Einstein Constants

and

Differential Topology

Claude LeBrun
Stony Brook University

Mathematics Colloquium,
University of Wisconsin, Madison,
April 11, 2025.

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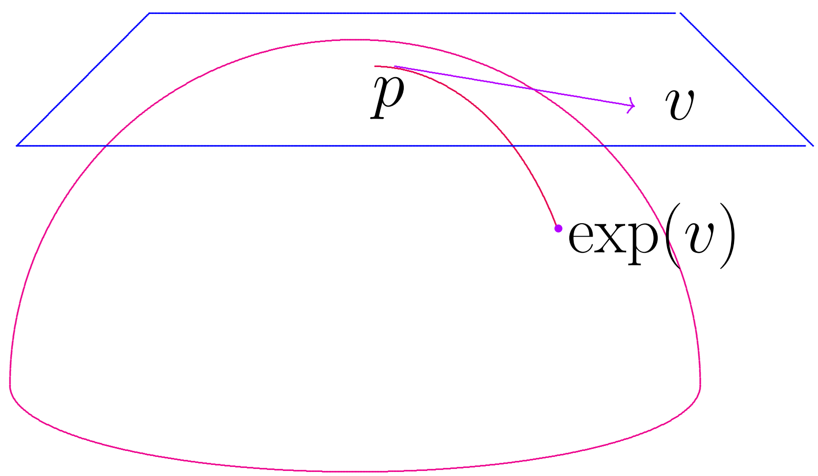

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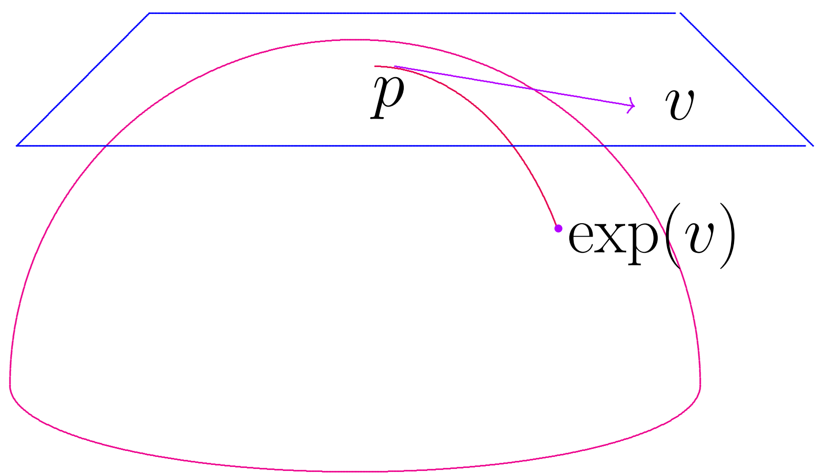

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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— A. Einstein, to G. Gamow

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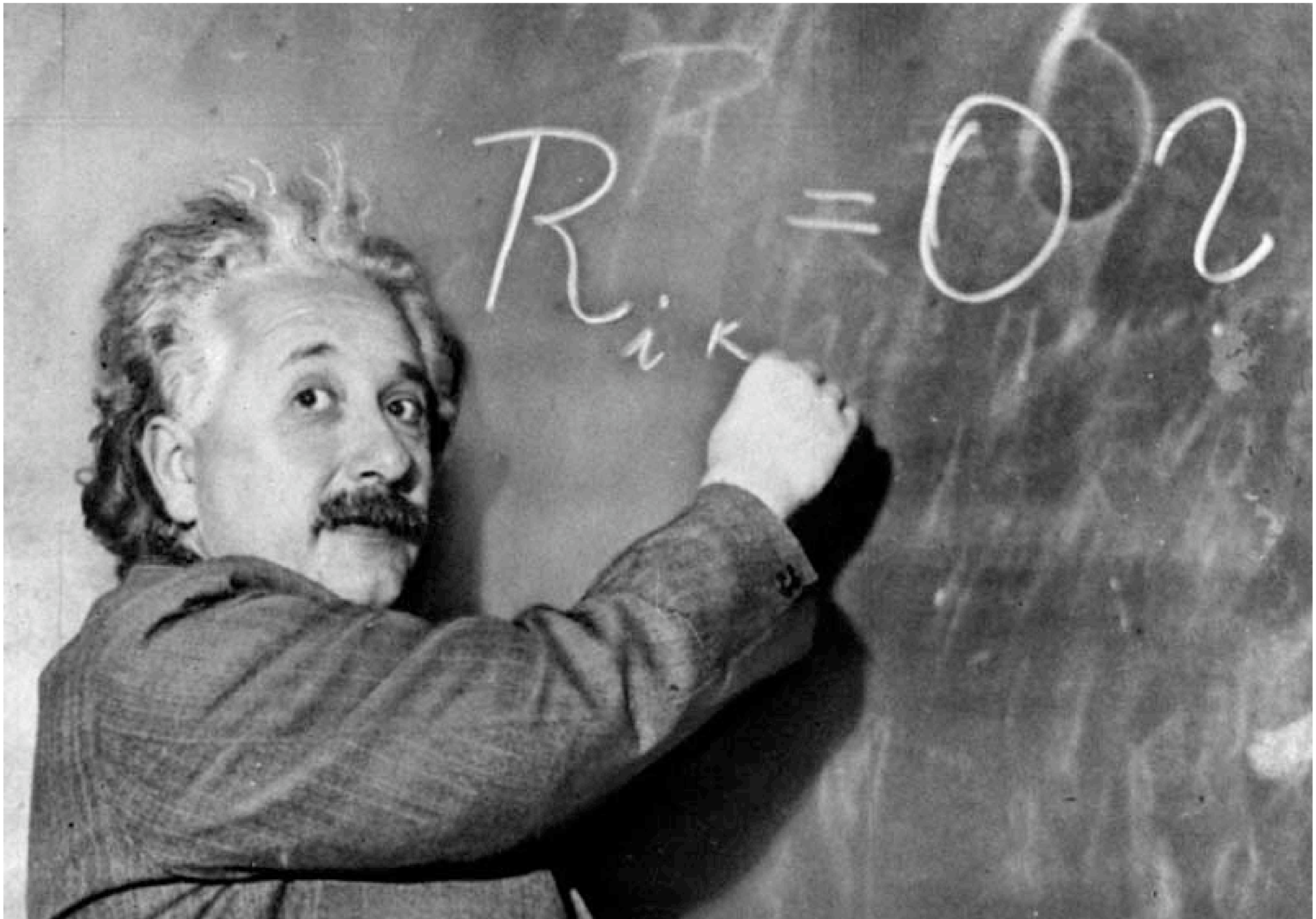

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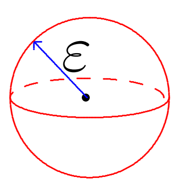

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Ergebnisse der Mathematik und ihrer Grenzgebiete
3. Folge · Band 10
A Series of Modern Surveys in Mathematics

Arthur L. Besse

Einstein Manifolds

Springer-Verlag Berlin Heidelberg GmbH

Marcel Berger

Besse en Chandesse

L'Auvergne

Acknowledgements

Pour rassembler les éléments un peu disparates qui constituent ce livre, j'ai dû faire appel à de nombreux amis, heureusement bien plus savants que moi. Ce sont, entre autres, Geneviève Averous, Lionel Bérard-Bergery, Marcel Berger, Jean-Pierre Bourguignon, Andrei Derdzinski, Dennis M. DeTurck, Paul Gauduchon, Nigel J. Hitchin, Josette Houillot, Hermann Karcher, Jerry L. Kazdan, Norihito Koiso, Jacques Lafontaine, Pierre Pansu, Albert Polombo, John A. Thorpe, Liane Valère.

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Enfin, qu'il me soit permis de saluer ici mon prédecesseur et homonyme Jean Besse, de Zürich, qui s'est illustré dans la théorie des fonctions d'une variable complexe (voir par exemple [Bse]).

Vôtre,

Arthur Besse

Le Faux, le 15 septembre 1986

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But it turns out that the answer is actually Yes!

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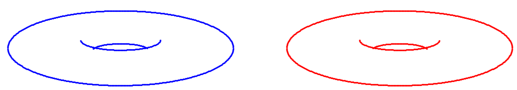

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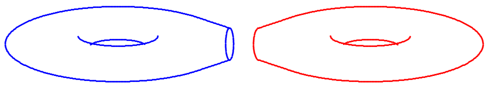

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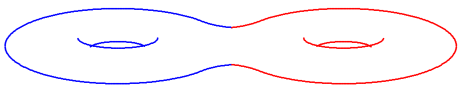

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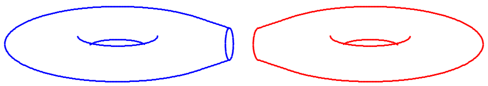

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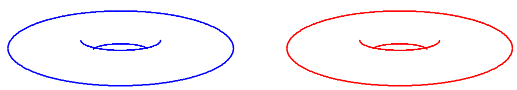

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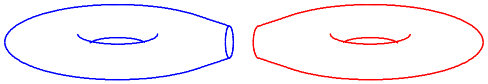

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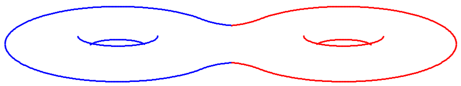

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of two Einstein manifolds is Einstein

$\Longleftrightarrow$ they have the same Einstein constant λ.

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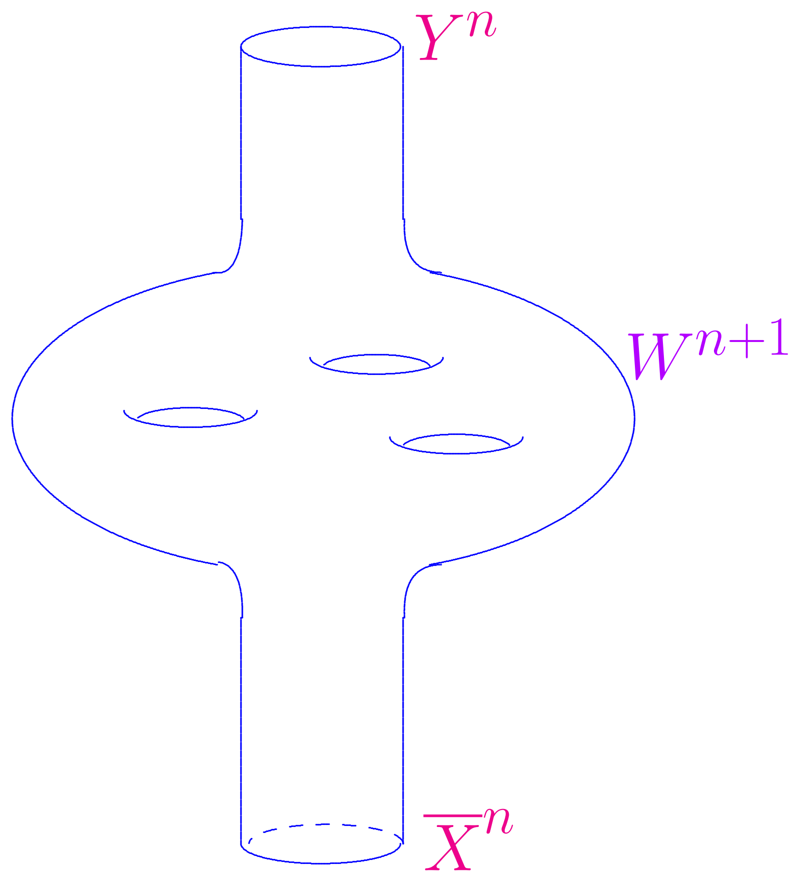
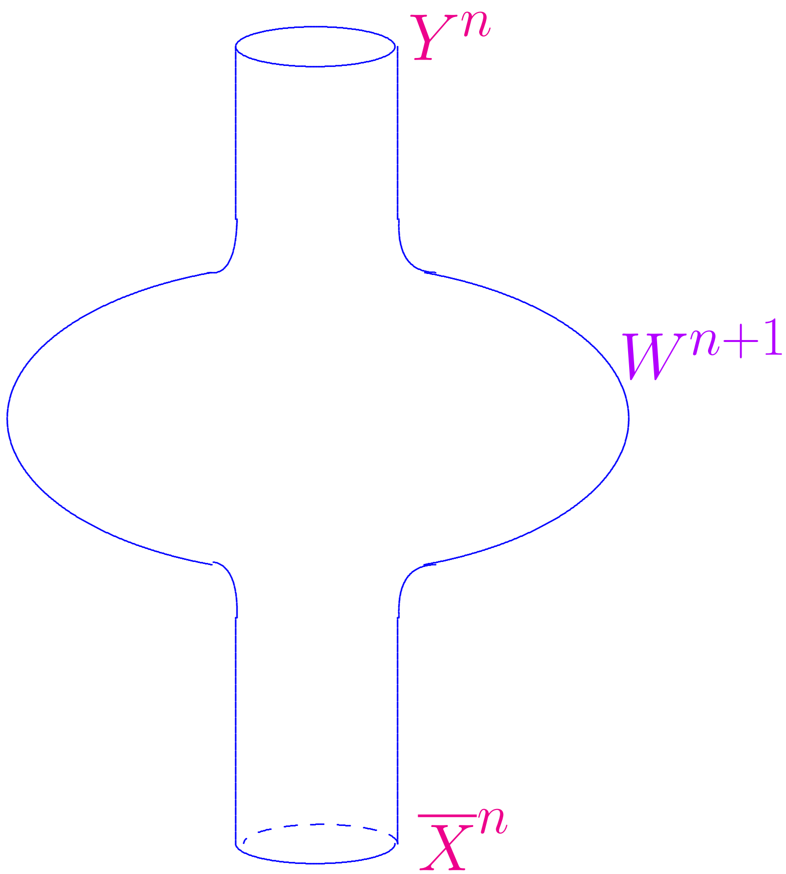
- Smale's h-cobordism theorem.

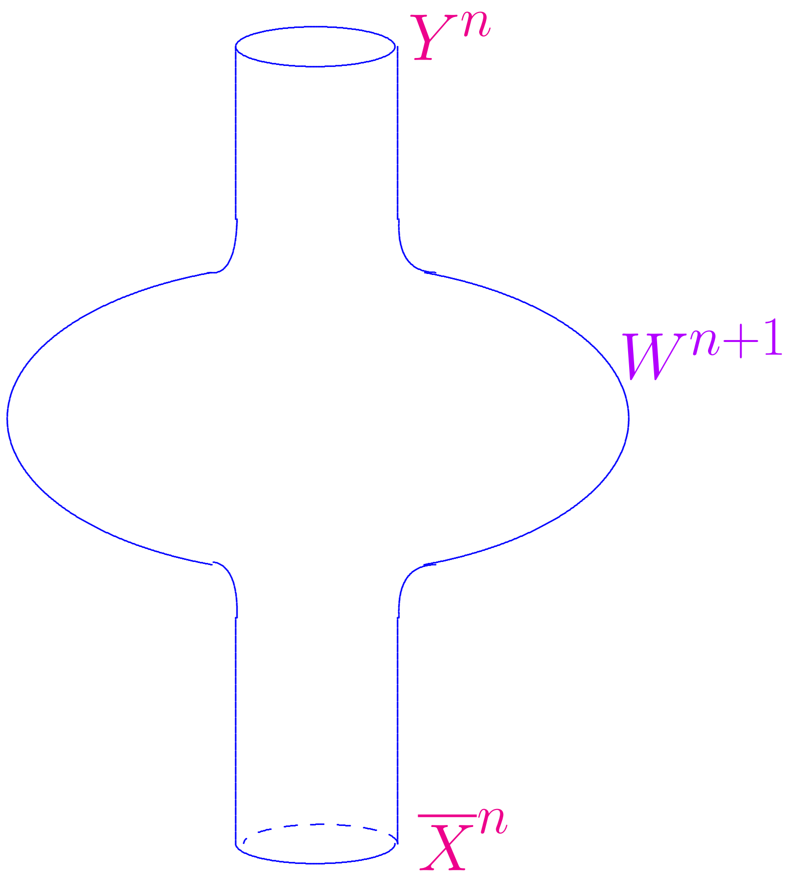


Cobordism

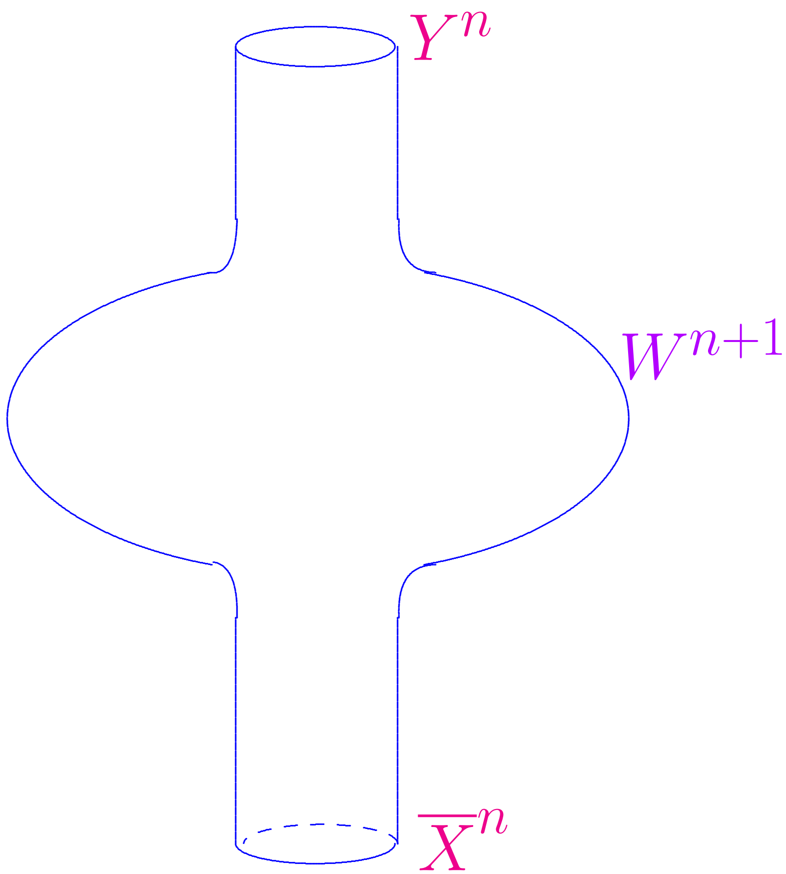


h-Cobordism

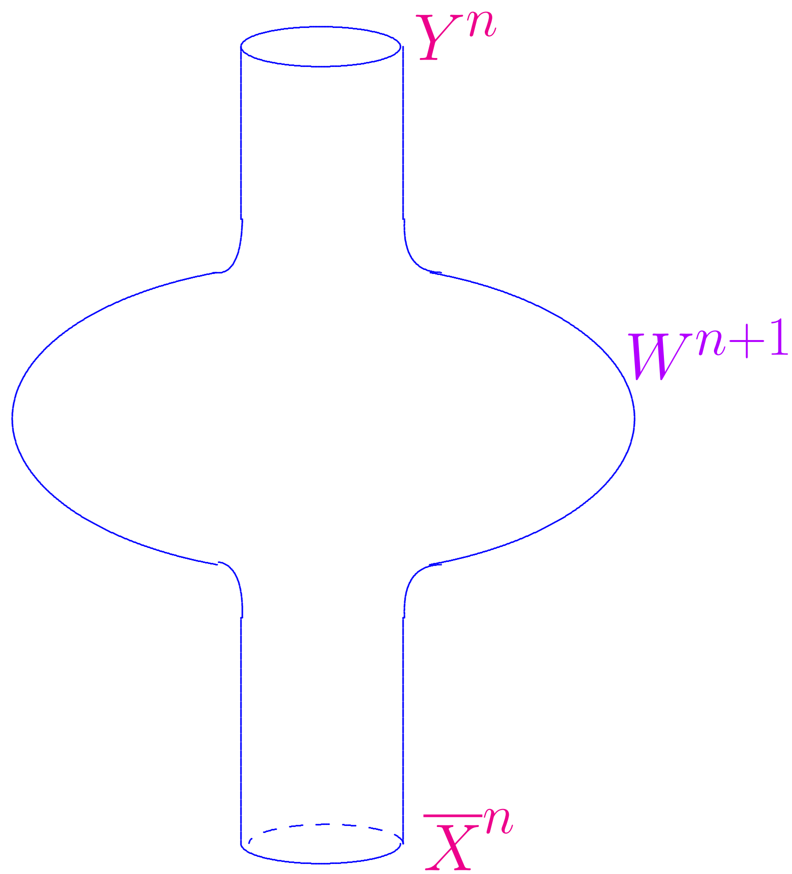
if $X \hookrightarrow W$, $Y \hookrightarrow W$ both homotopy equivalences



Smale: Suppose that X^n is h-cobordant to Y^n. If $\pi_1 = 0$ and $n > 4$, then X is diffeomorphic to Y.



Wall: Suppose that X^4 homotopy equivalent to Y^4.
If $\pi_1 = 0$, then X is h-cobordant to Y.



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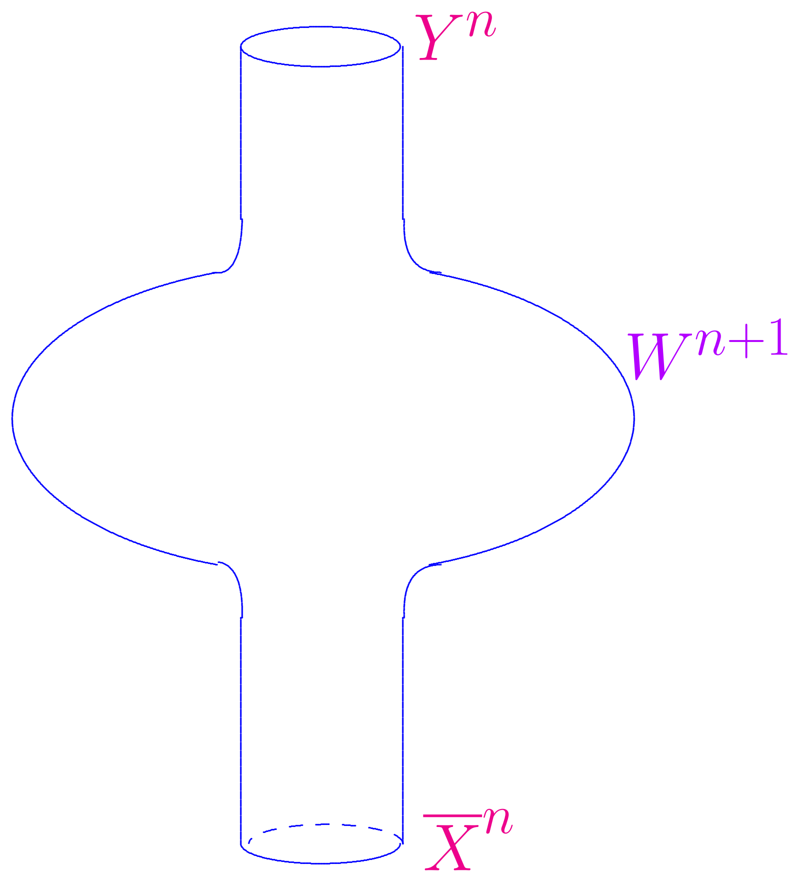
But Smale doesn't apply when $n = 4$!



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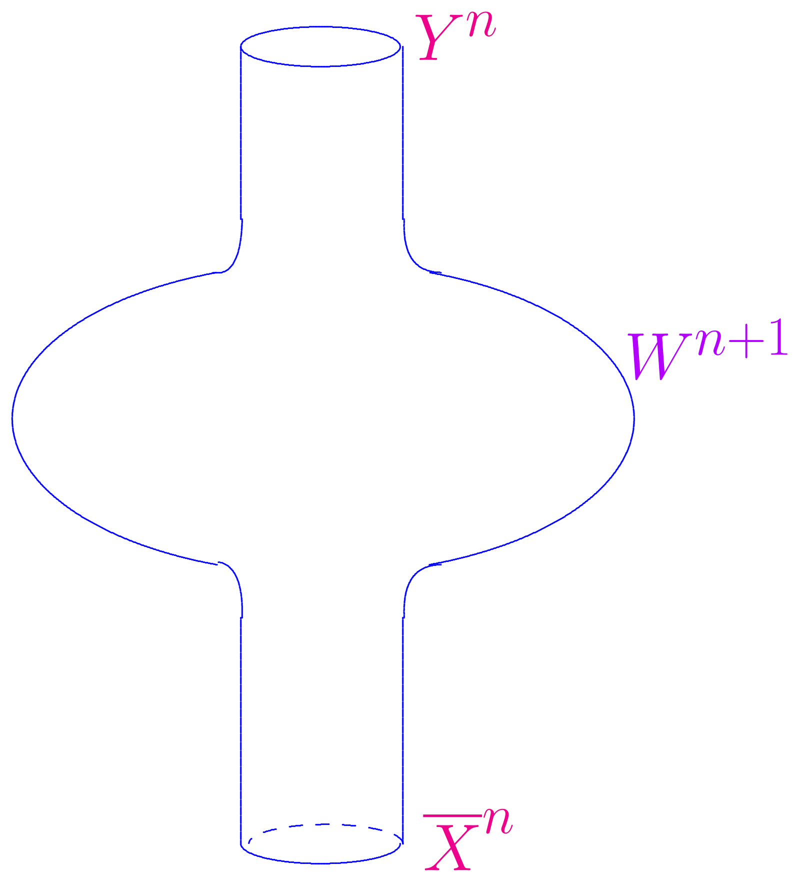

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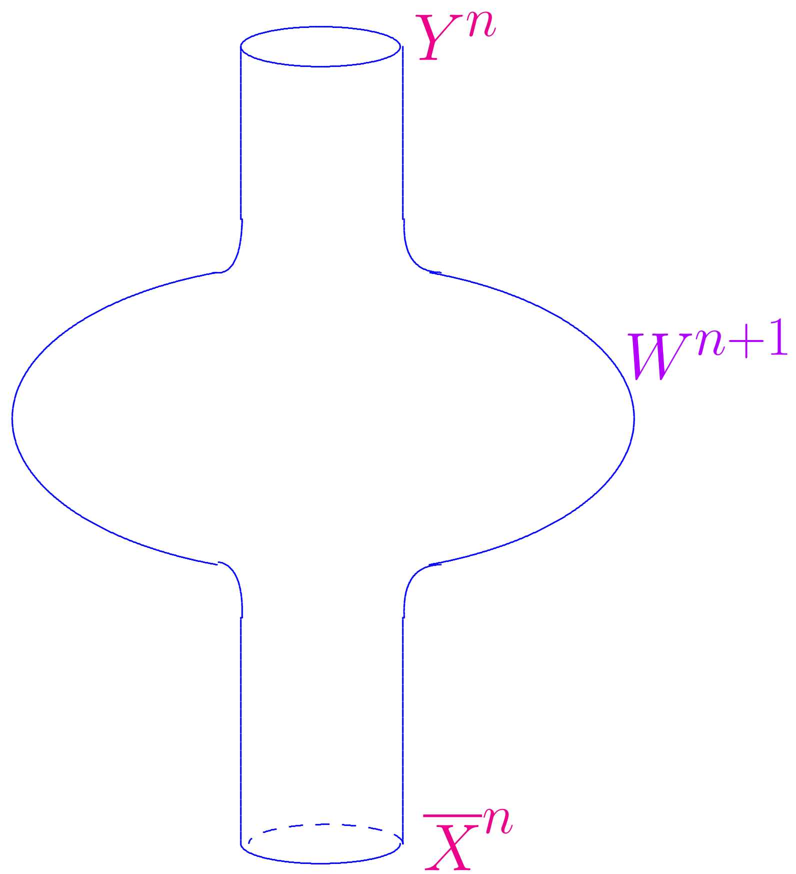
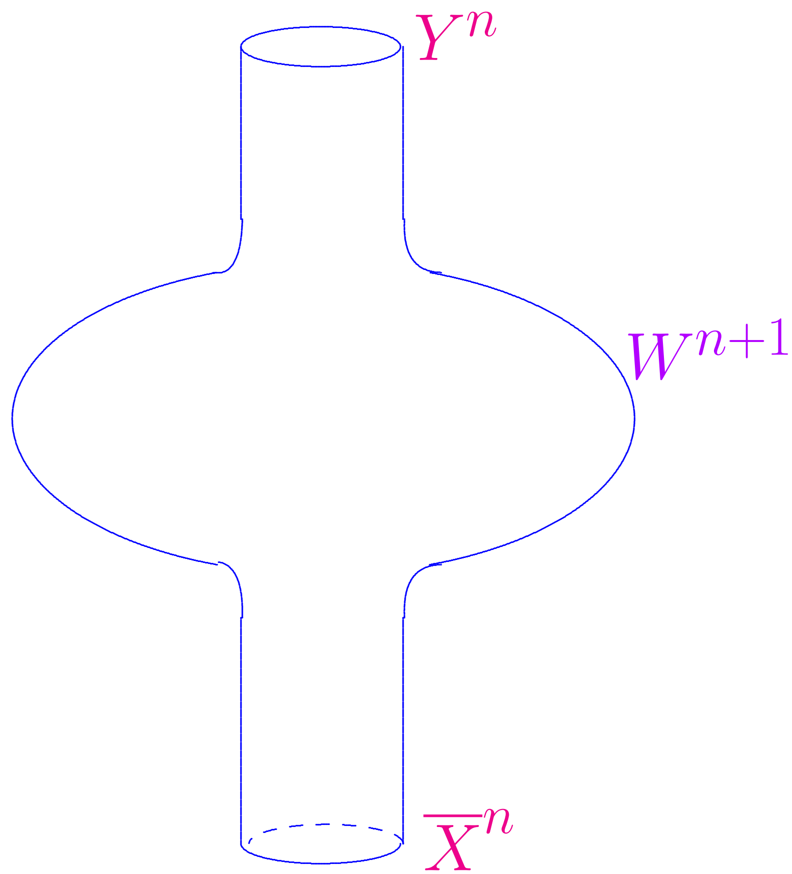
Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then $X \times X$ is actually diffeomorphic to $Y \times Y$.*



Indeed, if W is h-cobordism $X \sim Y$, then

$$(X \times W) \cup_{X \times Y} (W \times Y)$$

is an h-cobordism $(X \times X) \sim (Y \times Y)$.



Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then*

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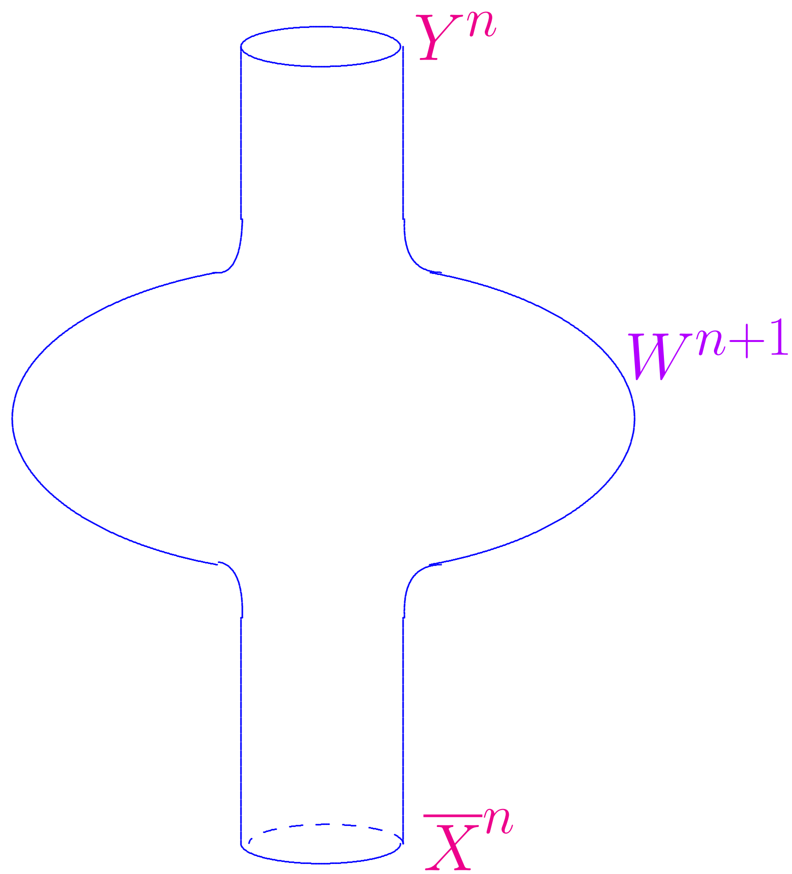

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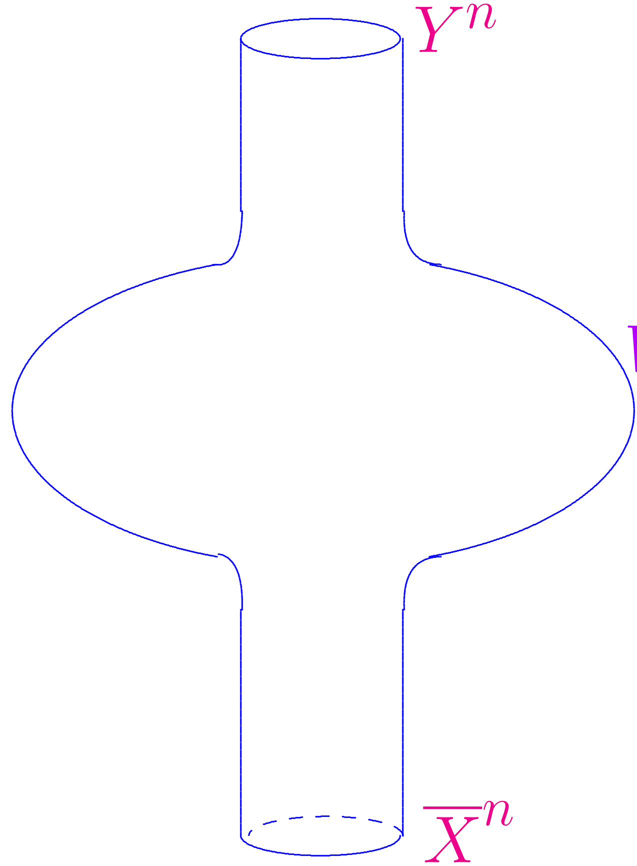

Lemma. *If* X^4 *and* Y^4 *are simply connected, non-spin, with* $\chi(X) = \chi(Y)$, $\tau(X) = \tau(Y)$, *then*

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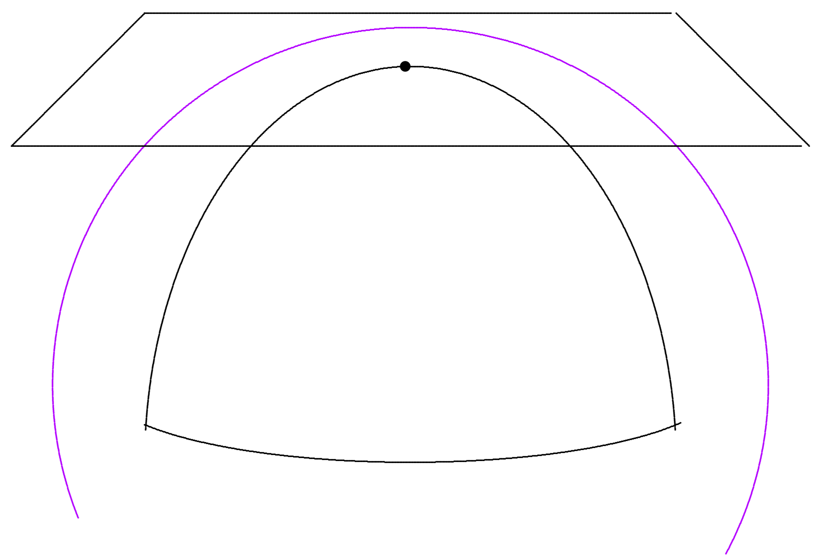

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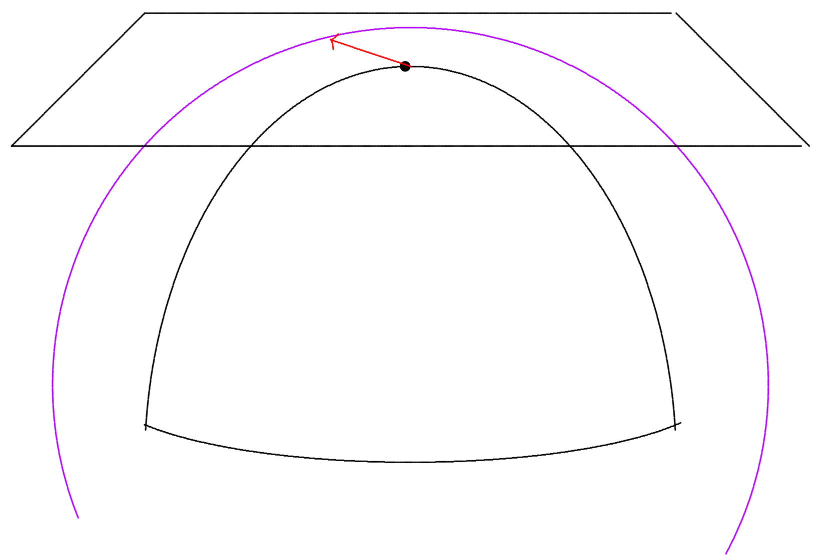

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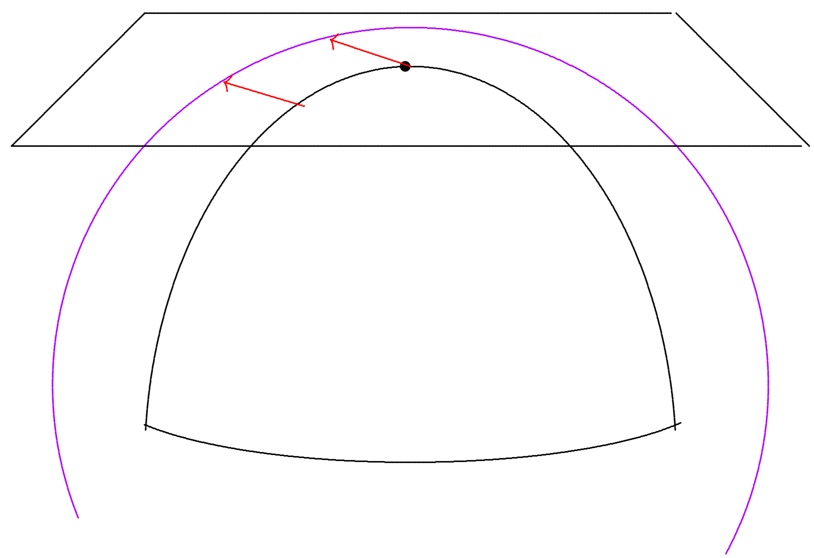

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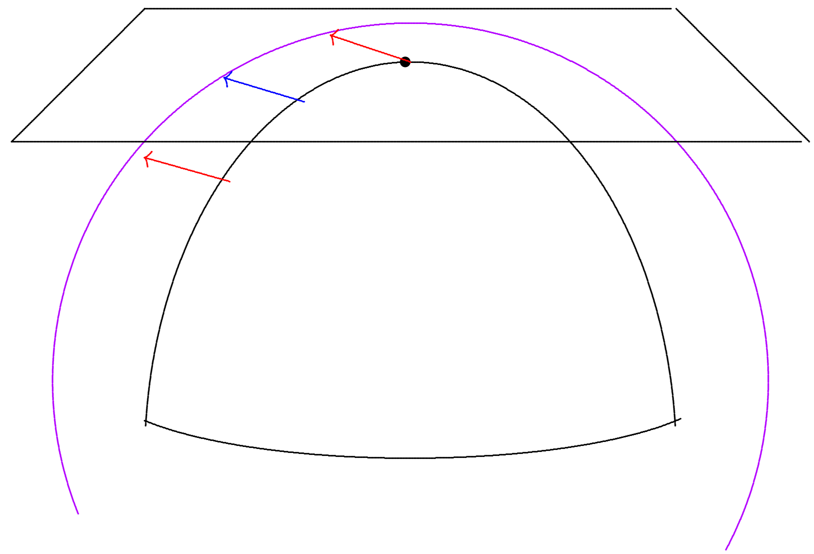

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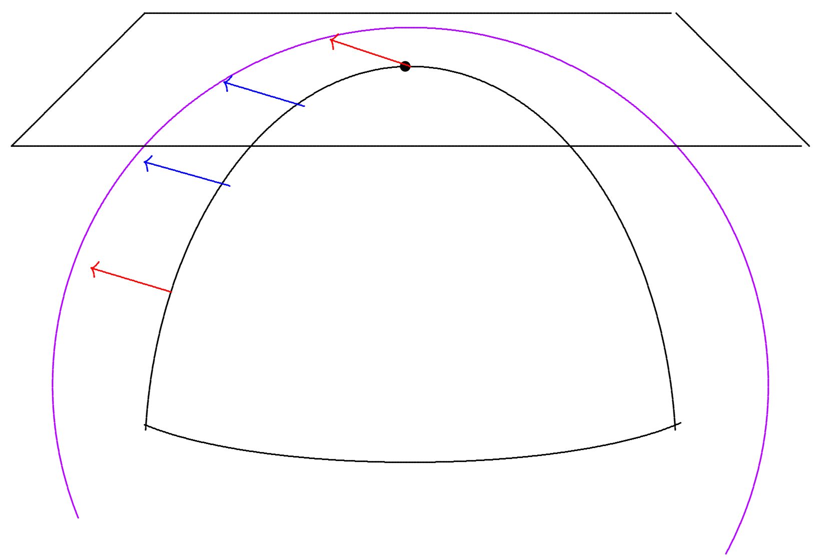

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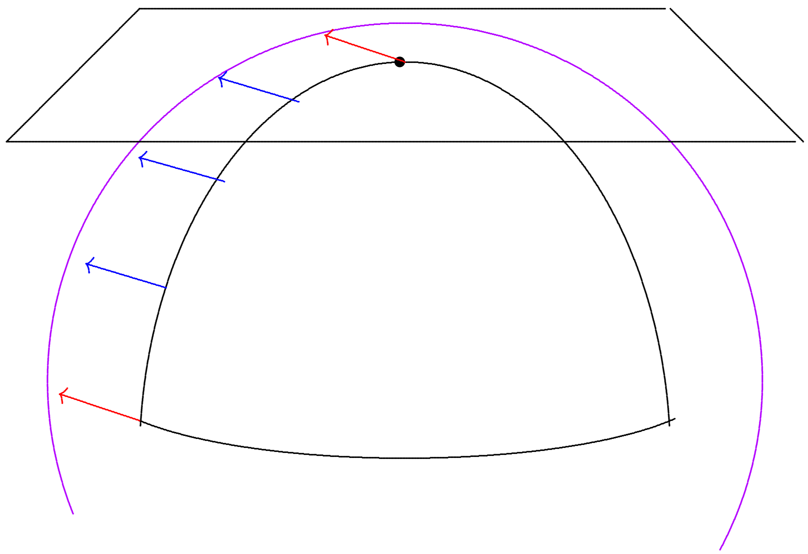

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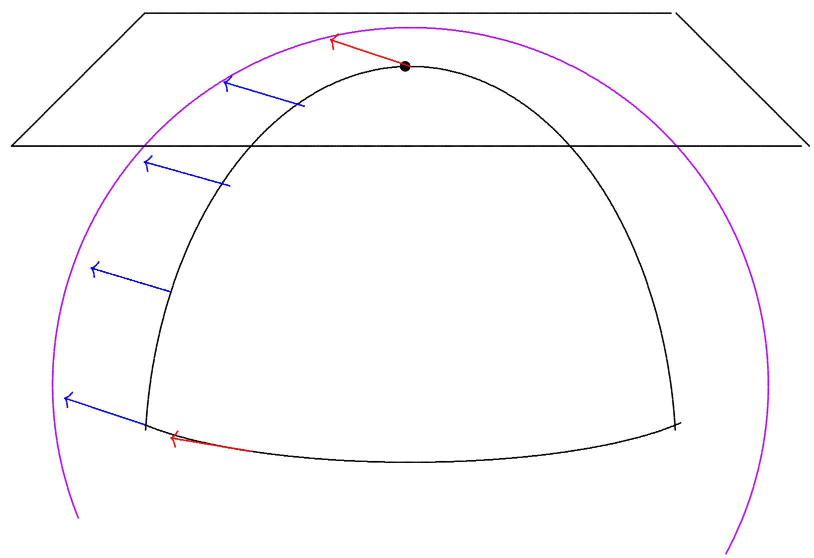

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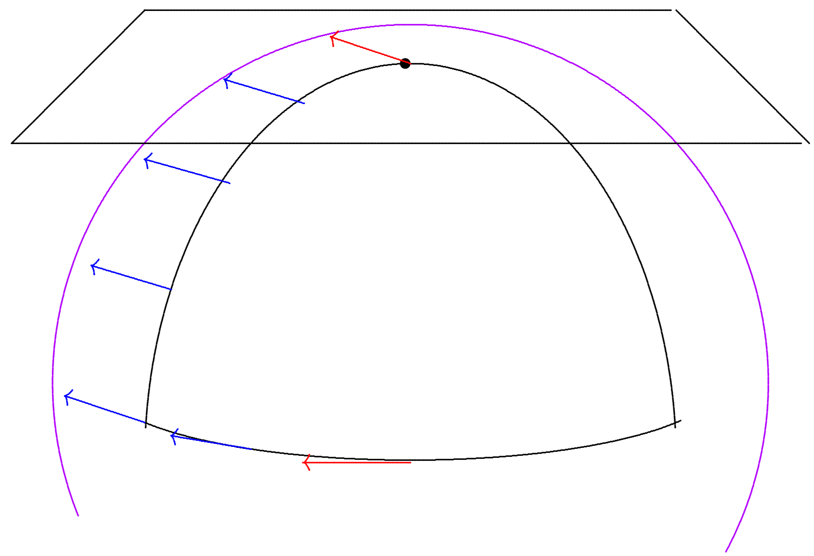

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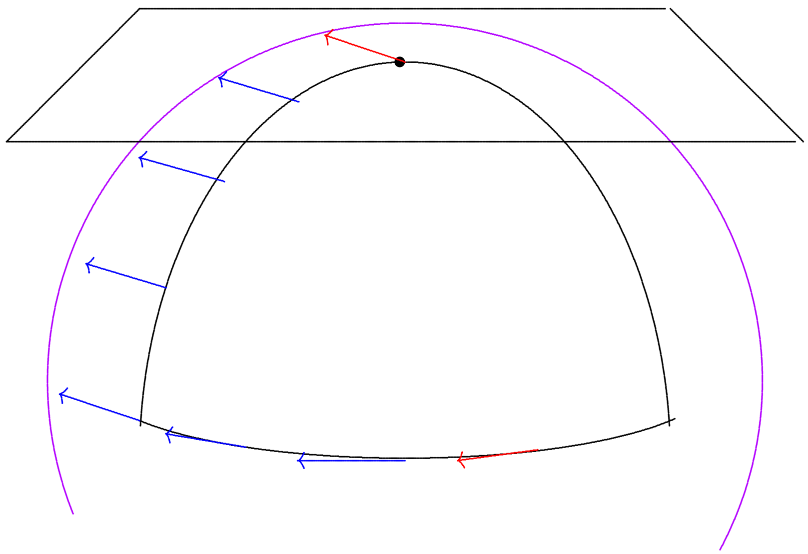

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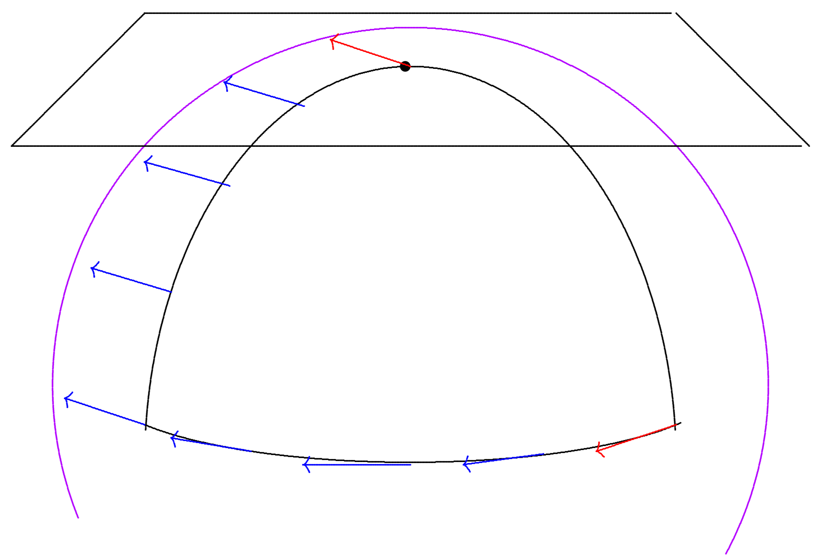

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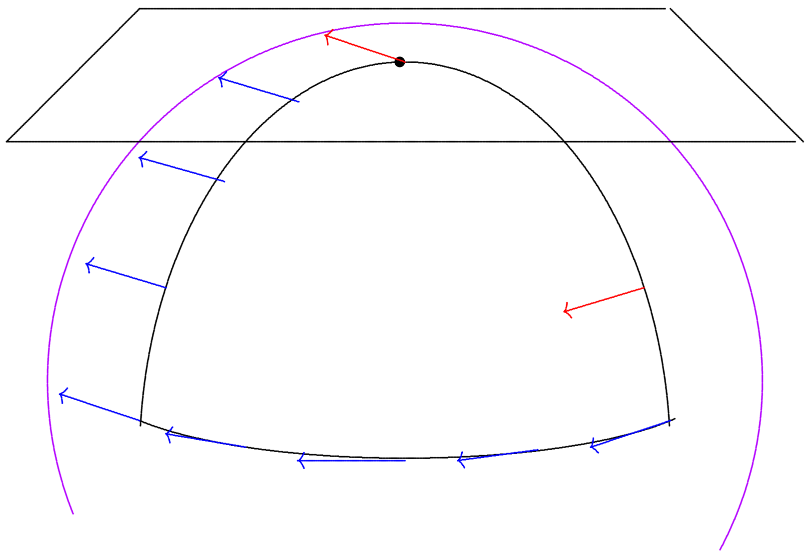

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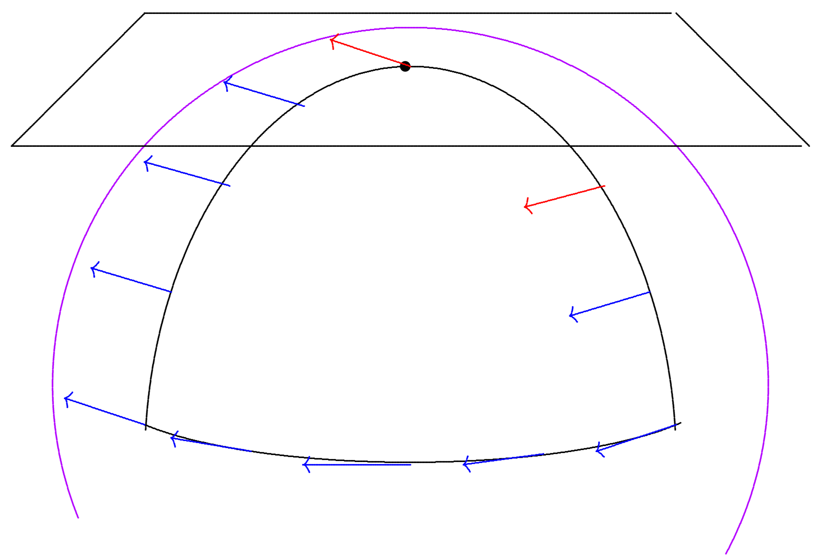

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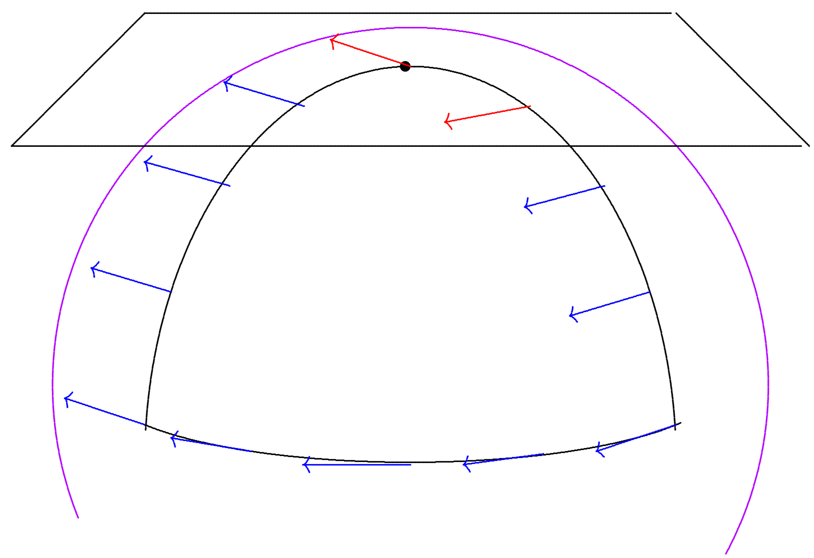

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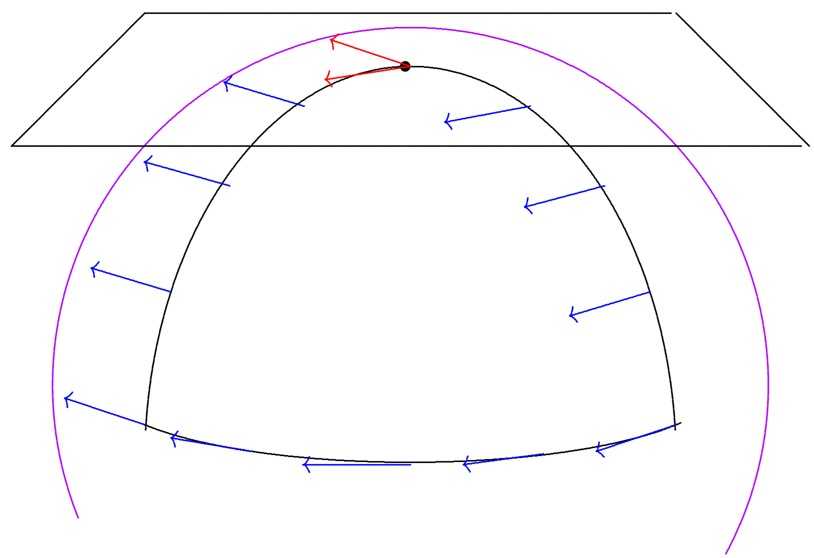

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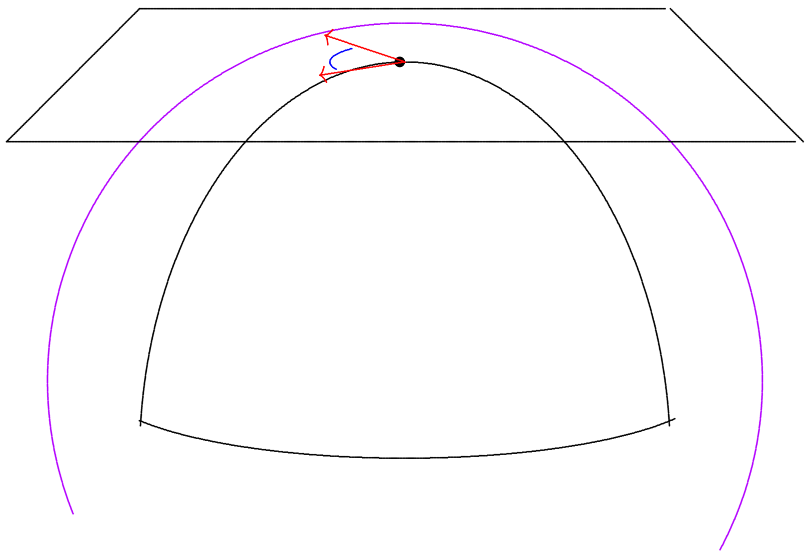
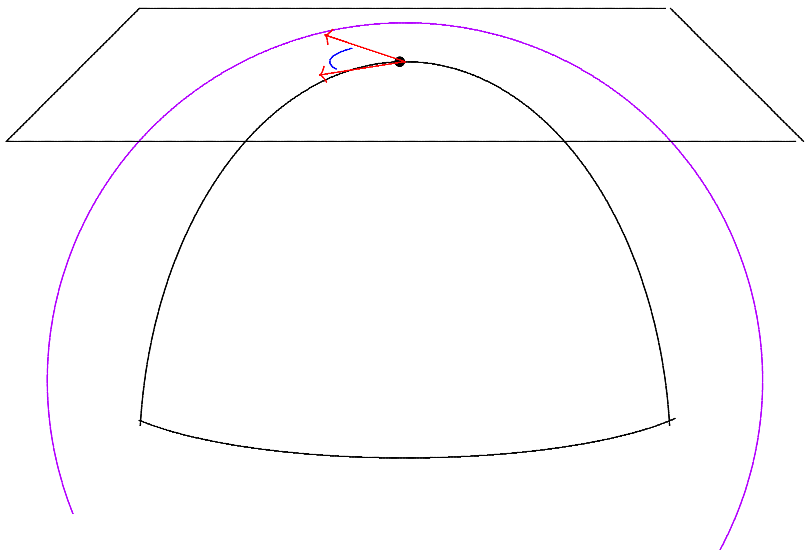


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

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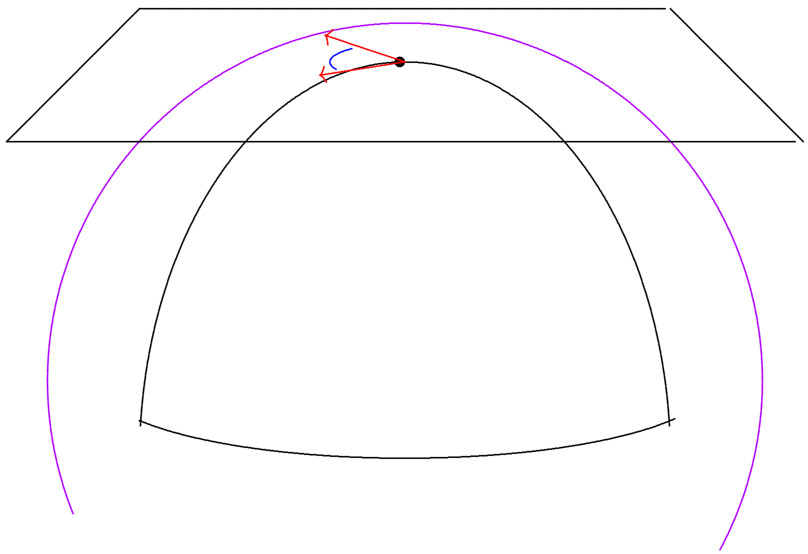

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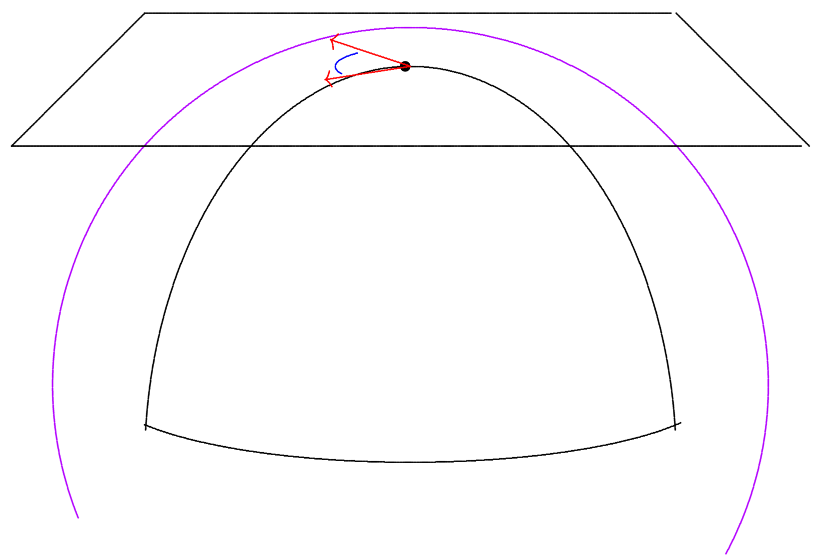

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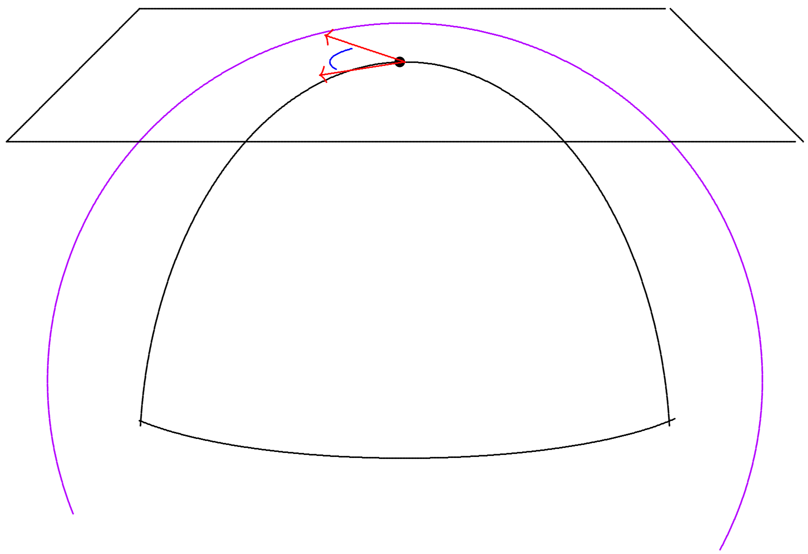

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Kähler, but for wildly different complex structures.

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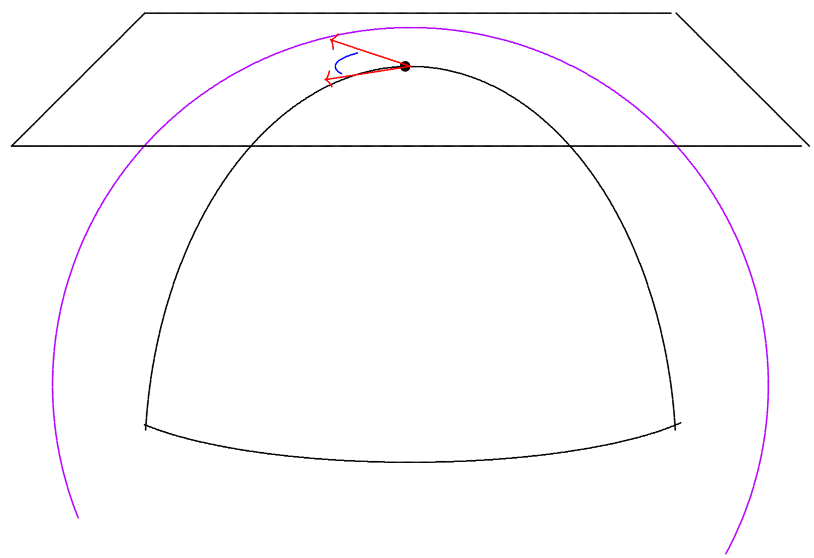

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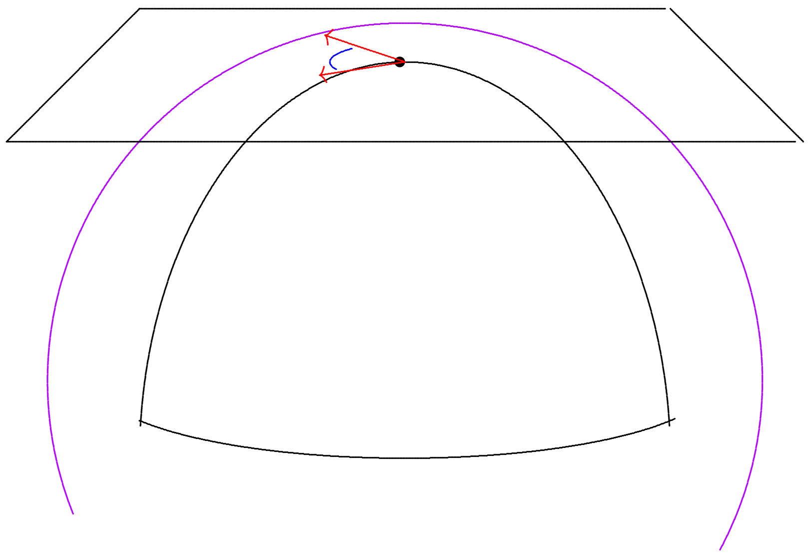

What are the possible holonomy groups?

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Theorem (Berger 1955...). *Let (M^n, g) be a complete, simply-connected Riemannian manifold. If (M, g) is neither a product of lower-dimensional manifolds nor a symmetric space, then its holonomy group appears on the following table:*

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n	$\mathbf{SO}(n)$	generic	$\mathbb{R}$
$2m$	$\mathbf{U}(m)$	Kähler	$\mathbb{C}$
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$$\mathbf{G}_2 := \mathrm{Aut}(\mathbb{O}) \subset \mathbf{SO}(7)$$

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Simons, Aleksevskii, Calabi, Hitchin, Bryant, Joyce…

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dimension n	$\mathrm{Hol}(M^n, g)$	geometry	Einstein?
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$2m$	$\mathbf{SU}(m)$	Calabi-Yau	✓
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n	$\mathbf{SO}(n)$	generic	?
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The latter seems to be especially delicate!

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Gromov-Lawson: $\widetilde{M}^7$ admits metrics with $s > 0$.

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Can such metrics coexist?

Alas, no!

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Theorem. *No smooth compact* M^7

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To make this plausible, will first illustrate assertion for prototypical examples due to S. Kobayashi '63.

Prototypical examples:

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$$\begin{array}{ccc} S^1 & \to & M^7 \\ & & \downarrow \\ & & X^6 \end{array}$$

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(X, h) Kähler-Einstein, $\lambda = 8$.

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Replace Hard Lefschetz on $H^*(X, \mathbb{R})$ with transverse version on $H^*_B(M, \mathfrak{F})$ due to El Kacimi-Alaoui.

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Theorem. *No smooth compact* M^7 *can admit both a Sasaki-Einstein metric* g_1 *and a metric* g_2 *with holonomy* $\subset \mathbf{G}_2$.

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But no one has found Calabi-Yau parters for these Fano manifolds!

Thanks for the invitation!

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