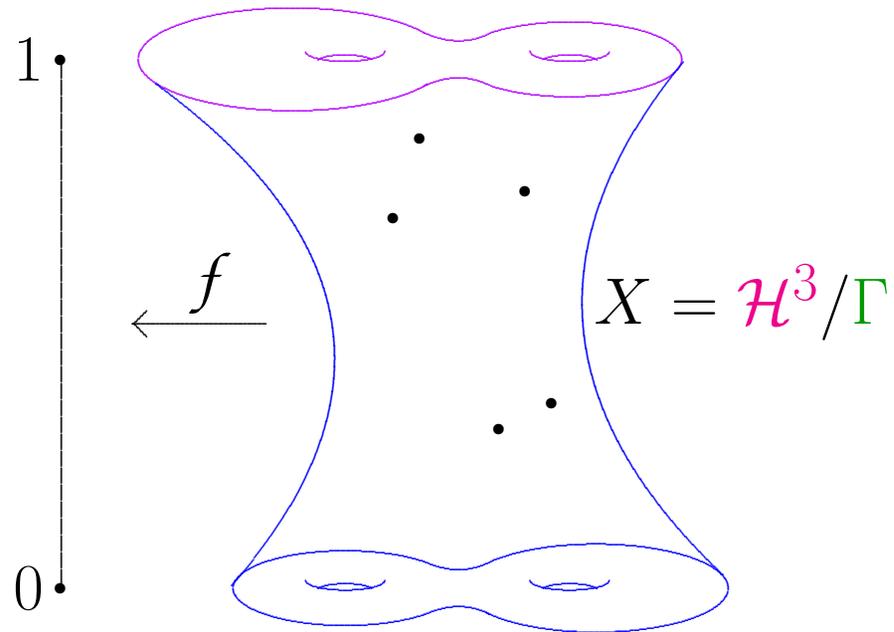


Construction of ASD 4-manifolds:



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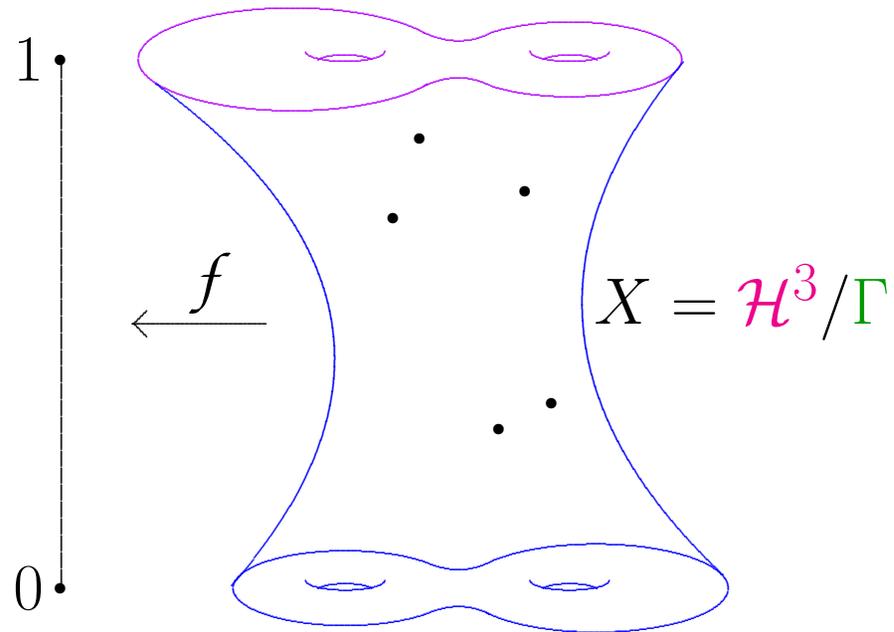
Choose  $k$  points  $p_1, \dots, p_k \in X$



Construction of ASD 4-manifolds:

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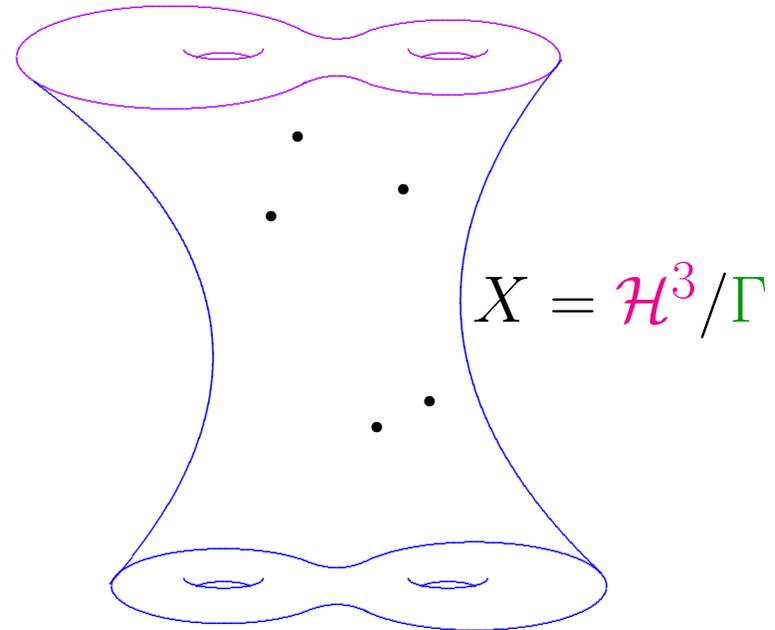


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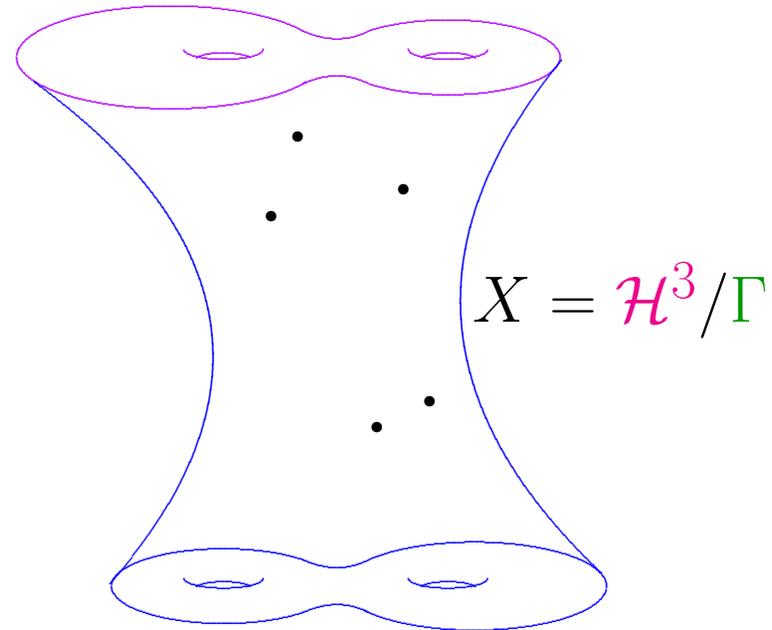
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Can do if  $k \neq 1$ .



Construction of ASD 4-manifolds:

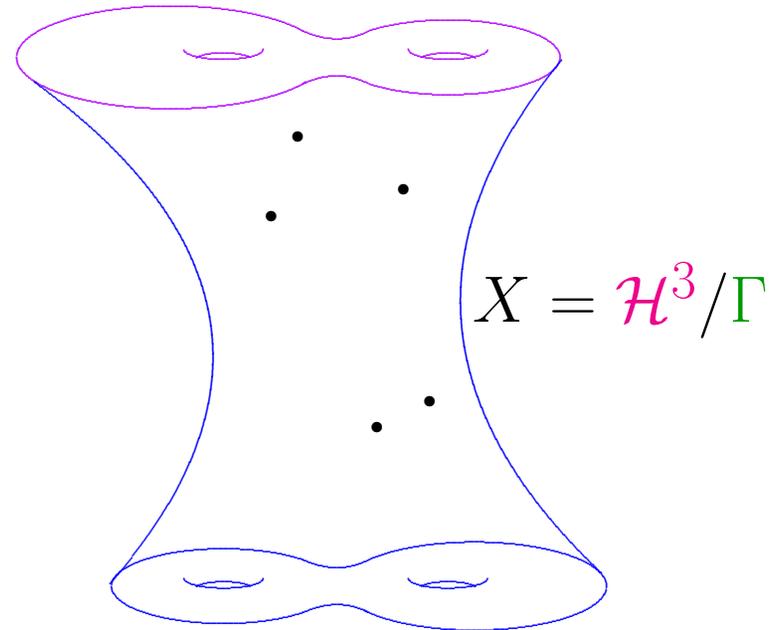
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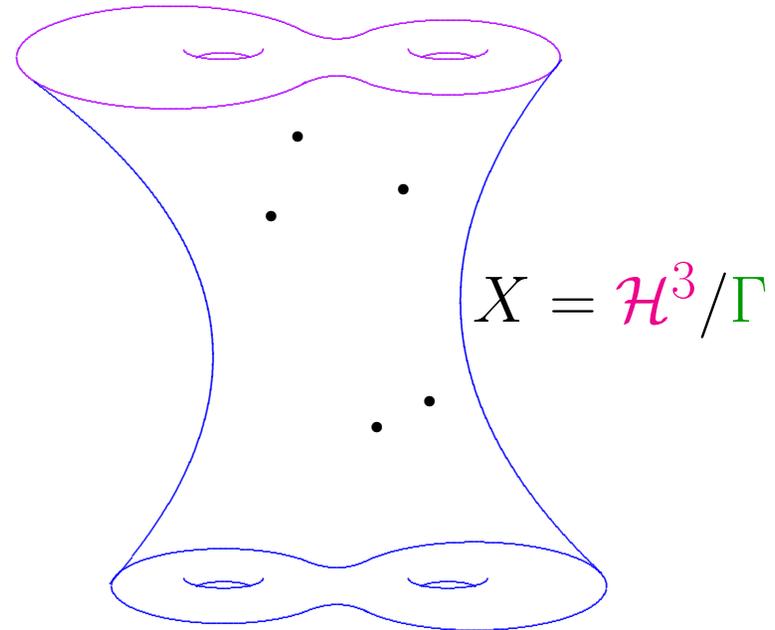
$$\Delta G_j = 2\pi \delta_{p_j}, \quad G_j \rightarrow 0 \text{ at } \partial \bar{X}$$



Construction of ASD 4-manifolds:

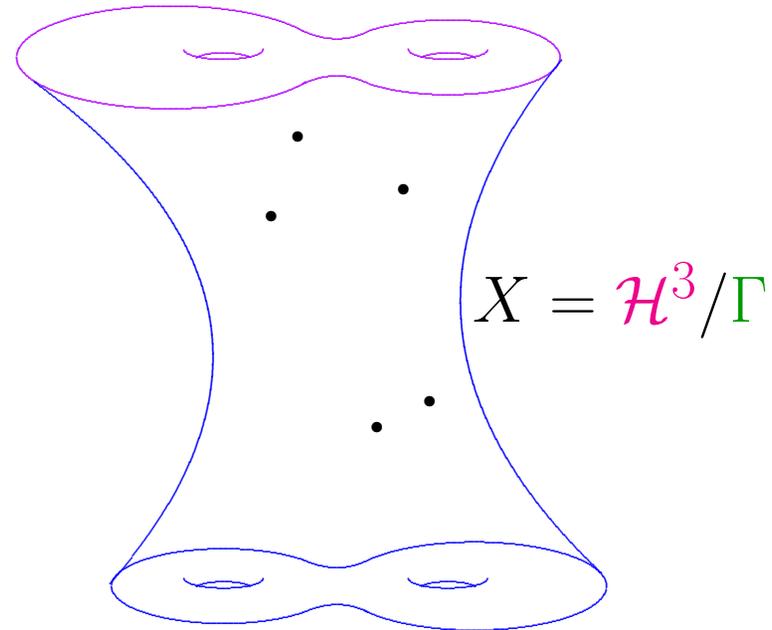
Let  $G_j$  be the Green's function of  $p_j$ , and set

$$V = 1 + \sum_{j=1}^k G_j.$$



Construction of ASD 4-manifolds:

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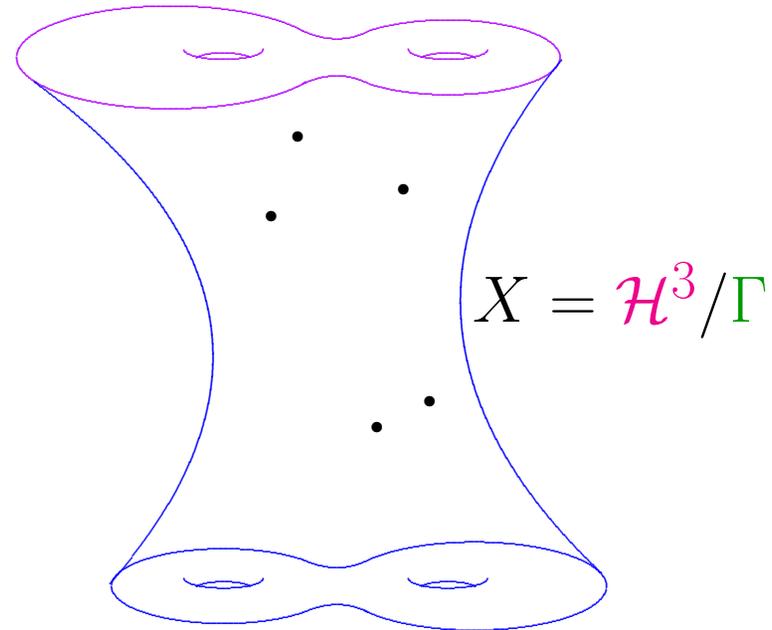


Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

Choose  $P \rightarrow (X - \{p_1, \dots, p_k\})$  circle bundle with connection form  $\theta$  such that

$$d\theta = \star dV.$$

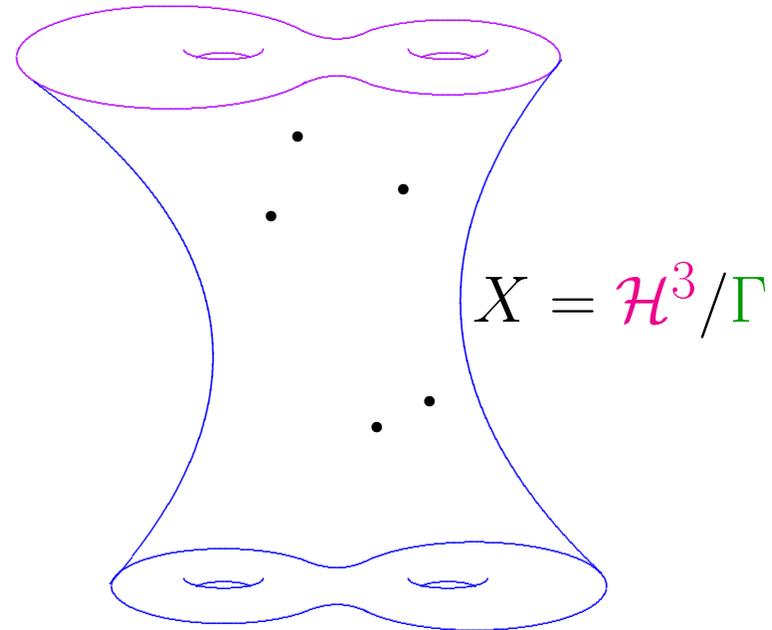


Construction of ASD 4-manifolds:

$$g = Vh + V^{-1}\theta^2$$

$$V = 1 + \sum_{j=1}^k G_j$$

$$d\theta = \star dV$$

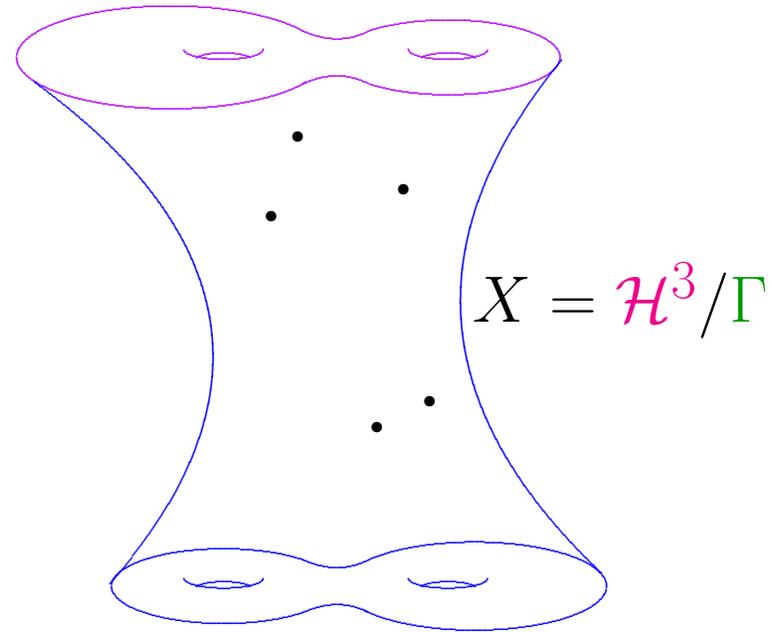


Construction of ASD 4-manifolds:

$$g = f(1 - f)[Vh + V^{-1}\theta^2]$$

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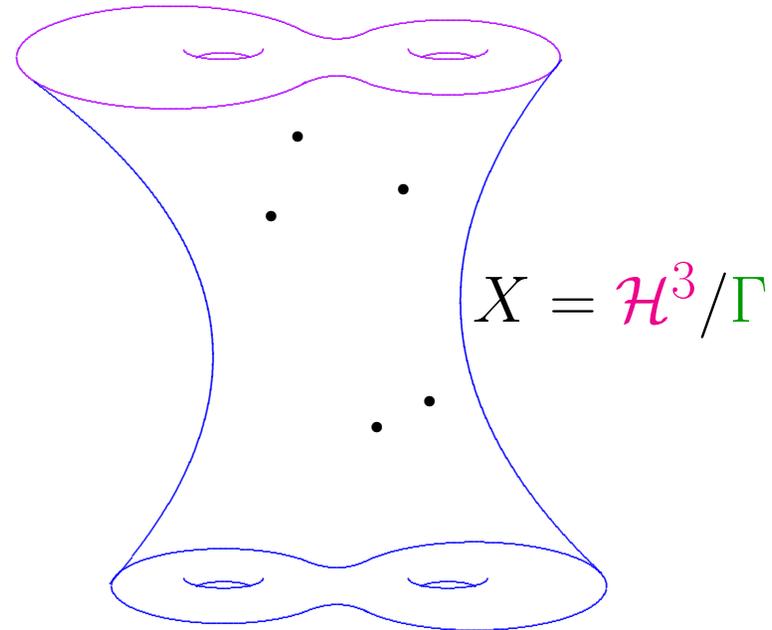
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Construction of ASD 4-manifolds:

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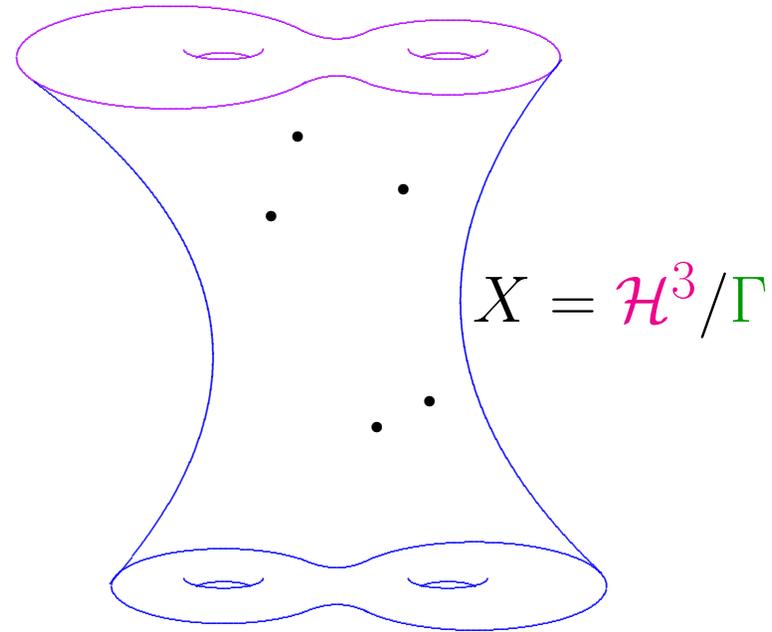
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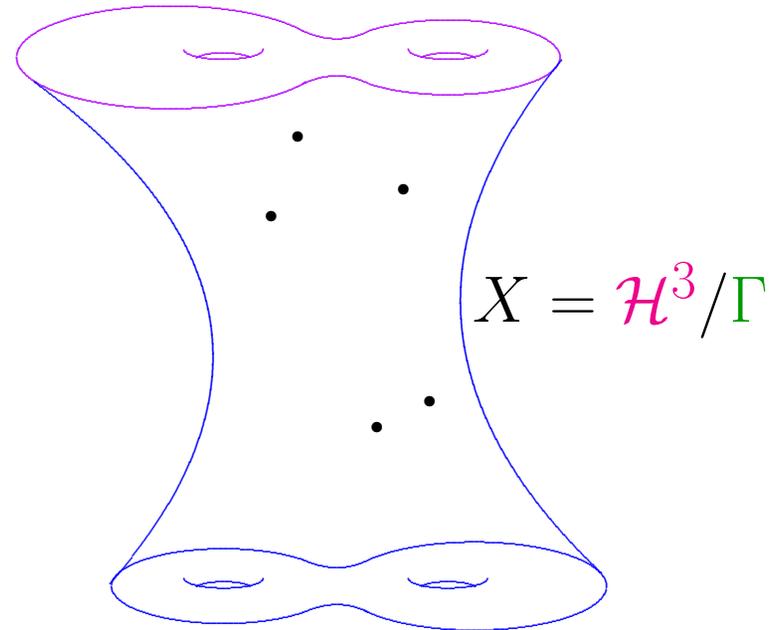
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 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
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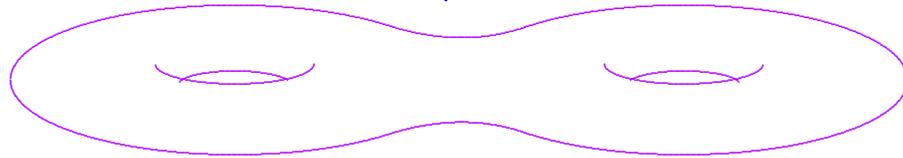
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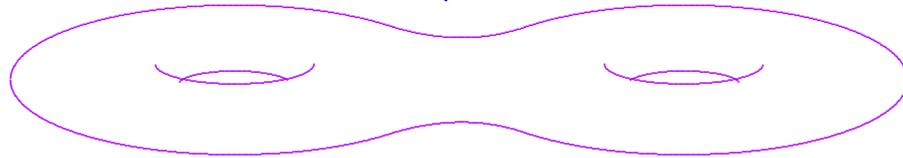
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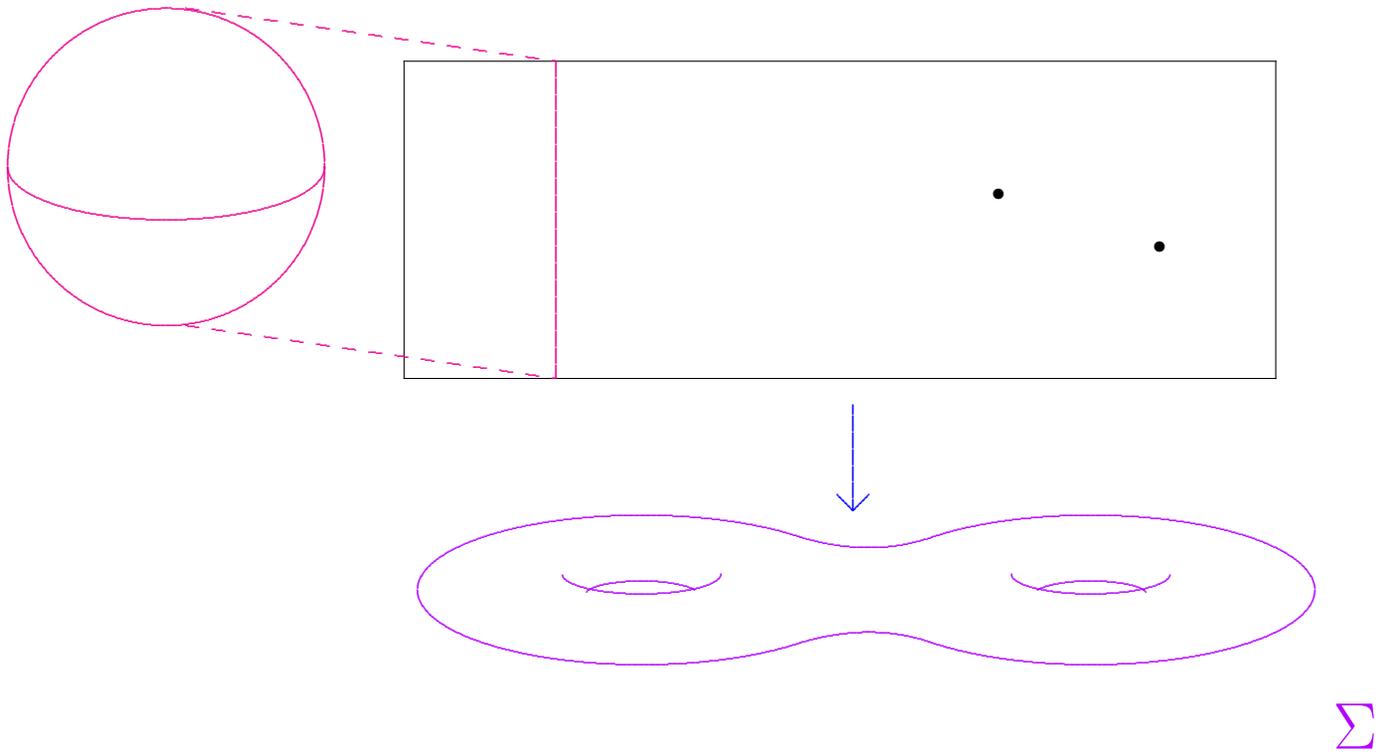
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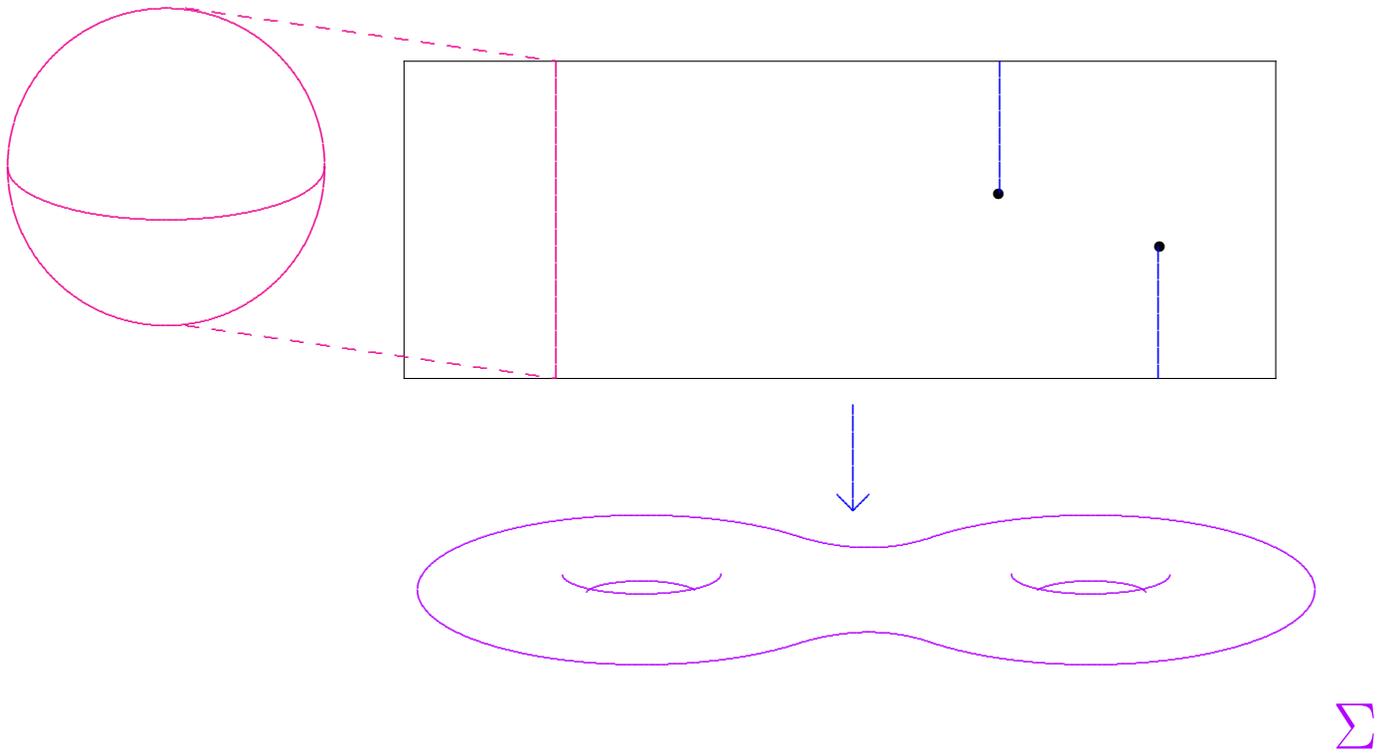


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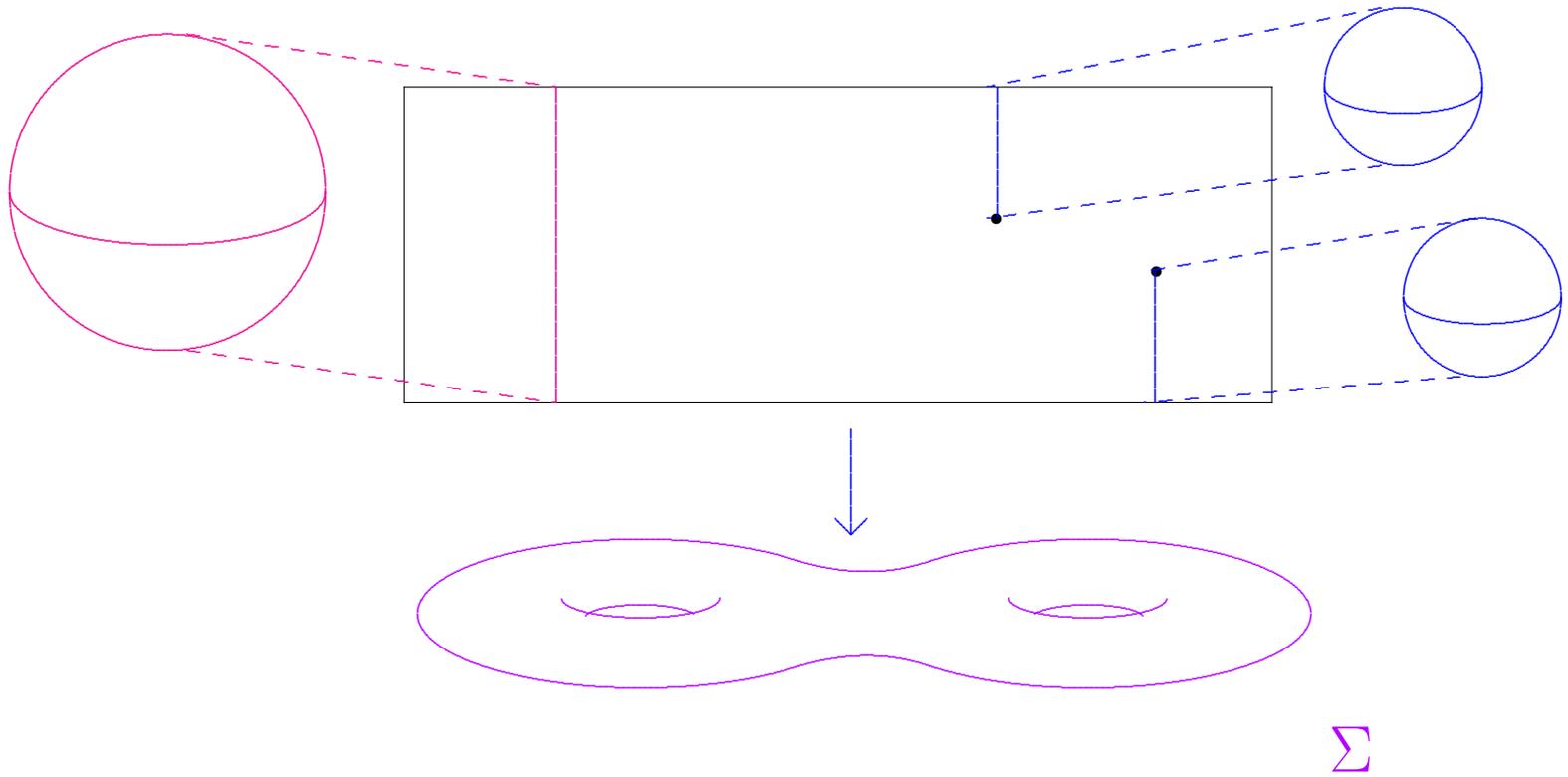
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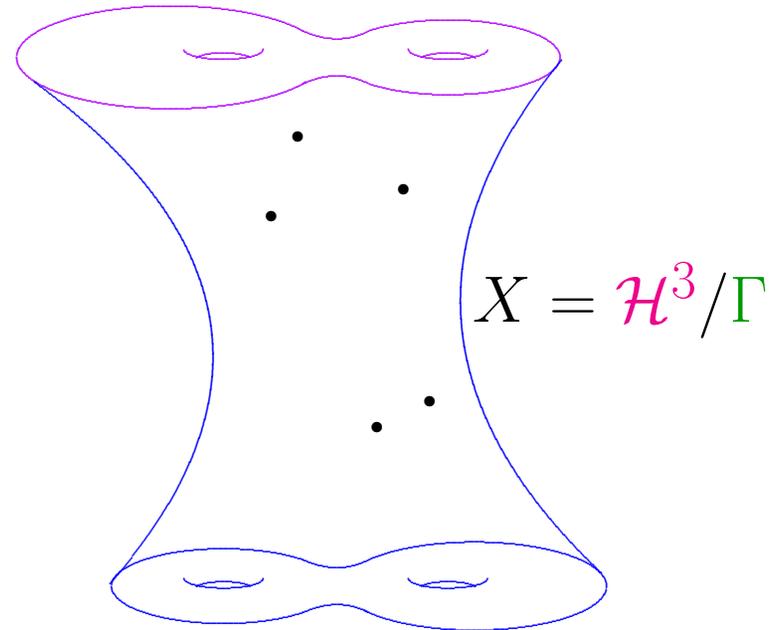


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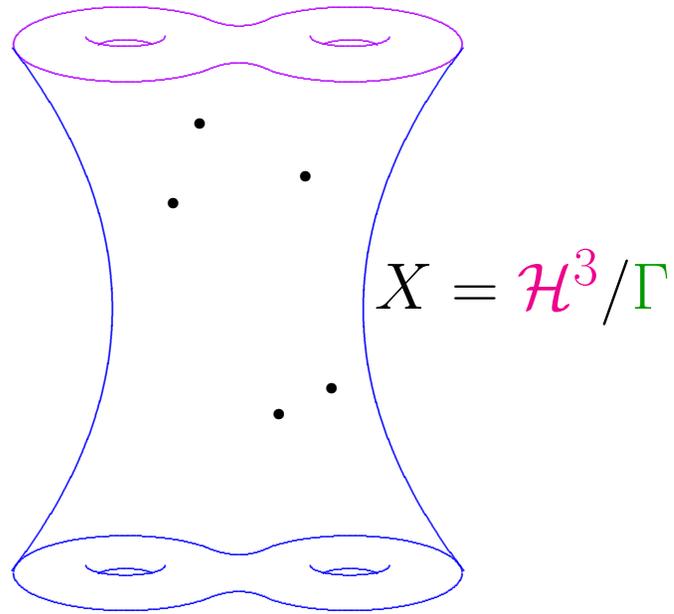


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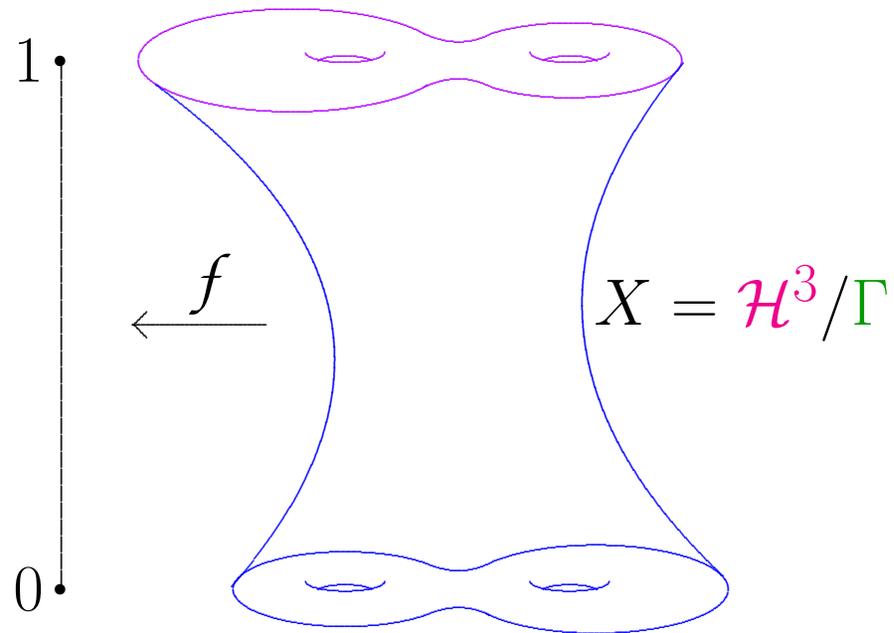


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Fuchsian case:  $(\Sigma \times S^2) \# k\overline{\mathbb{C}\mathbb{P}}_2$  scalar-flat Kähler



$\Gamma$  quasi-Fuchsian

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Tunnel-Vision function:

$$f : \overline{X} \rightarrow [0, 1]$$

$$\Delta f = 0$$

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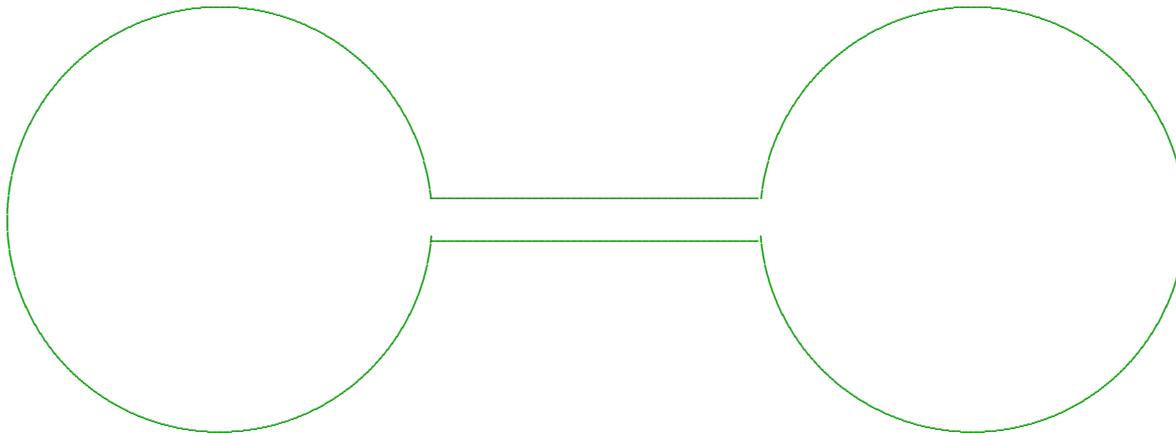
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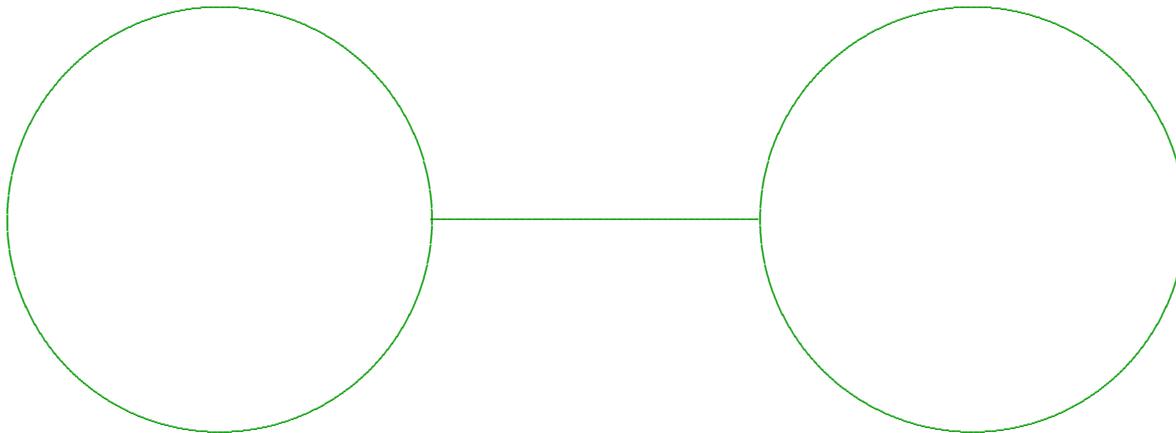
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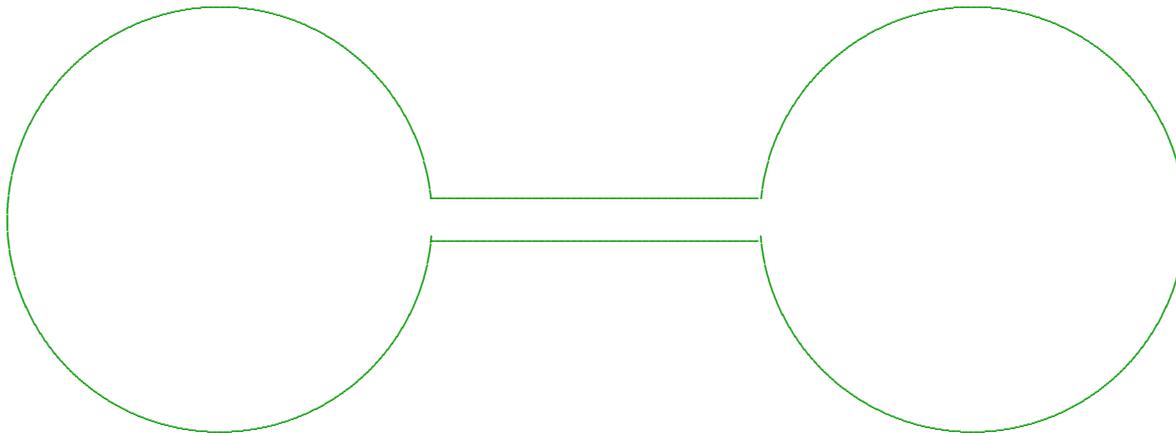
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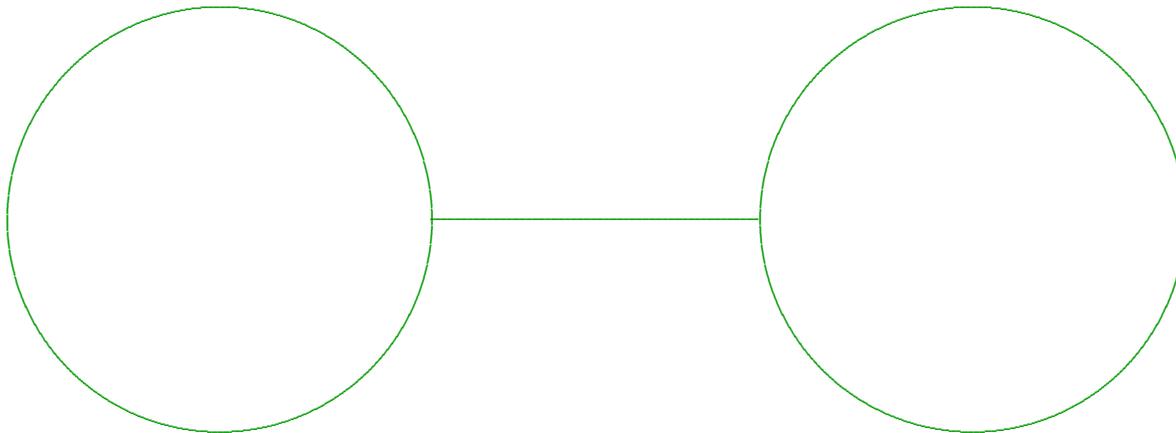
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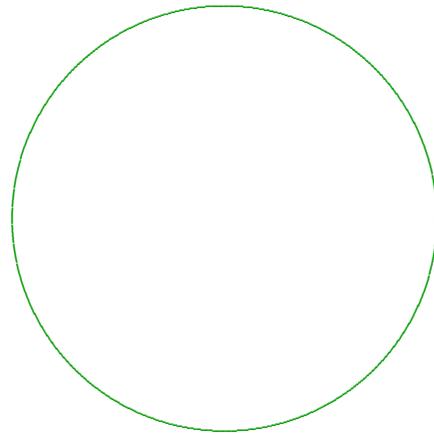
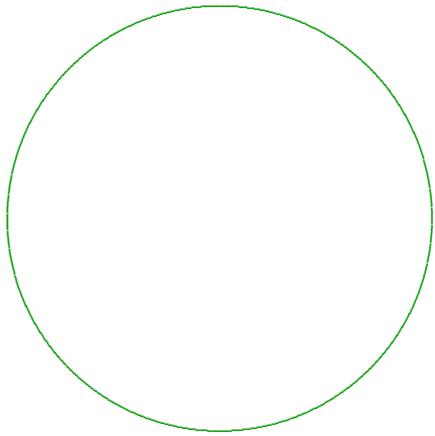
Ahlfors-Bers: Quasi-conformal mappings

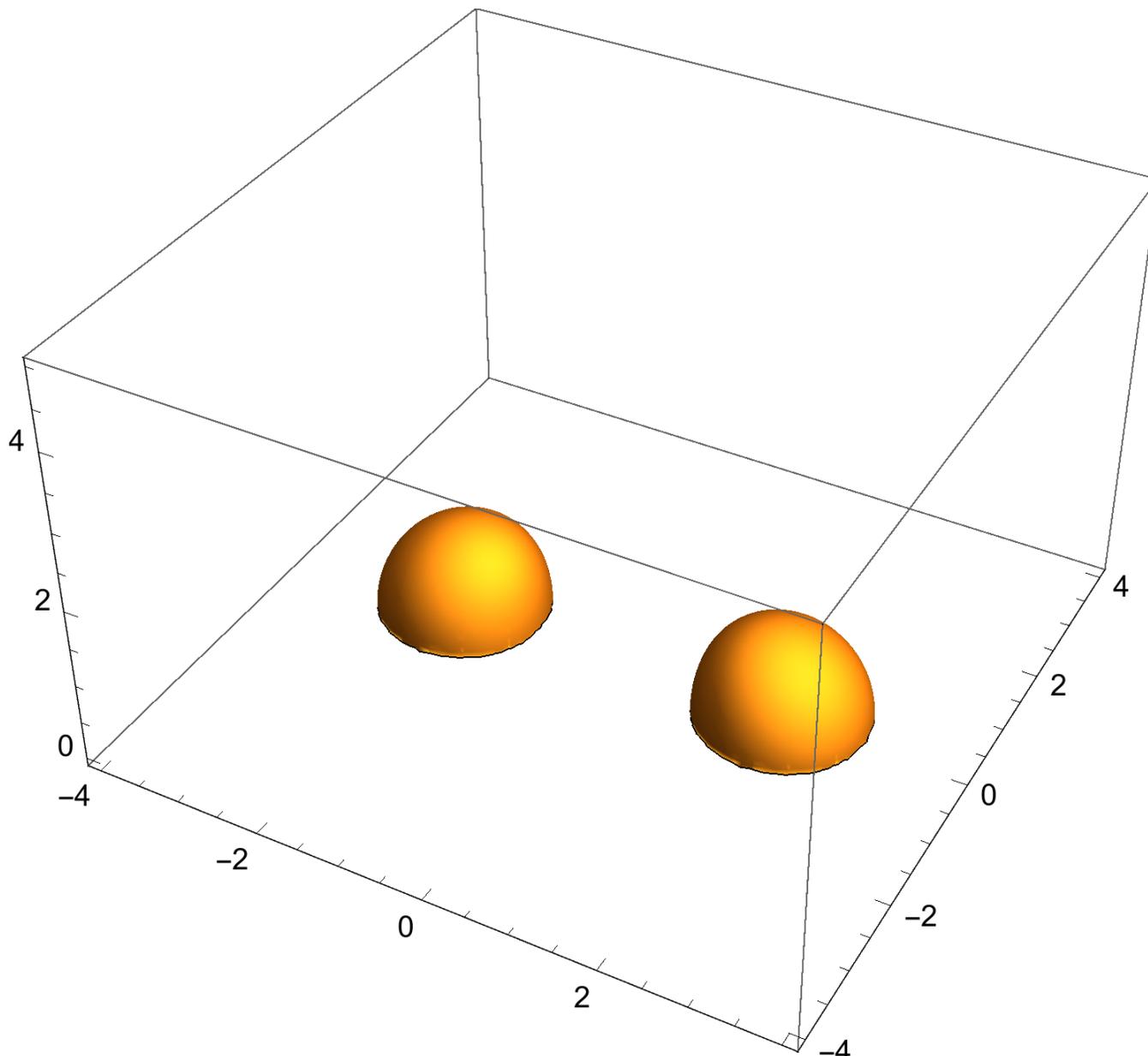


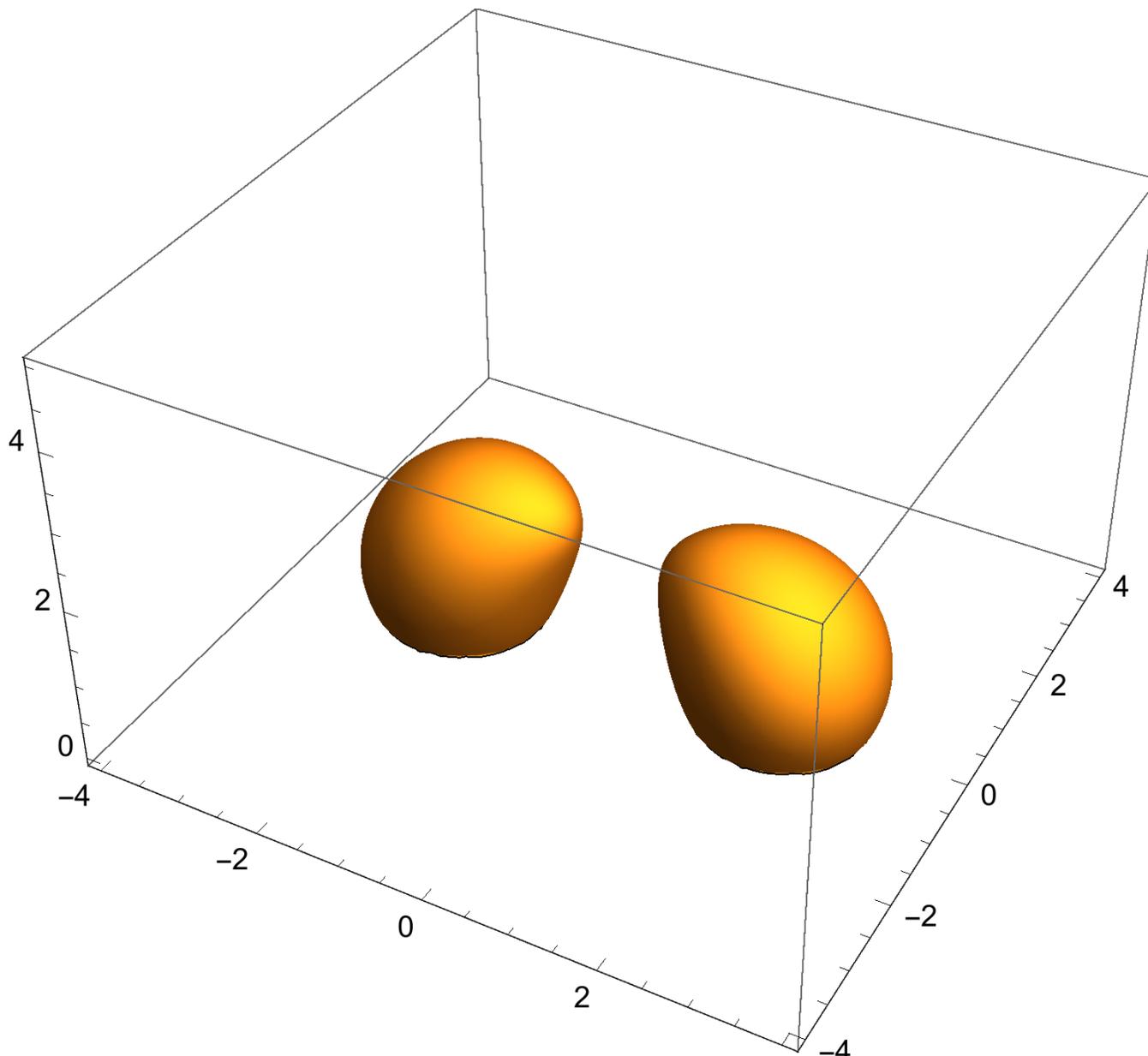


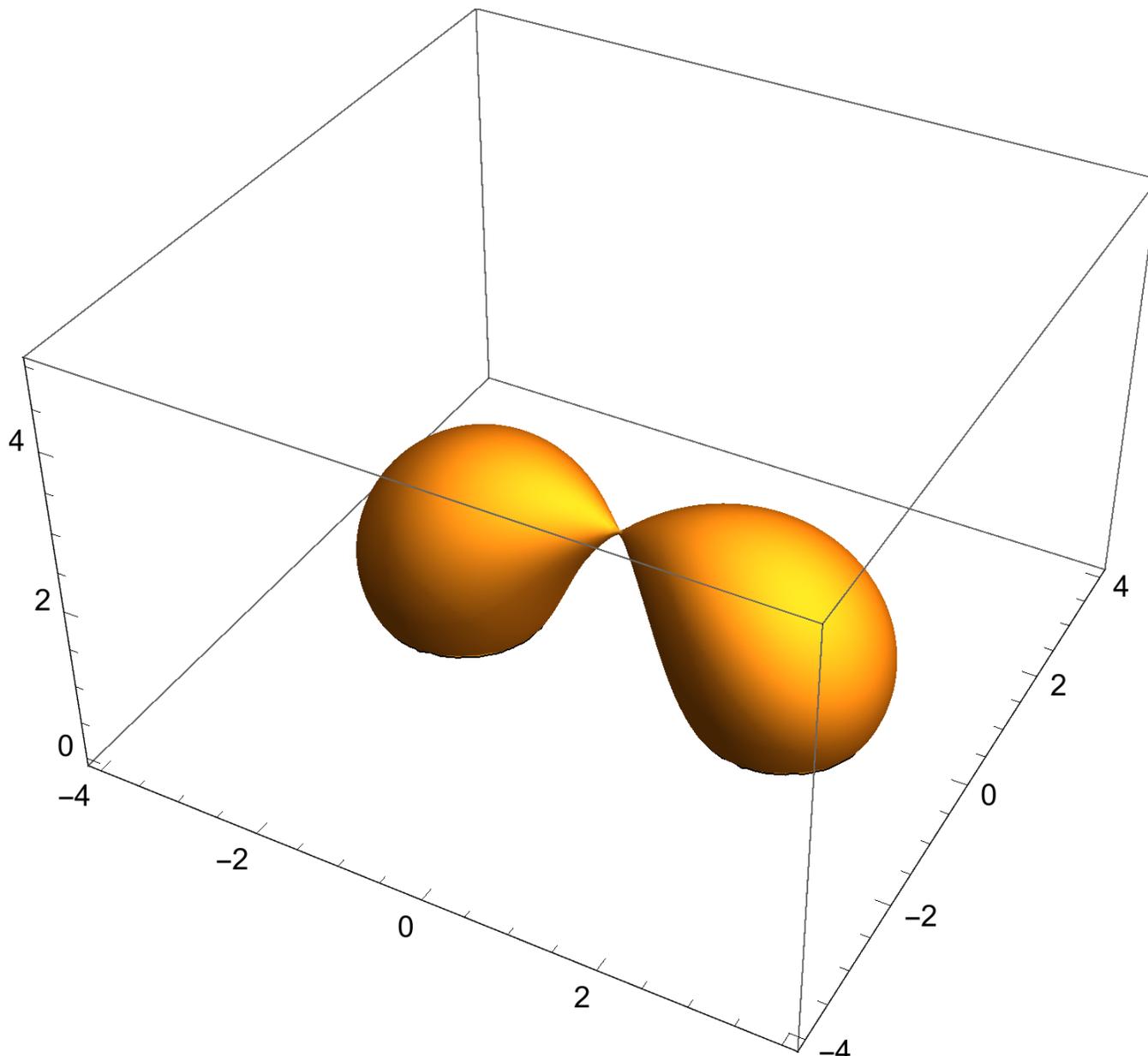


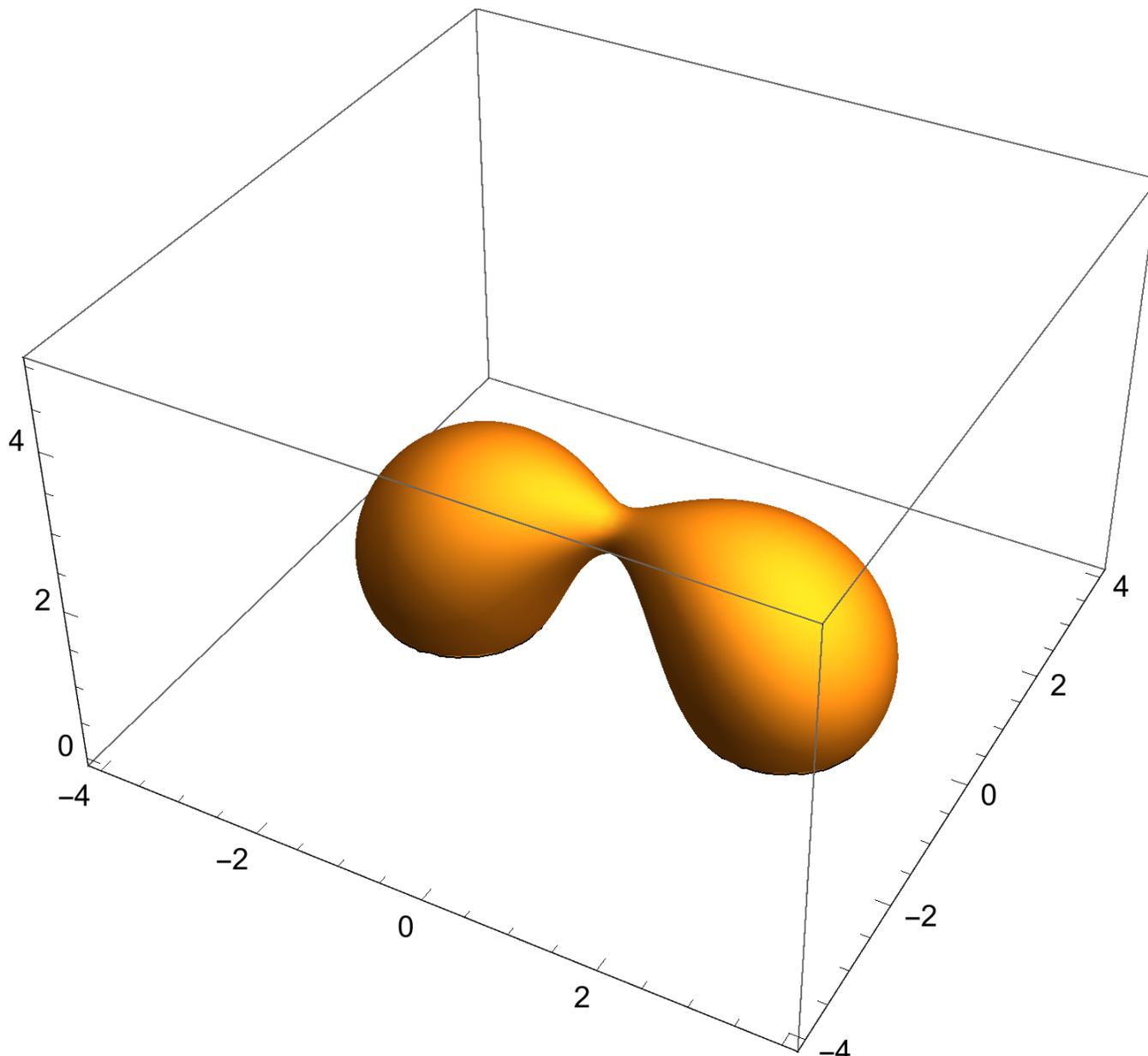


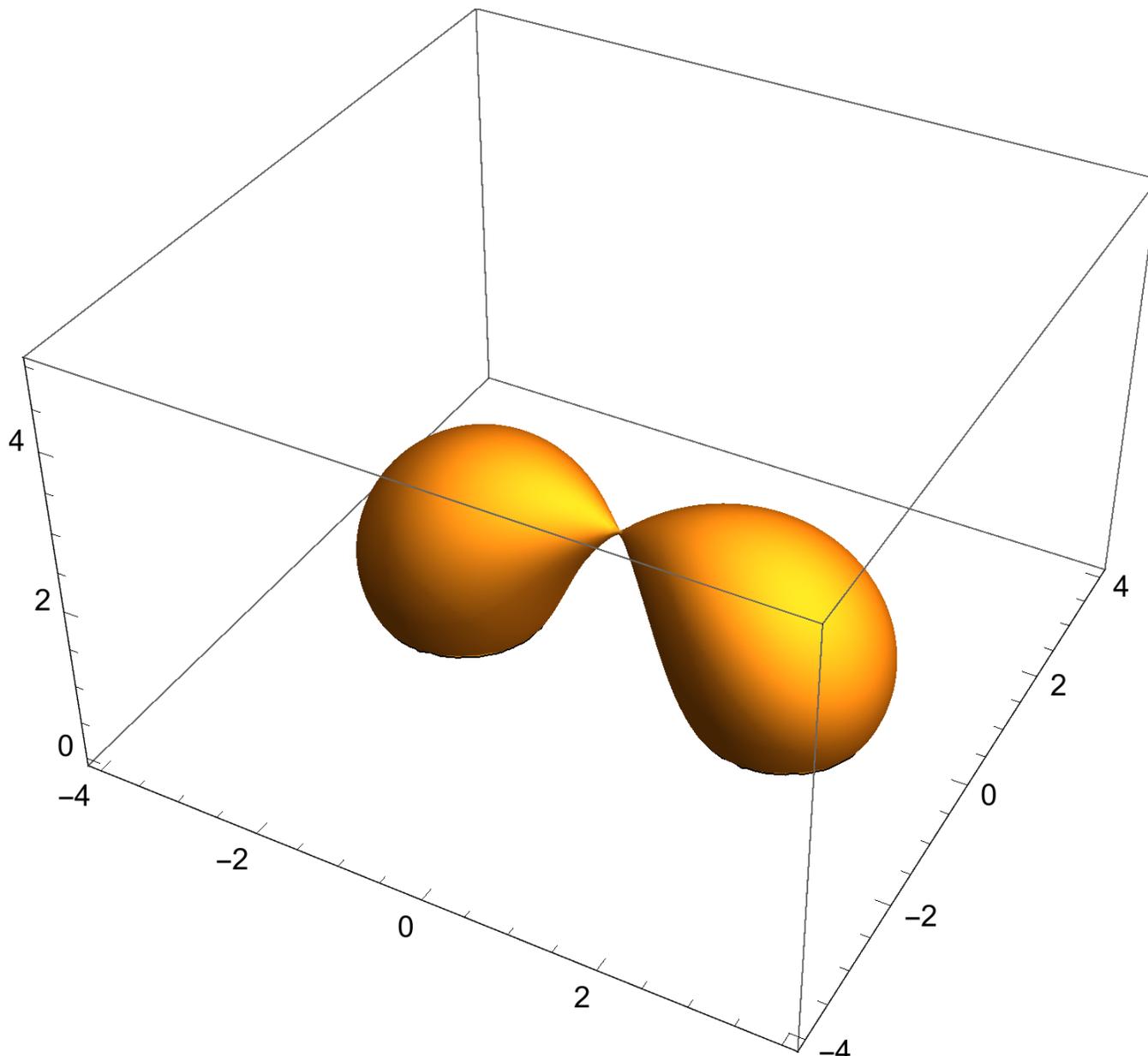


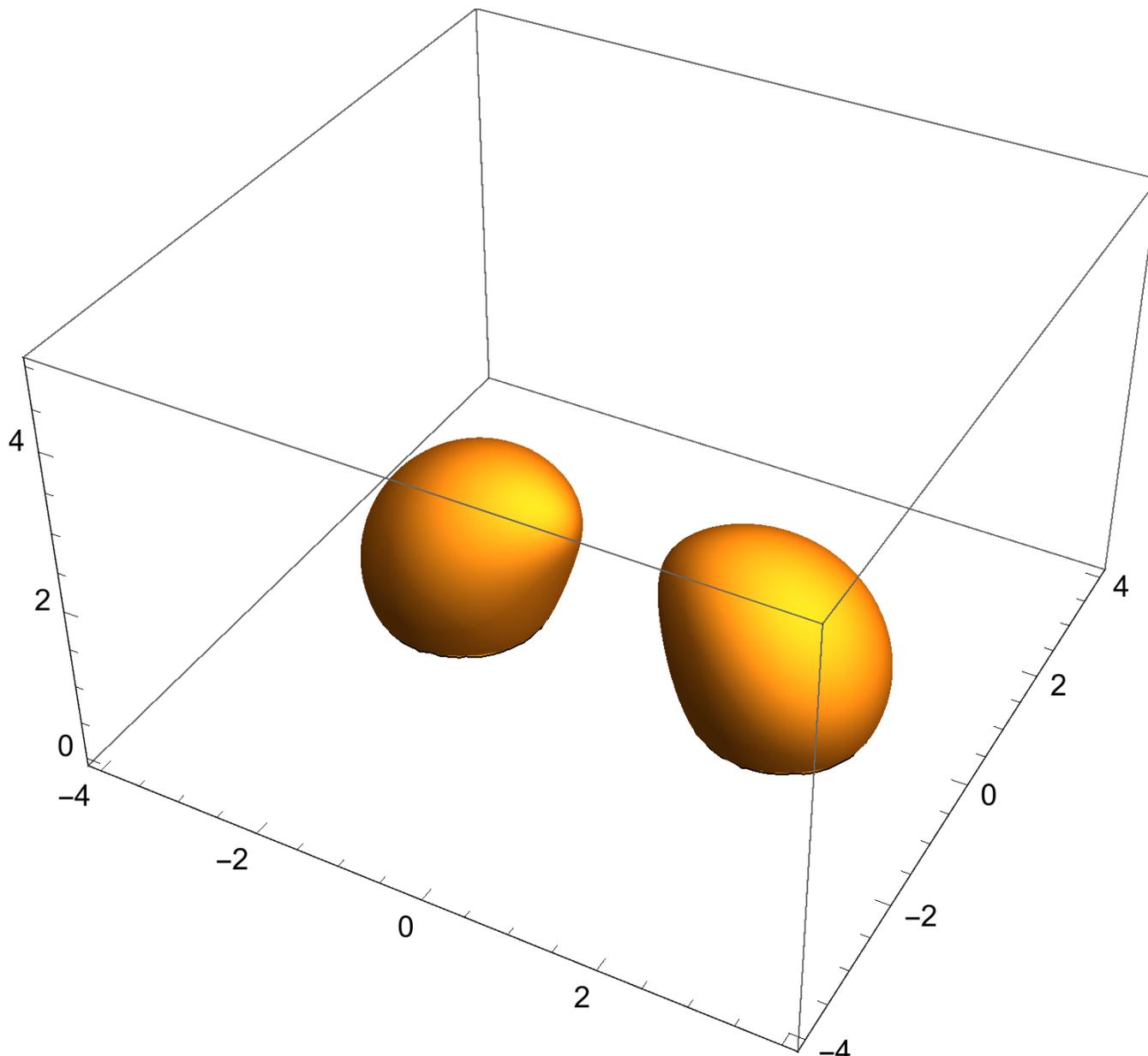


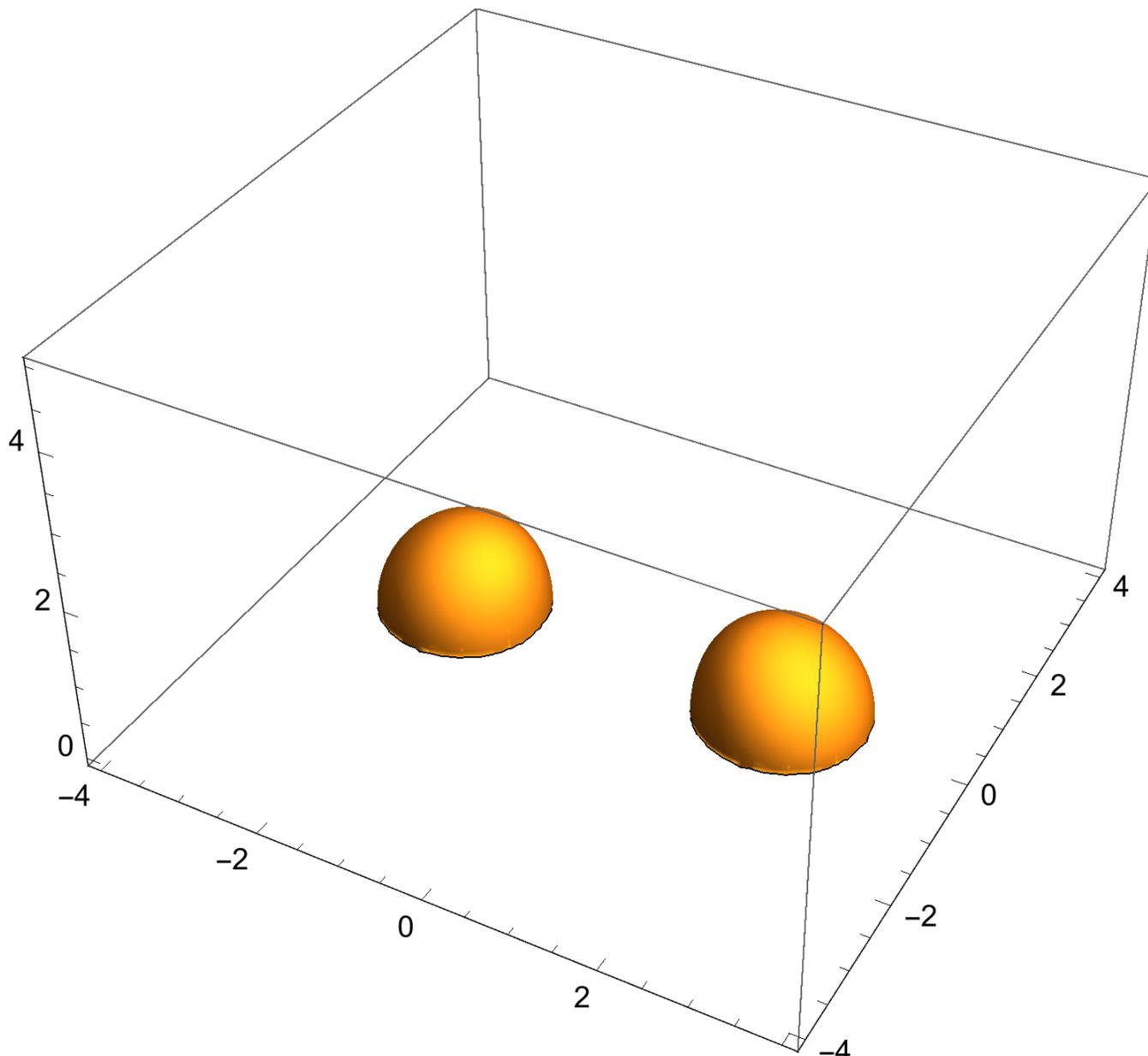


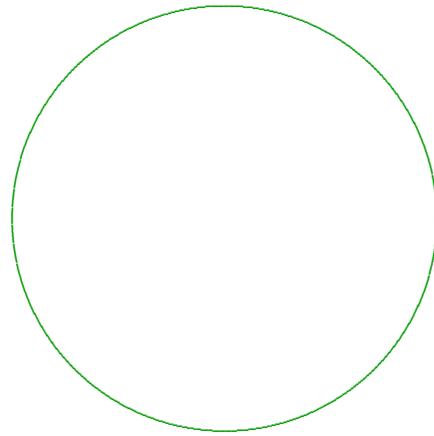
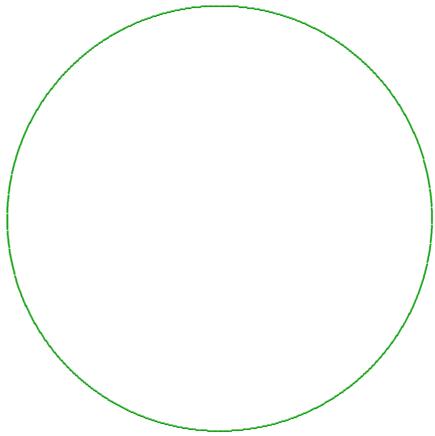


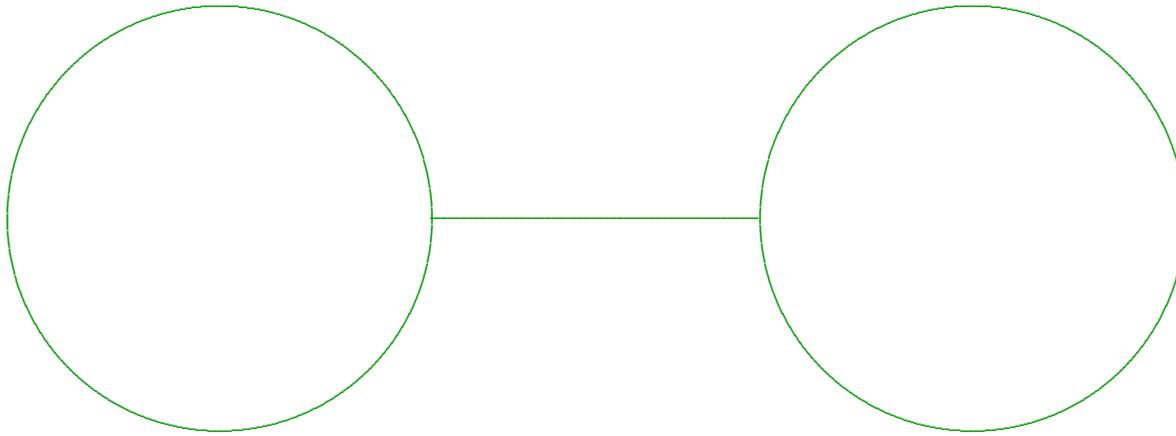


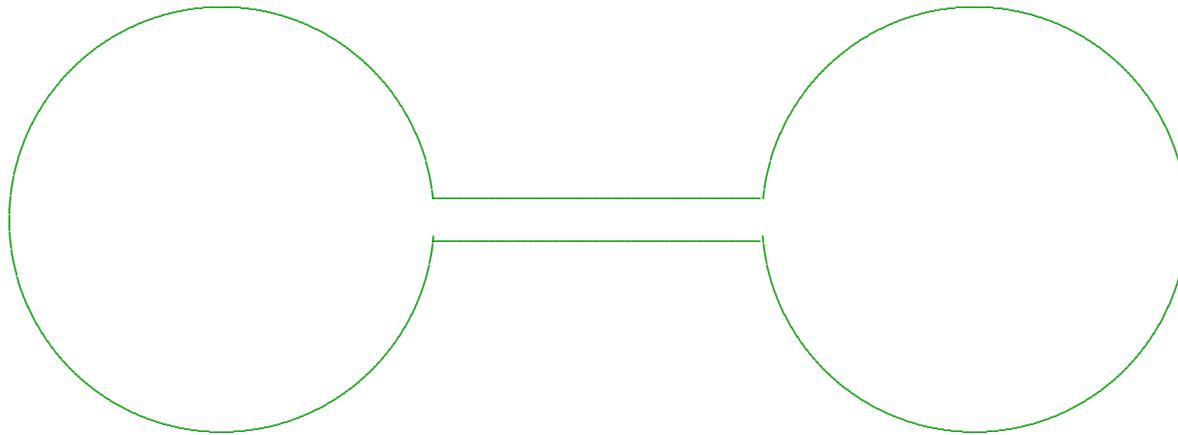


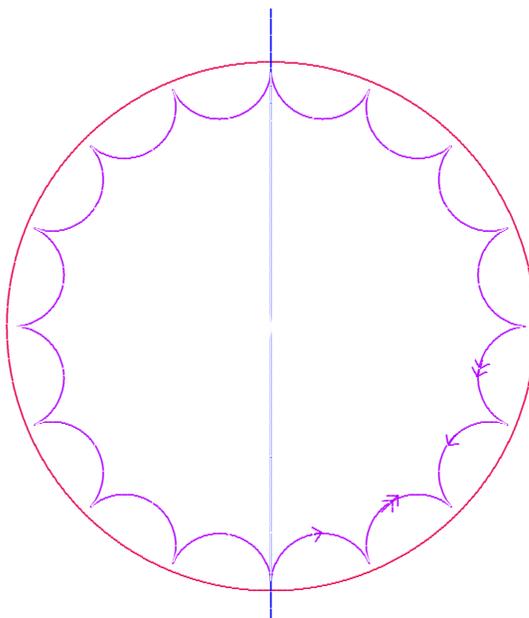
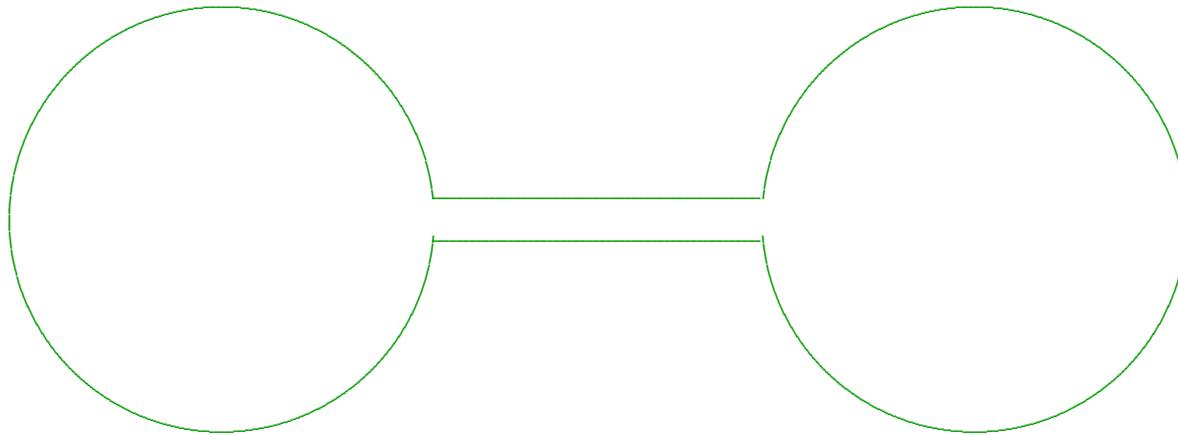


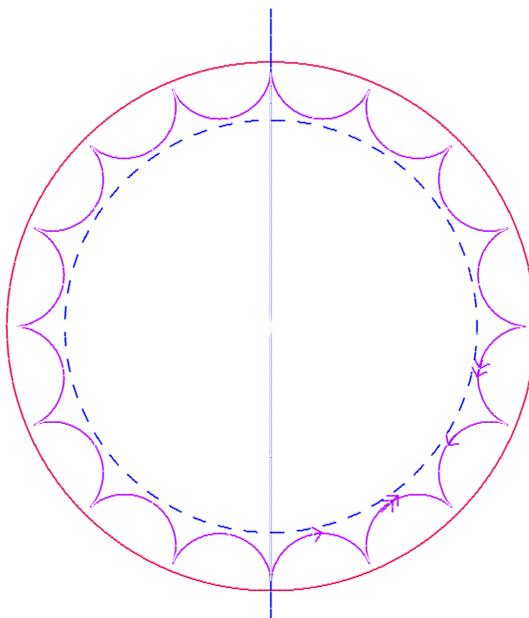
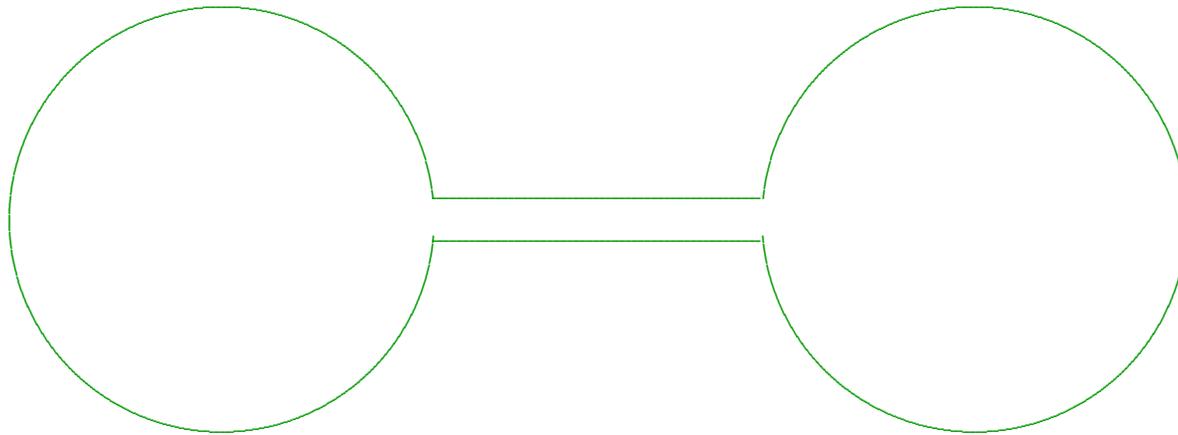


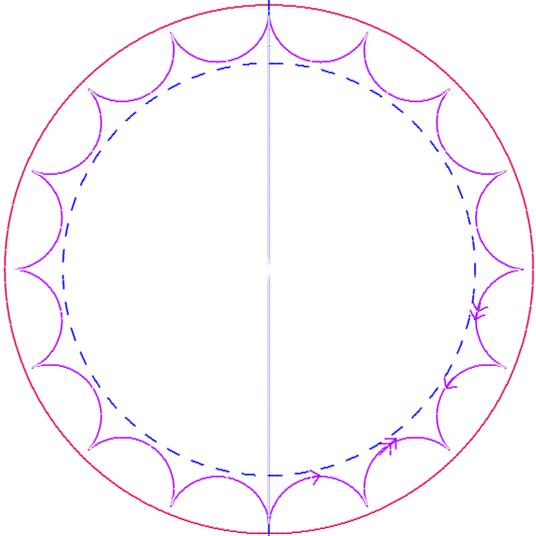
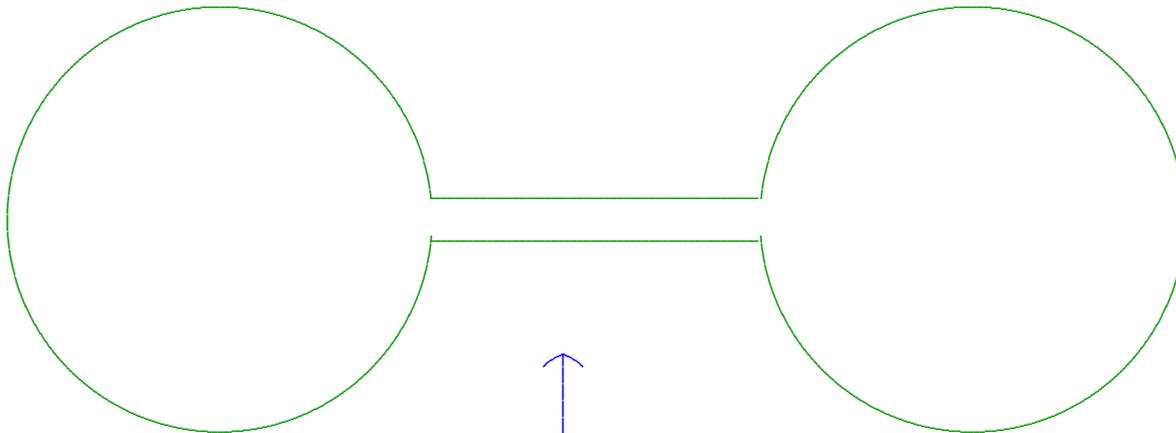


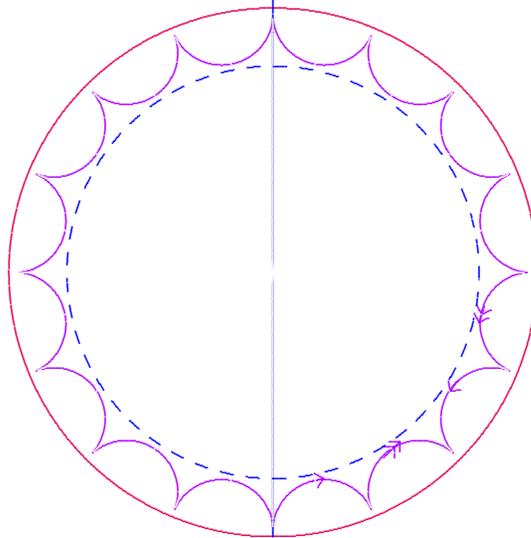
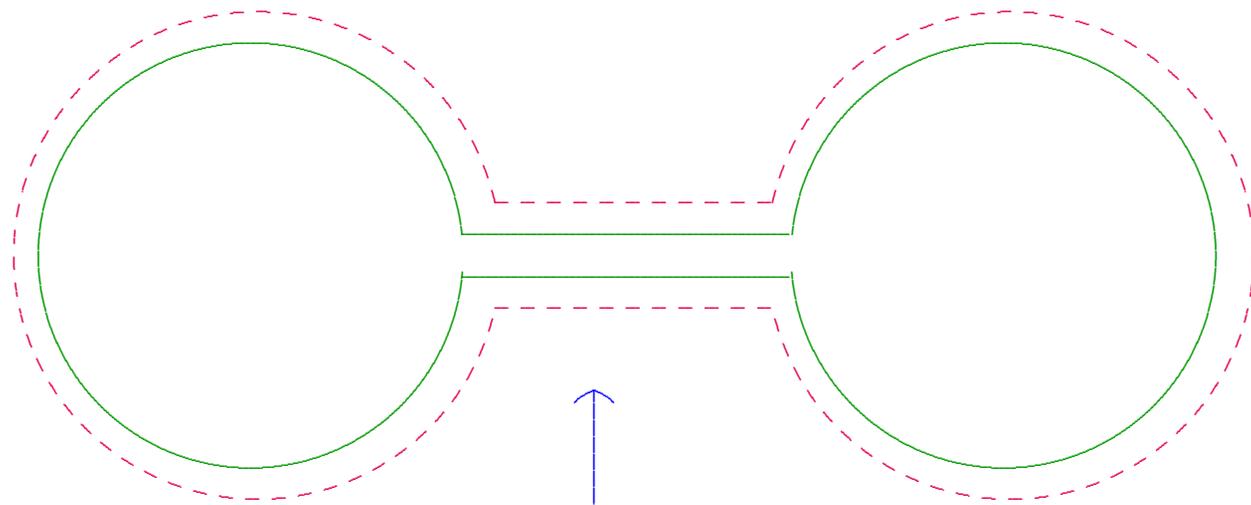


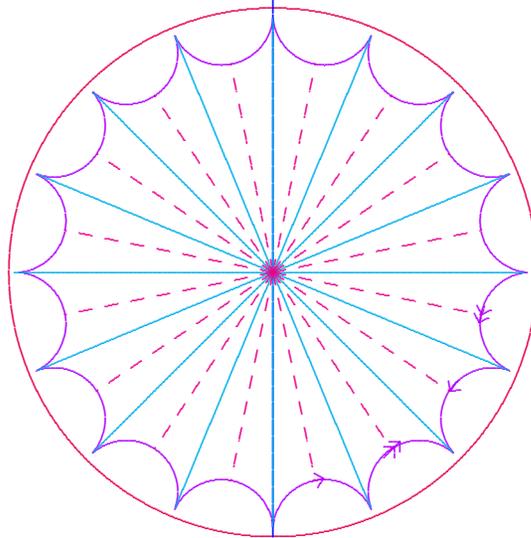
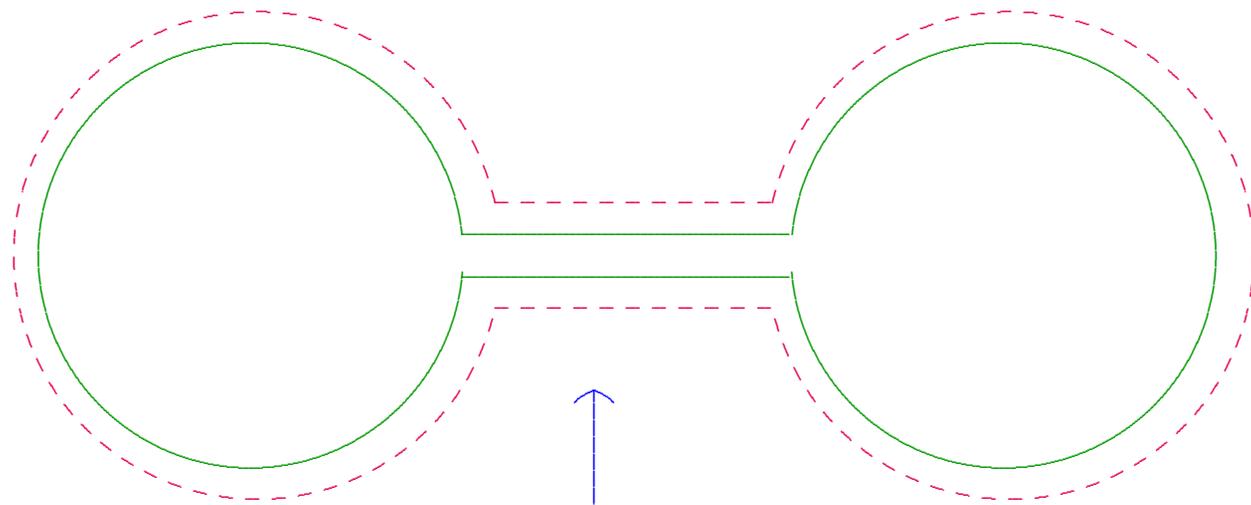


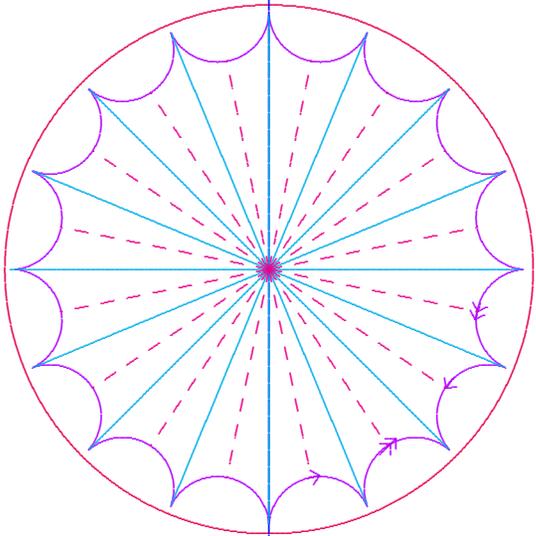
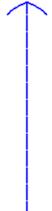
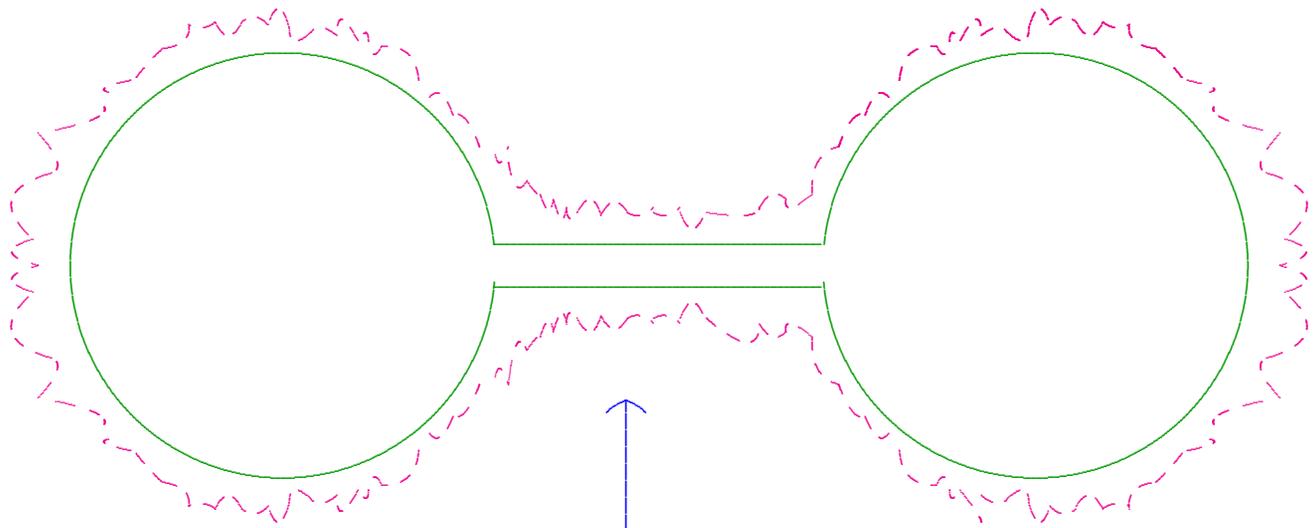








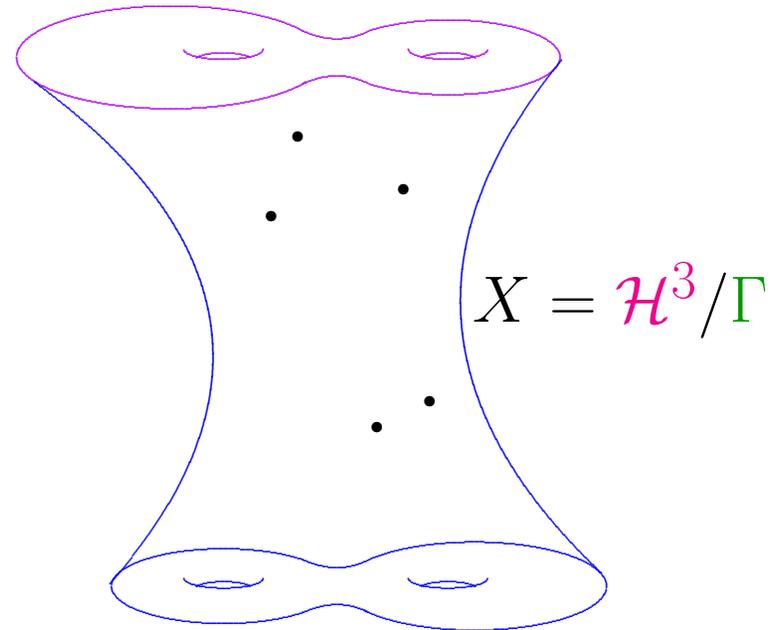




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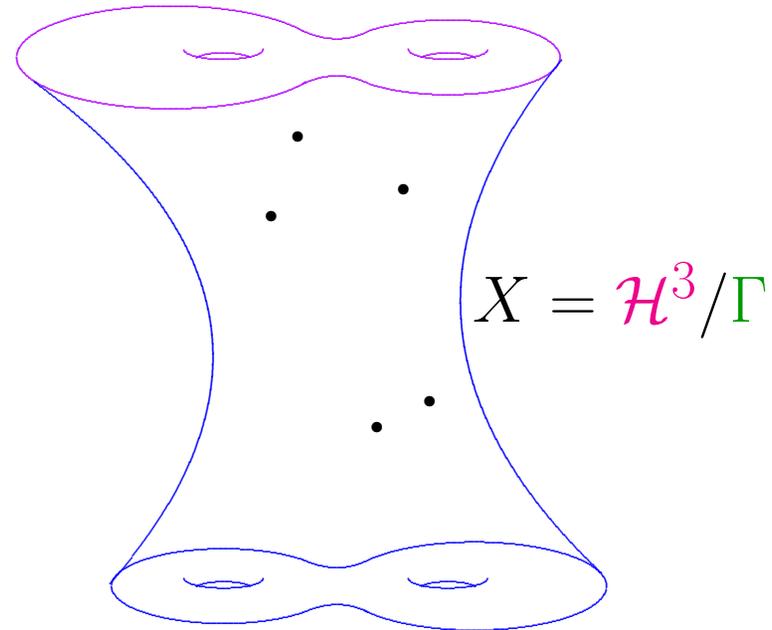
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Hitchin '87 component  $PSL(2, \mathbb{C})$

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It's a pleasure to be here!

