Einstein Metrics,

Four-Manifolds, &

Gravitational Instantons

Claude LeBrun
Stony Brook University

Canonical Metrics in Differential Geometry,
University of Wisconsin, Madison,
April 12, 2025.

Most recent results joint with

Most recent results joint with

Olivier Biquard
Sorbonne Université

Most recent results joint with

Olivier Biquard
Sorbonne Université

and

Paul Gauduchon
École Polytechnique

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(2024) 13295-13311.

Will also briefly mention

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work in progress

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Tristan Ozuch
MIT

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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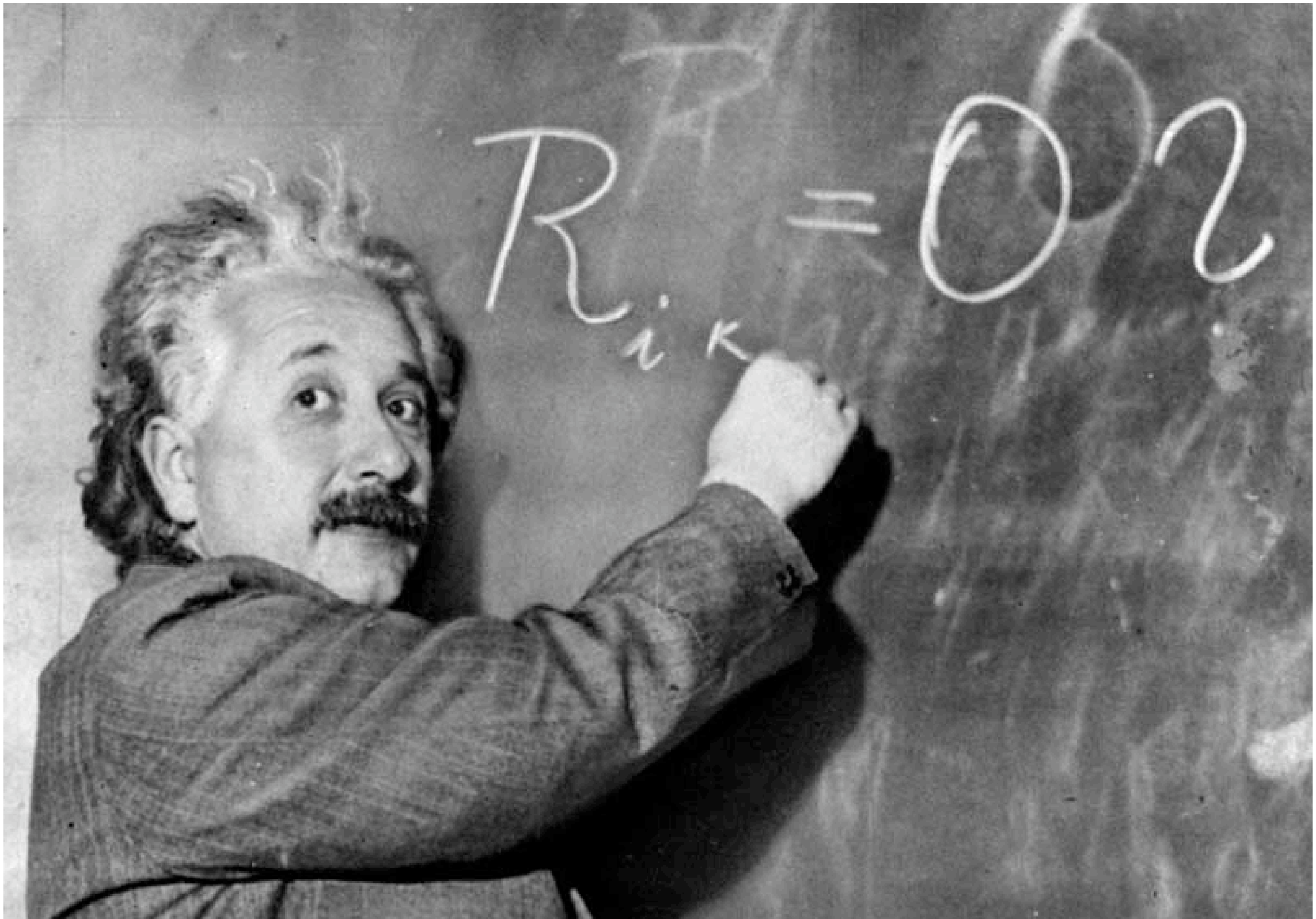

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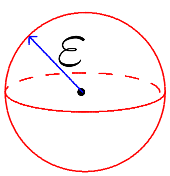
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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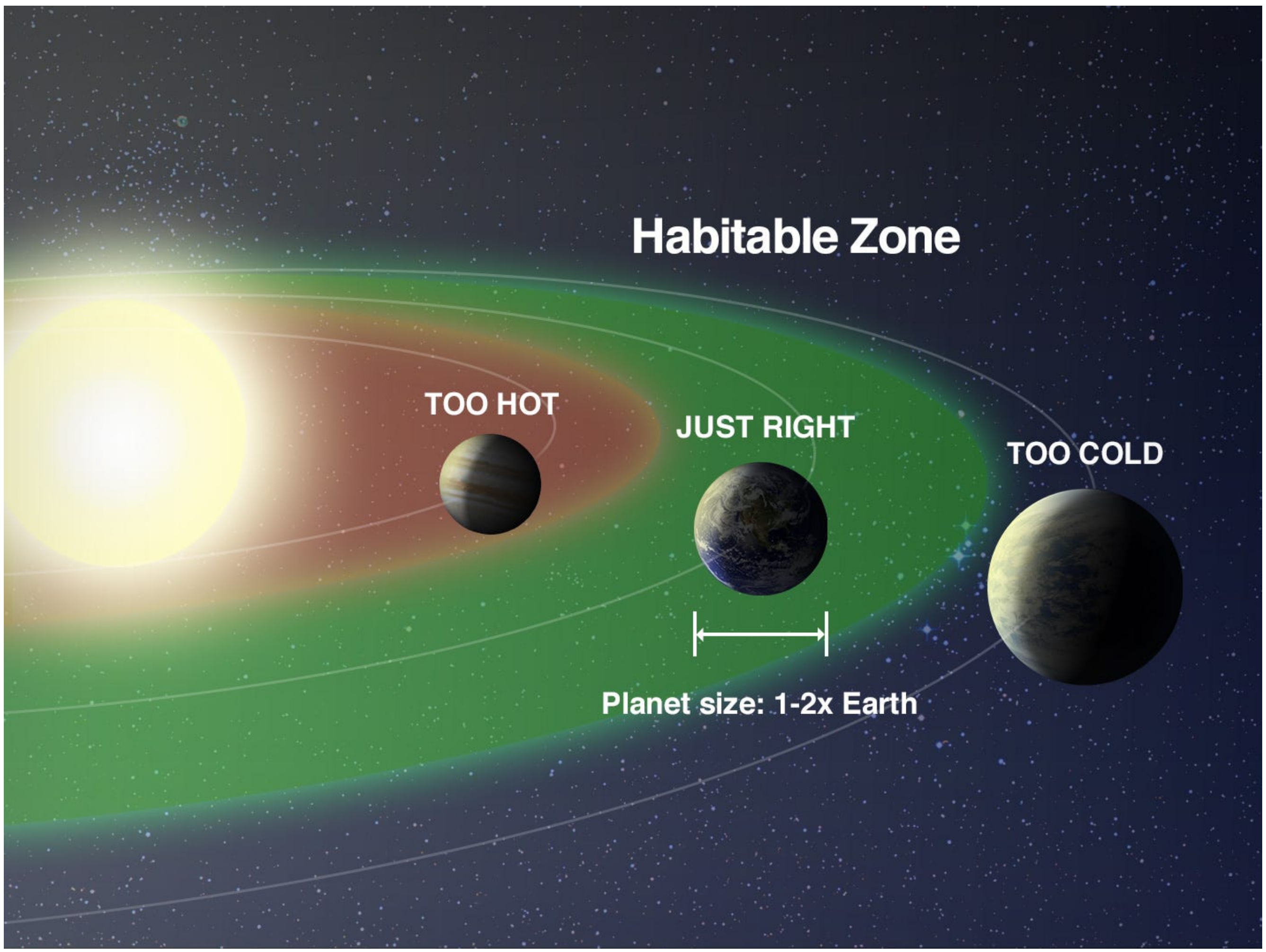

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Λ^+ self-dual 2-forms.
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Which 4-manifolds admit Einstein metrics?

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$$d\omega = 0, \qquad \lrcorner\omega : TM \xrightarrow{\cong} T^*M.$$

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$$\omega = dx \wedge dy + dz \wedge dt$$

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)? What if we also require $\lambda > 0$?*

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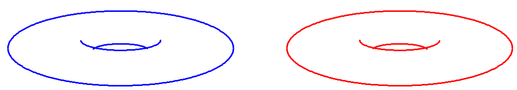

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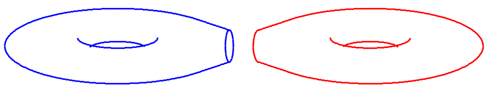

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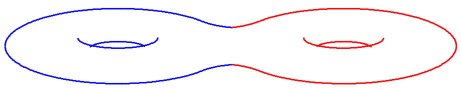

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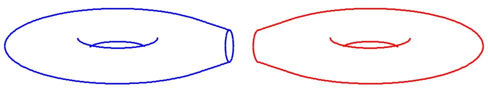

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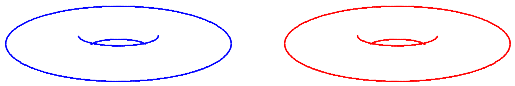

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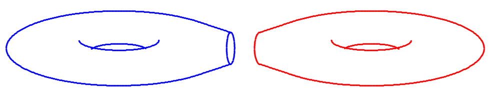

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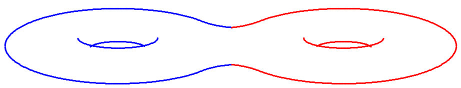

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Diffeotypes: exactly the Del Pezzo surfaces.

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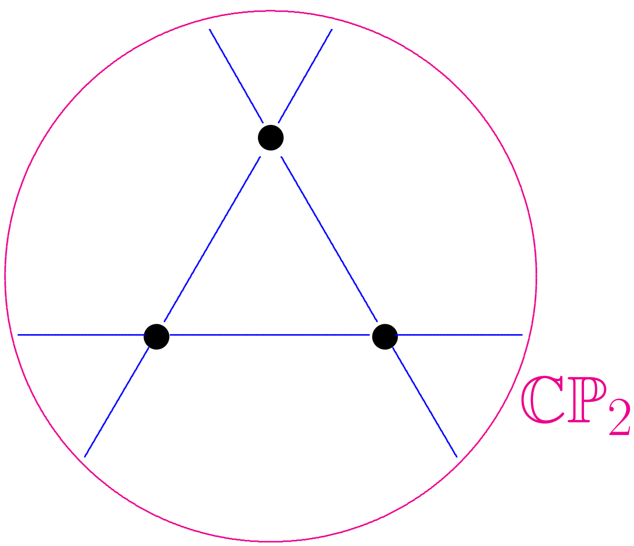

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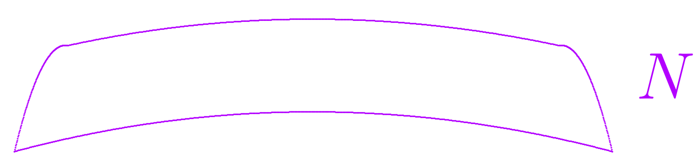

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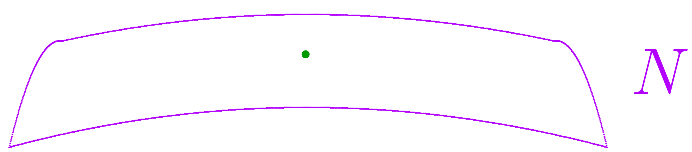

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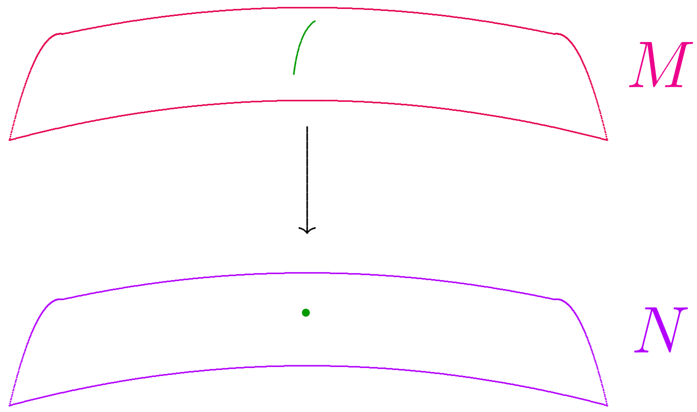

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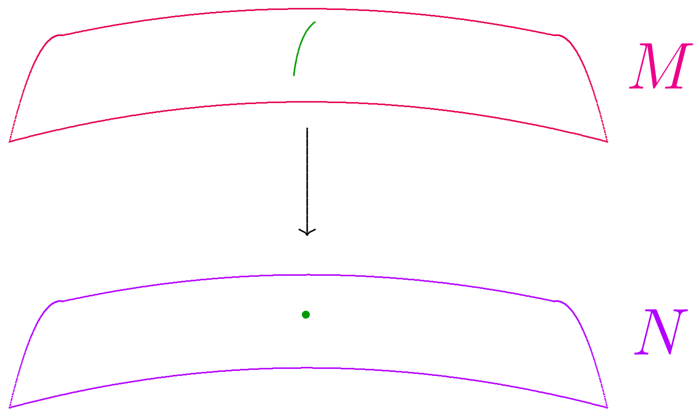

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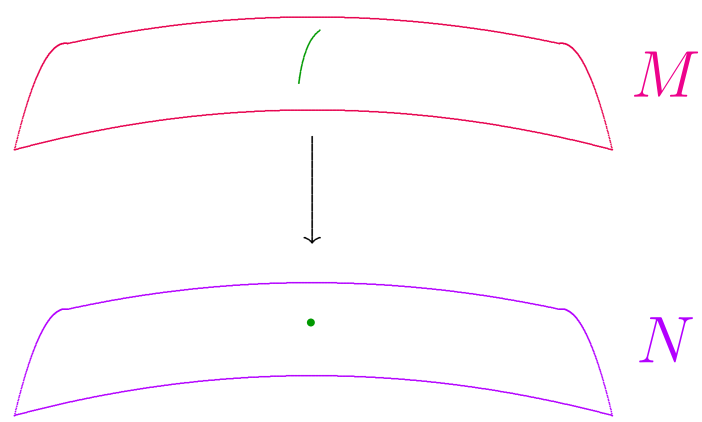

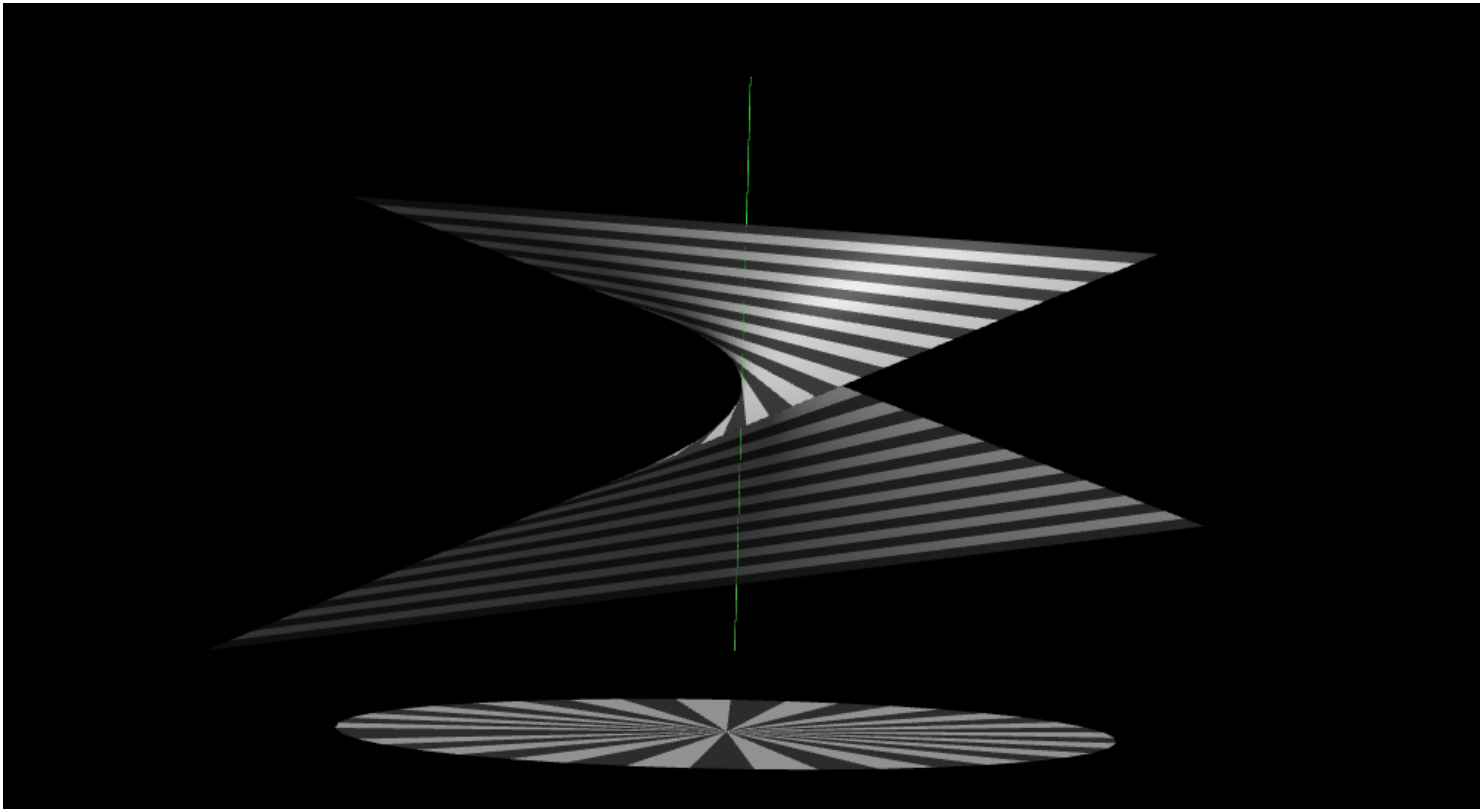

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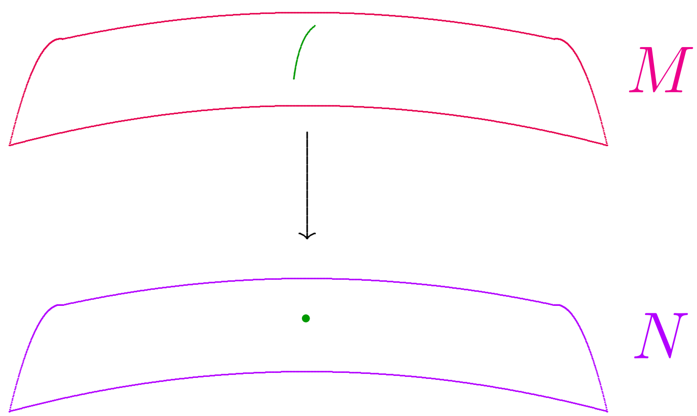

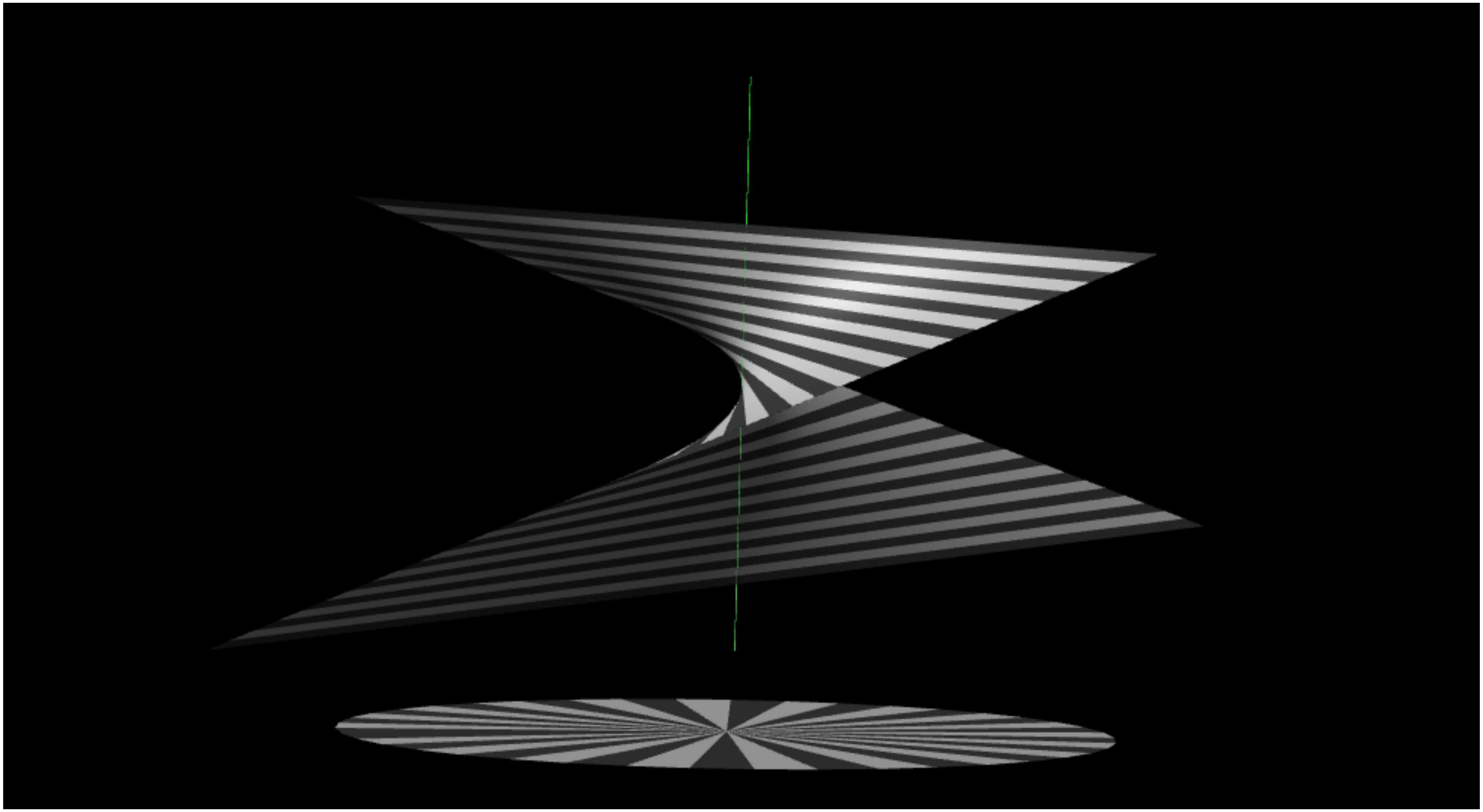

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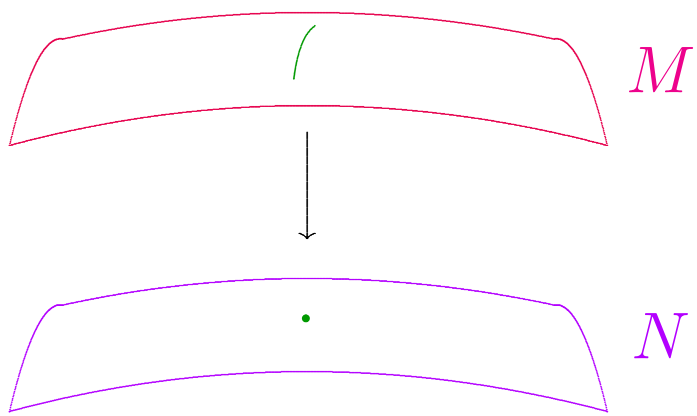
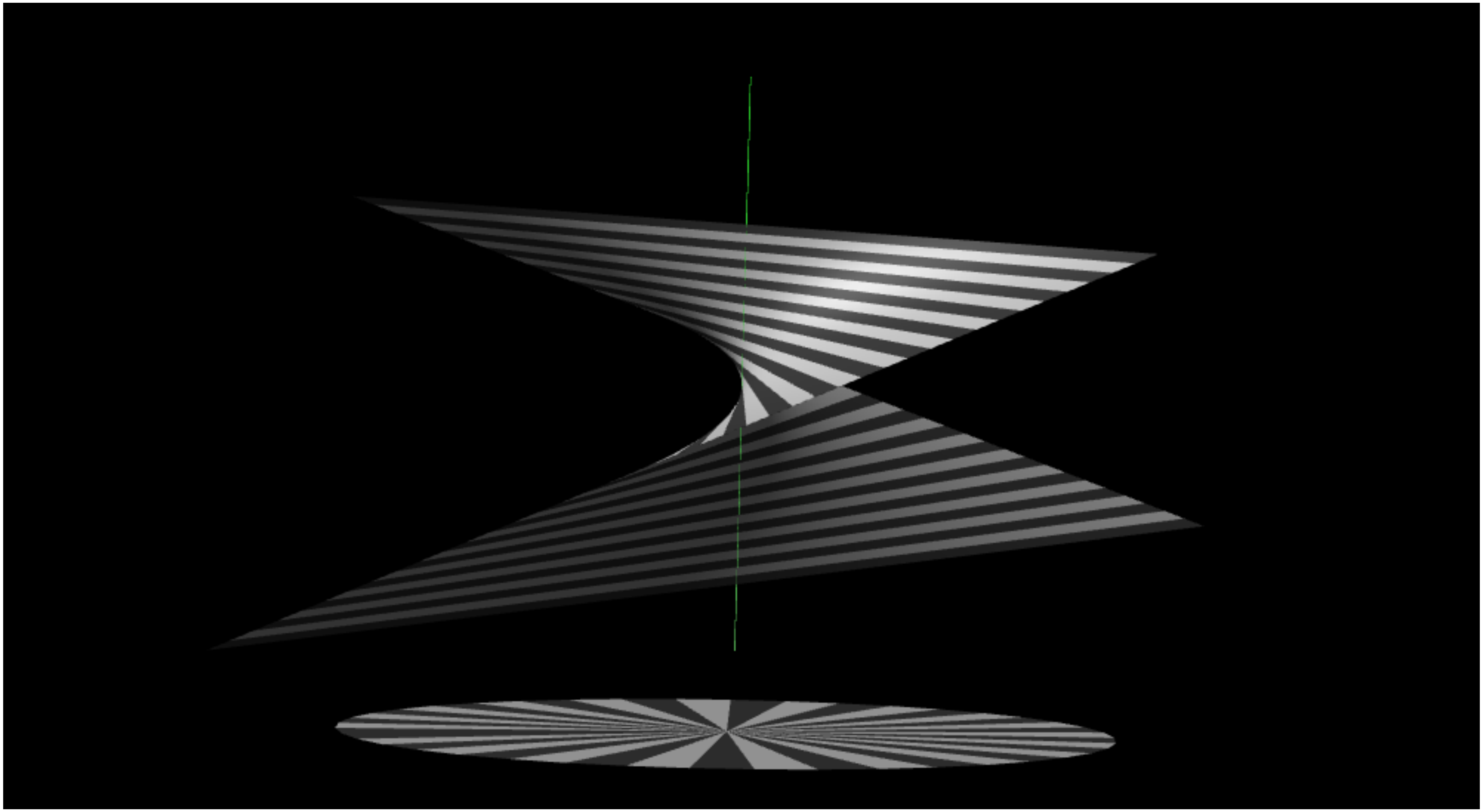


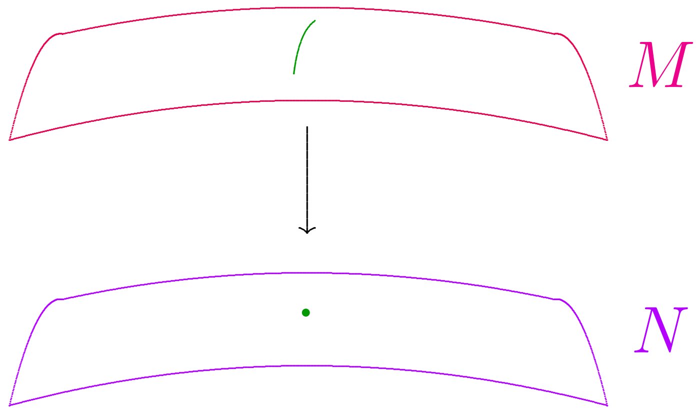


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

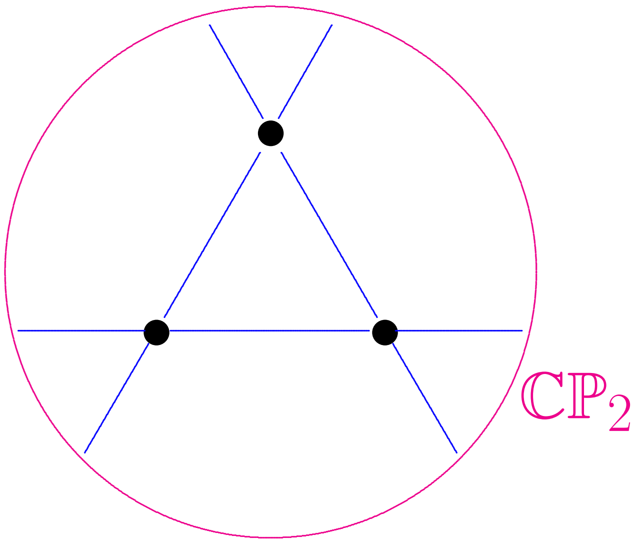
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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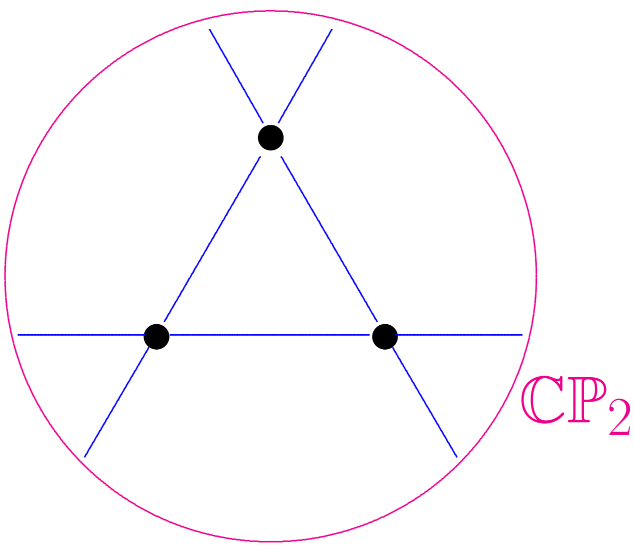

No 3 on a line,

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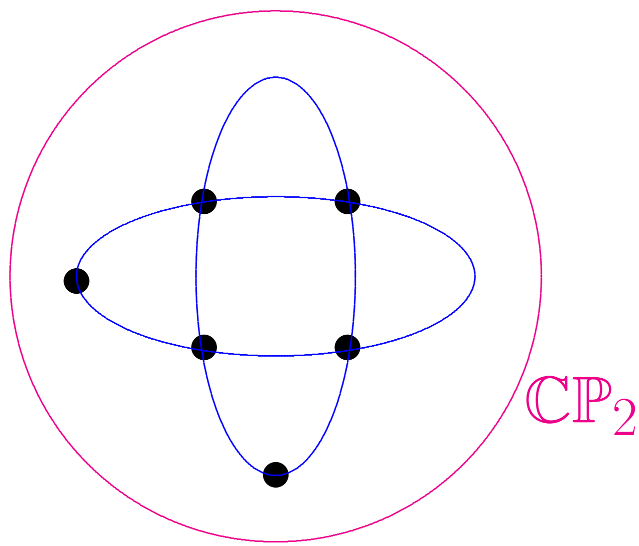

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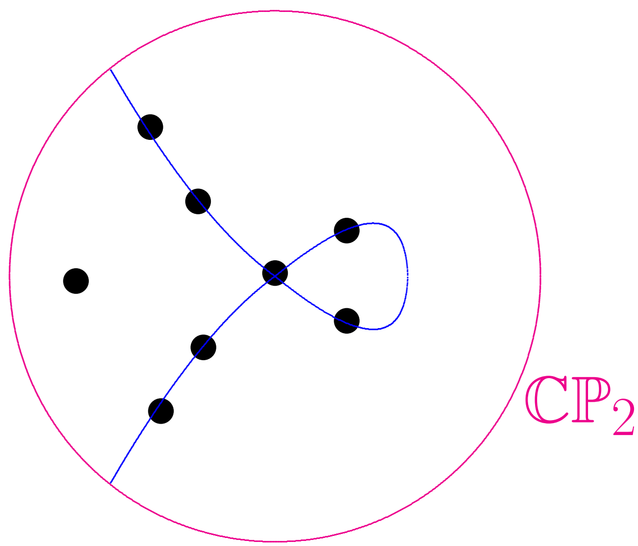

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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$$g = g(J\cdot, J\cdot)$$

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L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi.

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Existence: Page

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

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Understand all Einstein metrics on del Pezzos.

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Wu's criterion:

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Top eigenspace $L \subset \Lambda^+$ of W_+ is a line bundle.

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface.*

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W_+) > 0$*, and these sweep out exactly one connected component of* moduli space $\mathscr{E}(M)$*.*

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This illustrates how gravitational instantons play a crucial role, even when studying compact case.

Gravitational Instantons?

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Definition. *A gravitational instanton is a complete, non-compact, non-flat,*

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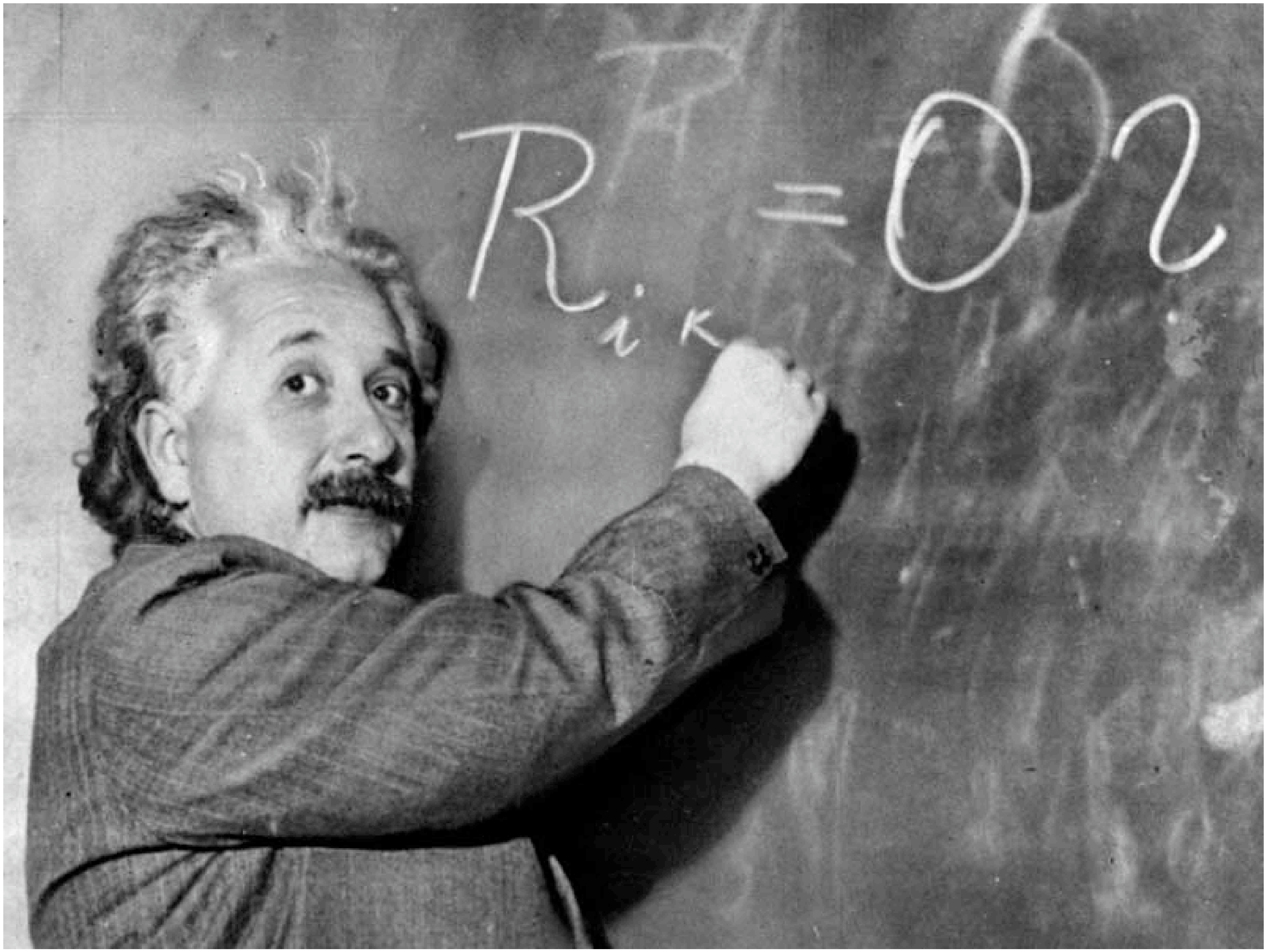

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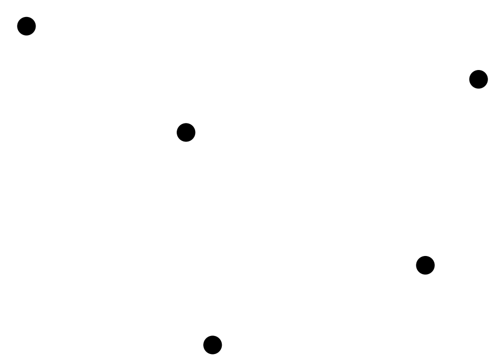

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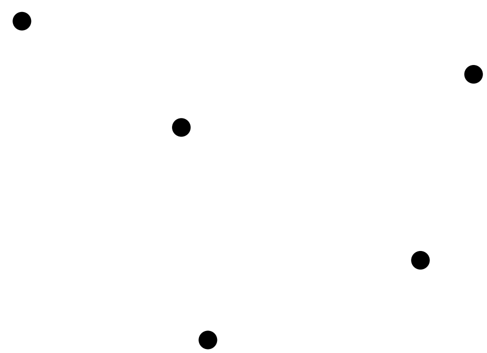

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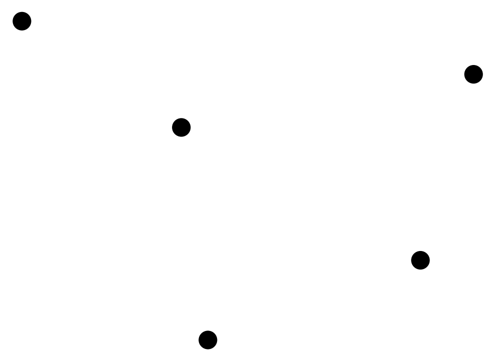

Data: ℓ points in $\mathbb{R}^3$ and $\kappa^2 \Longrightarrow V$ with $\Delta V = 0$

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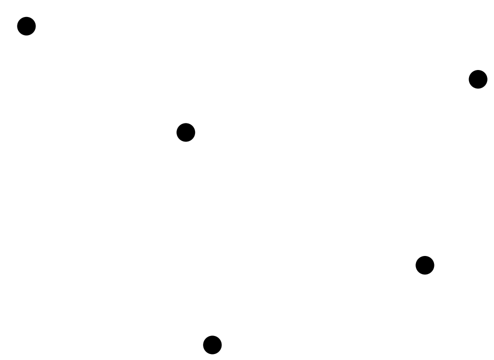

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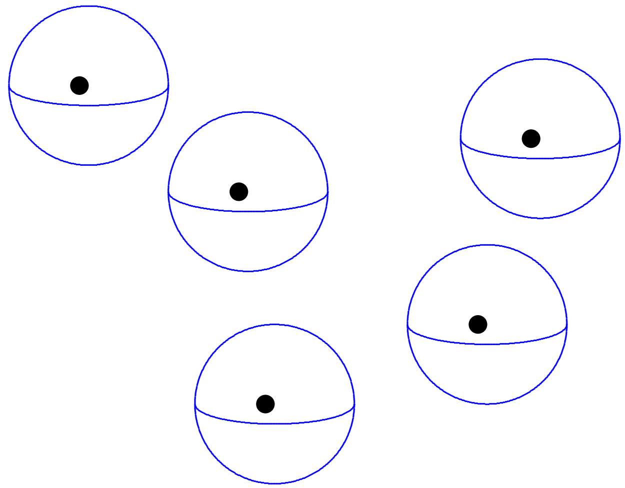

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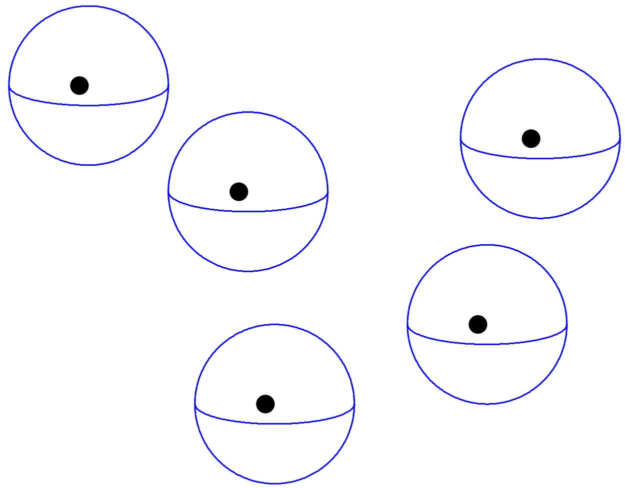

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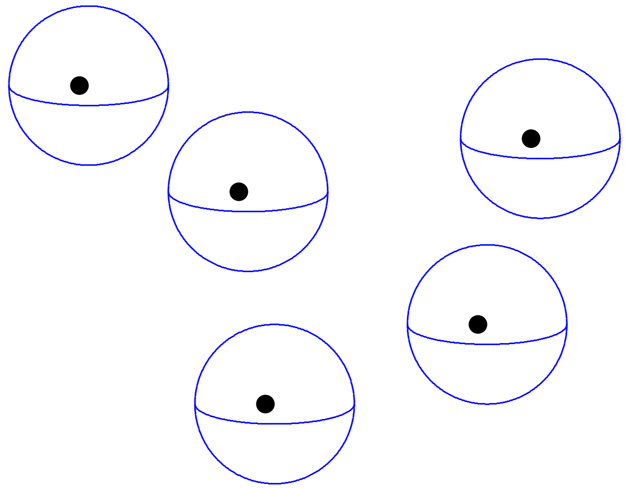

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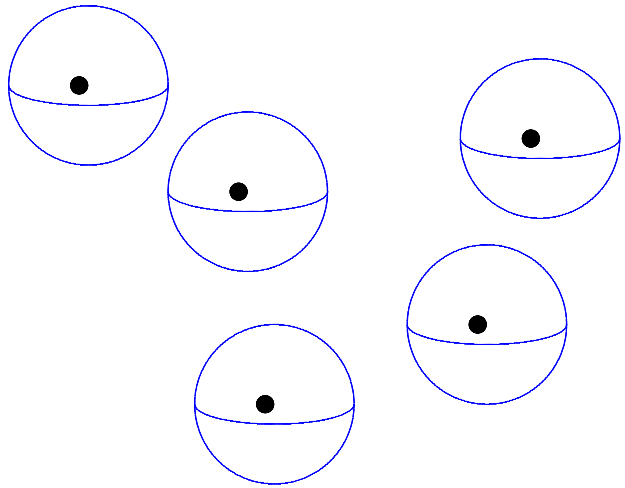

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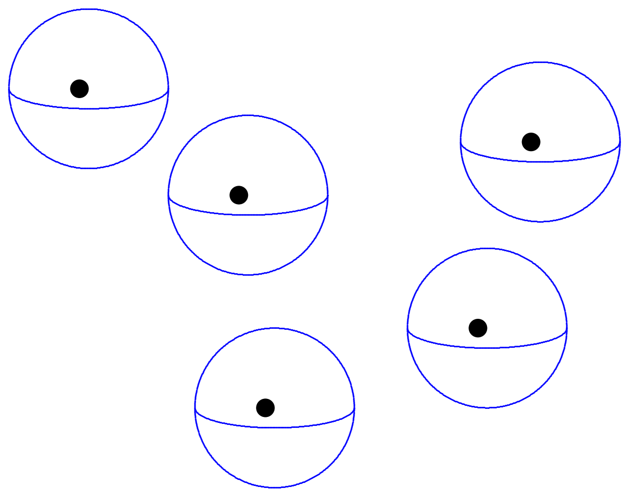

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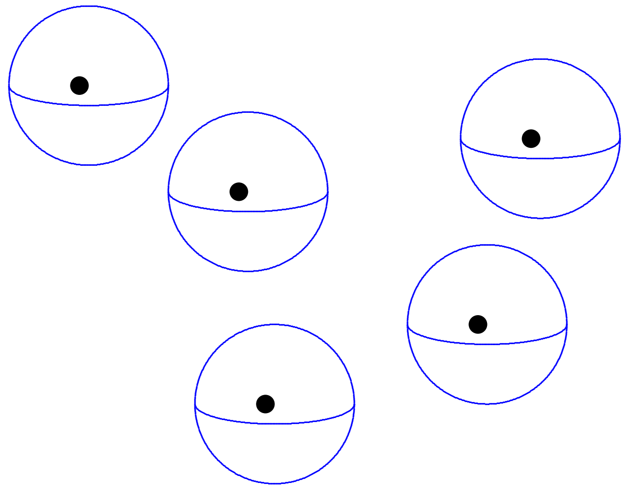

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

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This J determines opposite orientation from the hyper-Kähler complex structures.

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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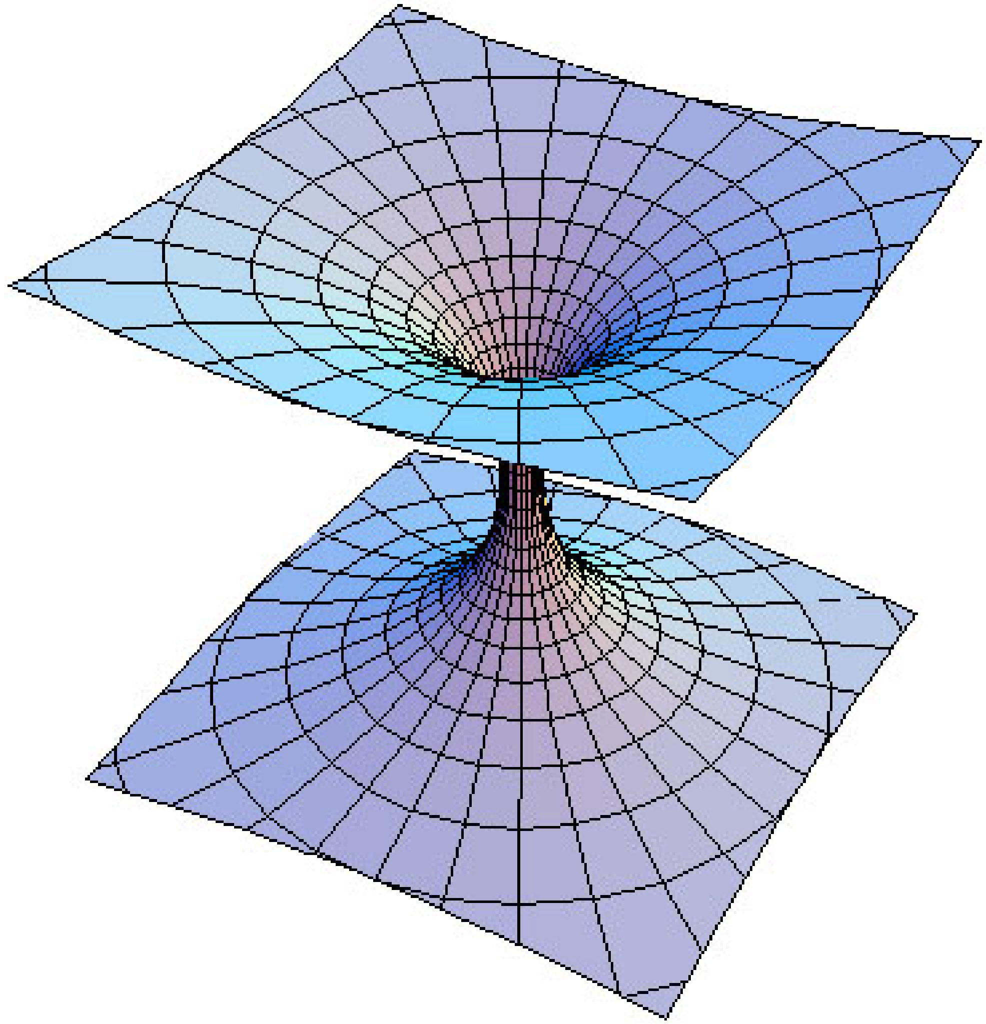

$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$

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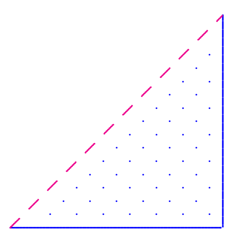

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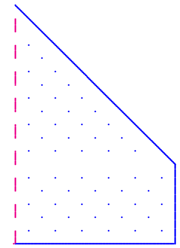

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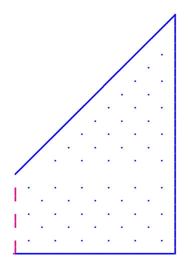

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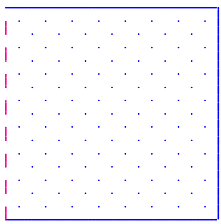

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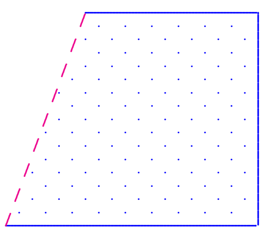

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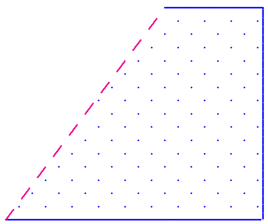

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Y. Chen & E. Teo, 2011

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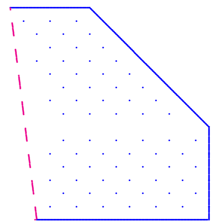

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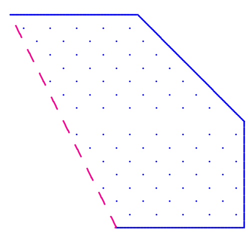

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allowing one to generalize methods first explored in the compact case.

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Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. Then

$$0 \geq |\nabla\omega|^2 + 3\langle\omega, (d + d^*)^2\omega\rangle$$

at every point, with respect to h.

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This optimal result combines **Theorem A** with a result of Mingyang Li, `arXiv:2310.13197`.

Thanks for the invitation!

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