EXPLICIT SELF-DUAL METRICS ON $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$

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Abstract. We display explicit half-conformally-flat metrics on the connected sum of any number of copies of the complex projective plane. These metrics are obtained from magnetic monopoles in hyperbolic 3-space by an analogue of the Gibbons-Hawking ansatz, and are conformal compactifications of asymptotically-flat, scalar-flat Kähler metrics on n-fold blow-ups of $\mathbb{C}^2$. The corresponding twistor spaces are also displayed explicitly, and are observed to be Moishezon manifolds— that is, they are bimeromorphic to projective varieties.

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§1. Introduction

Motivated by examples due to Poon [P_1], Donaldson and Friedman [DF] have proved the existence of self-dual conformal metrics on the connected sum

$$n\mathbb{CP}_2 := \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{n}$$

of any number of copies of the complex projective plane. (Here a Riemannian metric on an oriented 4-manifold is called *self-dual* if its Weyl curvature, considered as a bundle-valued 2-form, is in the +1 eigenspace of the Hodge star operator; an orientable Riemannian 4-manifold is called *half-conformally-flat* if this holds for at least one orientation). Their method involves a delicate desingularization of a singular model of the desired twistor space. An analytic argument for this existence theorem has also been given by Floer [Fl].

In this paper, we will obtain stronger results by more elementary methods. In fact, we will write down such metrics explicitly for each value of n by looking only for metrics with an S^1-symmetry, and observe that, in contrast to their generic deformations, the twistor spaces of the constructed metrics are *Moishezon*, meaning that they are bimeromorphically equivalent to projective-algebraic varieties, and are thus themselves abstract-algebraic. This shows that a recent result of Campana [Cp], to the effect that a compact self-dual 4-manifold with Moishezon twistor space must be homeomorphic to $n\mathbb{CP}_2$, $n \geq 0$, is essentially sharp. An elementary computation also shows that these self-dual conformal classes have representatives of positive scalar curvature, as conjectured by Floer. This dove-tails nicely with an elegant argument of Poon [P_2], which shows that this positivity is actually a consequence of the fact that the twistor space is Moishezon.

Our starting point is the study of Kähler surfaces with vanishing scalar curvature; it has long been noted ([D], cf. [L_3]) that such scalar-flat complex 2-manifolds have anti-self-dual Weyl curvature (with respect to the complex orientation). We will begin by giving a reformulation of the problem of constructing such metrics with the additional hypothesis of the existence of an isometric circle action. This construction generalizes the ansatz for hyper-Kähler 4-manifolds found by Gibbons and Hawking [GH] on their quest for a quantum theory of gravity. After examining a key example (due to Dan Burns) through this new pair of spectacles, we are then able to exploit a surprising linearity of the construction in order to "superpose" various copies of Burns' metric; in fact, the metrics so constructed are close cousins of the A_k gravitational instantons, with the difference that the moment-map rôle played by Euclidean 3-space in the Gibbons-Hawking construction is now usurped by hyperbolic 3-space. In particular, the fact that the Gibbons-Hawking metrics are hyper-Kähler is echoed by the fact that the metrics we will construct herein are conformally Kähler for a variety of different complex structures.

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§2. A Generalized Gibbons-Hawking Ansatz

Gibbons and Hawking [GH] found a way of generating all Ricci-flat Kähler surfaces with a triholomorphic circle action in terms of solutions of Laplace's equation on $\mathbb{R}^3$. We now generalize this construction to deal with scalar-flat Kähler surfaces.

Proposition 1. *Let $w > 0$ and u be smooth real-valued functions on an open set $\mathcal{V} \subset \mathbb{R}^3$ which satisfy*

$$\boxed{\begin{array}{ll} (1) \qquad & u_{xx} + u_{yy} + (e^u)_{zz} = 0 \\ (2) \qquad & w_{xx} + w_{yy} + (we^u)_{zz} = 0. \end{array}}$$

Suppose, moreover, that the deRham class of the closed 2-form

$$\frac{1}{2\pi}\alpha = \frac{1}{2\pi}(w_x \, dy \wedge dz + w_y \, dz \wedge dx + (we^u)_z dx \wedge dy)$$

is integral— that is, contained in the image of $H^2(\mathcal{V}, \mathbb{Z}) \to H^2(\mathcal{V}, \mathbb{R})$. Let $M \to \mathcal{V}$ be a circle bundle such that $[c_1(M)]^{\mathbb{R}} = [\frac{1}{2\pi}\alpha]$, and let ω be a connection 1-form on M whose curvature is α. (Thus, if $\mathcal{V}$ is simply connected, M and ω are determined up to gauge equivalence). Then

$$g = e^u w(dx^2 + dy^2) + w dz^2 + w^{-1}\omega^2$$

is a Kähler metric on M whose scalar curvature vanishes.

Conversely, every scalar-flat Kähler surface with S^1-symmetry locally arises by this construction.

Remarks:
- If $u = 0$, this is the Gibbons-Hawking ansatz.
- The metric is Ricci-flat iff $u_z = cw$ for some constant c; cf. [BF].
- Equation (1) occurs in the physics literature [Pa] under the rubric of the "$\mathbf{SU}(\infty)$ Toda-lattice equation," while equation (2) is just its linearization.

Proof of the Proposition.

We define an almost complex structure J on M by the prescription

$$dz \mapsto w^{-1}\omega, \qquad dx \mapsto dy.$$

Since our connection form ω satisfies

$$d\omega = (w_x \, dy \wedge dz + w_y \, dz \wedge dx + (e^u w)_z dx \wedge dy),$$

it follows that J is integrable; indeed,

$$\begin{aligned} d[wdz+i\omega] &= dw\wedge dz + i d\omega \\ &= w_x dx\wedge dz + w_y\, dy\wedge dz \\ &\quad + i(w_x\, dy\wedge dz + w_y\, dz\wedge dy + (e^u w)_z dx\wedge dy) \\ &= (dx+idy)\wedge[(w_x - iw_y)dz + i(e^u w)_z dy], \end{aligned}$$

so that $dx+idy$ and $wdz+i\omega$ generate a closed differential ideal.

The metric g is Hermitian with respect to this metric, and its associated 2-form is

$$\Omega = dz\wedge\omega + e^u w\ dx\wedge dy.$$

Since

$$\begin{aligned} d\Omega &= -dz\wedge d\omega + (e^u w)_z dz\wedge dx\wedge dy \\ &= -dz\wedge[(e^u w)_z dx\wedge dy] + (e^u w)_z\, dz\wedge dx\wedge dy \\ &= 0, \end{aligned}$$

our metric is Kähler.

Equation (1) now becomes the assertion that the scalar curvature vanishes. Indeed, let us choose a local trivialization of M with fiber coordinate t, so that $\omega = dt+\theta$ for θ some 1-form on $\mathcal{V}\subset\mathbb{R}^3$. The volume form of M is then given by

$$\frac{1}{2}\Omega\wedge\Omega = e^u w\ dx\wedge dy\wedge dz\wedge dt.$$

On the other hand, since

$$\frac{\partial}{\partial t} - iJ\left(\frac{\partial}{\partial t}\right) \equiv (1-iw^{-1}\theta_3)\frac{\partial}{\partial t} + iw^{-1}\ \frac{\partial}{\partial z}$$

is a vector field of type (1,0) whose real part preserves J, it is a *holomorphic* vector field, and so there locally exists a holomorphic (1,0)-form

$$\Psi \equiv wdz + i\omega \mod dx+idy$$

whose contraction with it is identically i. (Here $\theta_3 := \partial/\partial z \lrcorner\, \theta$.)
Thus

$$\begin{aligned} &\frac{(i)^2}{4}\Psi\wedge\bar\Psi\wedge(dx+idy)\wedge(\overline{dx+idy}) \\ &\qquad = w\ dx\wedge dy\wedge dz\wedge dt \end{aligned}$$

is the 4-form associated with a holomorphic frame, so that the Ricci form is given by

$$P = -i\partial\bar\partial\ \log\left(\frac{e^u\, w\ dx\wedge dy\wedge dz\wedge dt}{w\ dx\wedge dy\wedge dz\wedge dt}\right) = -i\partial\bar\partial u.$$

The condition that the scalar curvature vanish is just that $\Omega \wedge P = * < \Omega, P >= 0$, so that we require that

$$
\begin{aligned}
0 &= \Omega \wedge d(Jdu) \\
&= (dz \wedge \omega + e^u w \ dx \wedge dy) \wedge d(u_x \, dy - u_y dx + u_z w^{-1}\omega) \\
&= (u_{xx} + u_{yy} + we^u(w^{-1}u_z)_z) dx \wedge dy \wedge dz \wedge dt + u_z w^{-1} d\theta \wedge dz \wedge dt \\
&= (u_{xx} + u_{yy} + we^u(w^{-1}u_z)_z + w^{-1}u_z(we^u)_z) dx \wedge dy \wedge dz \wedge dt \\
&= \Big[u_{xx} + u_{yy} + (e^u)_{zz} \Big] dx \wedge dy \wedge dz \wedge dt,
\end{aligned}
$$

which is equation (1).

Conversely, suppose we are given a Kähler surface M with a Killing field X. If M has holonomy $U(2)$, the Killing field automatically preserves the complex structure; if the holonomy is smaller, the induced action on the S^2 of complex structures with the fixed orientation has a fixed point, so there is still an invariant parallel complex structure, although it may differ from the given one. We may therefore assume that X preserves the Kähler form, and thus, on any simply connected subset $\mathcal{U} \subset M$, define a function $z : \mathcal{U} \to \mathbb{R}$ as the Hamiltonian generating X; i.e. we require $dz = -X \lrcorner \, \Omega$, which defines z up to an additive constant. It then follows that JX is orthogonal to the level sets of z. On the other hand, $JX + iX$ is a holomorphic vector field, so that the leaf space of the foliation tangent to X and JX is locally a complex curve. Letting $x + iy$ be any holomorphic coordinate on this quotient for a suitable neighborhood in M, we have produced the coordinates x, y, and z of the construction. Taking t to be any function with $Xt \equiv 1$, the complex structure is of the form described above, and all the previous calculations apply. ∎

§3. The Burns Metric

Consider the following Kähler form on $\mathbb{C}^2 - \{\vec{0}\}$:

$$\Omega = -\frac{i}{2}\,\partial\overline{\partial}\left(\|\mathbf{z}\|^2 + m\log\|\mathbf{z}\|^2\right)$$

where $m > 0$ is a positive constant and where $\|\mathbf{z}\|^2 = \mathbf{z}_1\overline{\mathbf{z}_1} + \mathbf{z}_2\overline{\mathbf{z}_2}$. As was first pointed out by Dan Burns [Bu] [L$_4$] [L$_5$], this defines a zero-scalar-curvature Kähler metric on the blow-up $\tilde{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin. In this section we will examine this metric in terms of the ansatz developed in §2.

As our Killing field, we take the generator of clockwise rotation about the $\mathbf{z}_2$-axis:

$$X = Im\left(2\mathbf{z}_1\frac{\partial}{\partial\mathbf{z}_1}\right).$$

Note that the potential

$$\varphi = \frac{1}{2}\ \left(\|\mathbf{z}\|^2 + m\log\|\mathbf{z}\|^2\right)$$

is invariant under X, so that, letting $\xi = \mathbf{z}_1\frac{\partial}{\partial\mathbf{z}_1}$, we have $\xi\varphi = \overline{\xi}\varphi$, and

$$\begin{aligned} X \,\lrcorner\, \Omega &= -i(\xi - \overline{\xi}) \,\lrcorner\, (-i\,\partial\,\overline{\partial}\,\varphi) \\ &= -\left[\ \overline{\partial}(\xi\varphi) + \partial(\overline{\xi}\varphi)\ \right] \\ &= -d(\xi\varphi). \end{aligned}$$

Thus the Hamiltonian generating X may be taken to be

$$\begin{aligned} z &= \xi\varphi \\ &= \tfrac{1}{2}\,|\mathbf{z}_1|^2\left(1 + \frac{m}{|\mathbf{z}_1|^2 + |\mathbf{z}_2|^2}\right). \end{aligned}$$

On the other hand, the coordinate, x and y may be taken to be given by

$$\mathbf{z}_2 = x + iy,$$

since $\mathbf{z}_2$ is a holomorphic coordinate on the leaf space of the foliation tangent to X and JX.

The volume form of the Burns metric is

$$\tfrac{1}{2}\Omega\wedge\Omega = -\frac{1}{4}\left(1 + \frac{m}{|\mathbf{z}_1|^2 + |\mathbf{z}_2|^2}\right) d\mathbf{z}_1 \wedge d\overline{\mathbf{z}_1} \wedge d\mathbf{z}_2 \wedge d\overline{\mathbf{z}_2}.$$

On the other hand, the volume form whose contraction with X and JX yields $dx \wedge dy$ is

$$-\frac{1}{4|\mathbf{z}_1|^2} \quad d\mathbf{z}_1 \wedge d\overline{\mathbf{z}_1} \wedge d\mathbf{z}_2 \wedge d\overline{\mathbf{z}_2}.$$

Hence $e^u\, w\; dx \wedge dy \wedge dz \wedge dt = -\frac{1}{4}\left(1 + \frac{m}{|\mathbf{z}_1|^2+|\mathbf{z}_2|^2}\right) d\mathbf{z}_1 \wedge d\overline{\mathbf{z}_1} \wedge d\mathbf{z}_2 \wedge d\overline{\mathbf{z}_2}$ while $w\, dx \wedge dy \wedge dz \wedge dt = -\frac{1}{4|\mathbf{z}_1|^2}\; d\mathbf{z}_1 \wedge d\overline{\mathbf{z}_1} \wedge d\mathbf{z}_2 \wedge d\overline{\mathbf{z}_2}$ and

$$\boxed{e^u = |\mathbf{z}_1|^2 \left(1 + \frac{m}{|\mathbf{z}_1|^2 + |\mathbf{z}_2|^2}\right) = 2z.}$$

Thus we simply have $u = \log 2z$, which is invariant under translation in x and y. It is this simple observation which will allow us to construct our new metrics on n-fold blow-ups of $\mathbb{C}^2$.

We will also need to know the function w for the Burns metric. We may calculate this by

$$\begin{aligned}
w^{-1} \;:&= g(X,X) = \omega(X, JX) \\
&= 2i\omega(\xi, \overline{\xi}) \\
&= 2\xi\overline{\xi}\varphi \\
&= 2\mathbf{z}_1 \frac{\partial}{\partial \mathbf{z}_1}\, \overline{\mathbf{z}_1}\, \frac{\partial}{\partial \overline{\mathbf{z}_1}}\, \tfrac{1}{2}\left(\|\mathbf{z}\|^2 + m \log \|\mathbf{z}\|^2\right) \\
&= |\mathbf{z}_1|^2 \left(1 + \frac{m|\mathbf{z}_2|^2}{(|\mathbf{z}_1|^2 + |\mathbf{z}_2|^2)^2}\right).
\end{aligned}$$

Thus, since $2z = |\mathbf{z}_1|^2 \left(1 + \frac{m}{|\mathbf{z}_1|^2+|\mathbf{z}_2|^2}\right)$,

$$2zw = 1 + \frac{m|\mathbf{z}_1|^2}{(|\mathbf{z}_1|^2 + |\mathbf{z}_2|^2)^2 + m|\mathbf{z}_2|^2}\,.$$

Since $|\mathbf{z}_1|^2 = \varepsilon - \frac{r^2}{2} + \sqrt{(\varepsilon + \frac{r^2}{2})^2 + mr^2}$, where $r^2 := x^2 + y^2$ and $\varepsilon := z - \frac{m}{2}$, we obtain $w = \frac{1}{2z} + F_m$, where the function

$$F_m(x,y,z) = \frac{1}{4z}\left(\frac{m + \varepsilon + \frac{r^2}{2}}{\sqrt{(\varepsilon + \frac{r^2}{2})^2 + mr^2}} - 1\right)$$

is defined on the complement of the point $(x,y,z) = (0,0,\frac{m}{2})$, and is positive for $z > 0$.

§4. The Hyperbolic Ansatz

Our conclusions concerning the Burns metric are best understood by recasting them in a geometrical guise. If we consider those metrics arising in Proposition 1 when $u = \log 2z$, the key equation (2) becomes

$$w_{xx} + w_{yy} + (2zw)_{zz} = 0.$$

Let us introduce a new variable $V = 2zw$. Our equation then reads

$$\triangle V = \sum_{j,k=1}^{3} |h|^{-\frac{1}{2}} \frac{\partial}{\partial x^j} |h|^{\frac{1}{2}} h^{ij} \frac{\partial}{\partial x^k} V = 0,$$

where $\triangle$ is the Laplace-Beltrami operator of the Riemannian metric

$$h = \frac{dx^2 + dy^2}{2z} + \frac{dz^2}{4z^2}$$

on the upper half-space $H^3 := \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$. However, this metric really represents that of hyperbolic space $\mathcal{H}^3$; if we introduce a new coordinate $q = \sqrt{2z}$, the above metric becomes

$$h = \frac{dx^2 + dy^2 + dq^2}{q^2},$$

which is the usual conformal upper-half-space model for $\mathcal{H}^3$. Notice that the formula for the metric of Proposition 1 now reads

$$\begin{aligned} g &= 2z(Vh + V^{-1}\omega^2) \\ &= q^2(Vh + V^{-1}\omega^2). \end{aligned}$$

However, our model of hyperbolic space singles our a point at infinity in a purely arbitrary manner; thus any metric of the above form is conformal to Kähler for an entire 2-sphere's worth of different complex structures, one for each point at infinity! To summarize:

Proposition 2. *Let V be any solution of the Laplace-Beltrami equation*

$$\triangle V = 0$$

on a region $\mathcal{V} \subset \mathcal{H}^3$ of hyperbolic 3-space, and assume that the cohomology class of $\frac{1}{2\pi} \star dV$ is integral, where $\star$ is the Hodge star operator of $\mathcal{H}^3$. Then if ω is a connection form for a circle bundle whose curvature is $\star dV$, then

$$[g] = \left[Vh + V^{-1}\omega^2\right]$$

is half-conformally-flat. Moreover, if q is any horospherical height function for $\mathcal{H}^3$, the metric

$$g = q^2(Vh + V^{-1}\omega^2)$$

is Kähler, with scalar curvature zero.

Here a *horospherical height function* means the exponential of a Busemann function— i.e. q is a function on $\mathcal{H}^3$ whose restriction to some directed geodesic is the exponential of the affine parameter, and is constant on the forward-directed horospheres orthogonal to this geodesic.

Notice that this provides a hyperbolic analog of the fact that the Gibbons-Hawking metrics [GH] are hyper-Kähler. For other incarnations of the ansatz $Vh + V^{-1}\omega^2$, see [JT] [Pd].

We now re-examine the particular function w which arises for the Burns metric. Recall that the hyperbolic distance from the point $(x, y, q) = (0, 0, q_0)$ is given in the upper-half-space model by

$$\rho = \cosh^{-1}\left[\frac{x^2 + y^2 + q^2 + q_\circ^2}{2qq_\circ}\right].$$

Hence

$$\begin{aligned}\coth\ \rho &= \frac{x^2 + y^2 + q^2 + q_\circ^2}{\sqrt{(x^2 + y^2 + q^2 + q_\circ^2)^2 - 2q^2q_\circ^2}}\\ &= \frac{m + \varepsilon + \frac{r^2}{2}}{\sqrt{(\varepsilon + \frac{r^2}{2})^2 + mr^2}},\end{aligned}$$

where $m = q_\circ^2, \varepsilon = z - \frac{m}{2} = \frac{q^2 - m}{2}$, and $r^2 = x^2 + y^2$. Thus, for the Burns metric we have

$$\begin{aligned}V = 2zw &= 1 + \frac{1}{2}(\coth\ \rho - 1)\\ &= 1 + \frac{1}{e^{2\rho} - 1}.\end{aligned}$$

But $\frac{1}{e^{2\rho}-1}$ is precisely the fundamental solution G of the Laplace-Beltrami operator, with the normalization

$$\triangle G = -2\pi\delta$$

relative to the hyperbolic volume form. The Burns metric may therefore be thought of as corresponding to a single magnetic monopole in hyperbolic 3-space. In the next section we will consider solutions corresponding to multi-monopoles.

§5. Construction of the metrics

Let $\{p_j = (a_j, b_j, c_j)\}$ be an arbitrary collection of points in the upper half-space H^3, and let

$$G_j = \frac{1}{e^{2\rho_j} - 1} = f_{2c_j}\left((x-a_j)^2 + (y-b_j)^2, z - c_j\right)$$

be the normalized hyperbolic Green's function centered at p_j, where

$$f_m(s,\varepsilon) := \tfrac{1}{2}\left(\frac{m + \varepsilon + \frac{s}{2}}{\sqrt{\left(\varepsilon + \frac{s}{2}\right)^2 + ms}} - 1\right).$$

Let

$$V := 1 + \sum_{j=1}^{n} G_j, \tag{3}$$

and let $w = \frac{1}{2z}V$, which is a positive solution of the equation

$$w_{xx} + w_{yy} + (2zw)_{zz} = -2\pi \sum_{j=1}^{n} \delta_{p_j}$$

relative to the Euclidean volume form on the upper half-space on H^3, so that the cohomology class

$$\frac{1}{2\pi}[\alpha] = \frac{1}{2\pi}\Big[w_x dy \wedge dz + w_y dz \wedge dx + (2zw)_z dx \wedge dy\Big] \in H^2(H^3 - \{p_j\})$$

is an integral class, assigning -1 to a small sphere around any p_j. Let $M \to H^3 - \{p_j\}$ be the circle bundle whose Chern class is $\frac{1}{2\pi}[\alpha]$; by Chern-Weil theory, M has a connection 1-form ω, unique up to gauge equivalence, whose curvature is α. Define a Riemannian metric on M by

$$g = 2zw(dx^2 + dy^2) + w\,dz^2 + w^{-1}\omega^2 = 2z[Vh + V^{-1}\omega^2].$$

By Proposition 1, this is a Kähler metric of scalar curvature 0.

We now produce a larger manifold $\overline{M}$ by attaching an $\mathbb{R}^2$ at $z = 0$ and attaching points at each p_j. These are both well-defined operations, as near $z = 0$ the map $M \to H^3$ is diffeomorphically conjugate to $\mathbb{C}^2 - (\mathbb{C} \times \{0\}) \to H^3 : (\mathbf{z}_1, \mathbf{z}_2) \to (|\mathbf{z}_1|^2, \mathbf{z}_2)$, while near p_j it is diffeomorphically conjugate to

$$\begin{gathered}\mathbb{C}^2 - \{0\} \to \mathbb{R}^3 - \{0\} \\ (\mathbf{z}_1, \mathbf{z}_2) \longmapsto \left(\frac{|\mathbf{z}_1|^2 - |\mathbf{z}_2|^2}{2}, \Re(\mathbf{z}_1\overline{\mathbf{z}}_2), \Im(\mathbf{z}_1\overline{\mathbf{z}}_2)\right).\end{gathered}$$

This can be done, moreover, in such a way that g extends smoothly to $\overline{M}$.

Let us begin by considering the p_j. If we introduce exponential polar coordinates on $\mathcal{H}^3$ near such a point, our formula for the metric becomes

$$g = q^2[V(d\rho^2 + \sinh^2\rho\ g_{S^2}) + V^{-1}\omega^2],$$

where $V = \frac{1}{2\rho} + F$, for F some smooth function on a neighborhood of the origin in $\mathbb{R}^3$. (Here g_{S^2} denotes the standard metric on the unit 2-sphere.) For ρ small, the circle bundle $\pi: M \to \mathcal{H}^3 - \{p_j\}$ may be identified with

$$\begin{aligned} \wp: \mathbb{R}^+ \times S^3 &\to \mathbb{R}^+ \times S^2 \\ (r, \mathbf{x}) &\mapsto \left(\frac{r^2}{2}, \mu(\mathbf{x})\right), \end{aligned}$$

where $\mu: S^3 \to S^2$ is the Hopf map. Introduce an orthonormal coframe $\{\sigma_1, \sigma_2, \sigma_3\}$ on the unit 3-sphere S^3 such that $4\sigma_1 \wedge \sigma_2$ is the pull-back of the area form for (S^2, g_{S^2}). Since

$$d\sigma_3 = 2\sigma_1 \wedge \sigma_2,$$

we may, after a gauge transformation, take

$$\omega = -\sigma_3 + \wp^*\theta$$

where θ is a smooth 1-form on a neighborhood of $\mathbf{0} \in \mathbb{R}^3$. Our formula for the metric now reads

$$\begin{aligned} \frac{g}{q^2} = (1 + r^2F)dr^2 + (1 + r^2\widetilde{F})r^2(\sigma_1^2 + \sigma_2^2) \\ + (1 + r^2F)^{-1}r^2(\sigma_3 - \wp^*\theta)^2, \end{aligned}$$

where $\widetilde{F}$ is the smooth function $F + \left(\frac{\sinh\left(\frac{r^2}{2}\right)}{r^2/2}\right)^2 + r^2F\left(\frac{\sinh\left(\frac{r^2}{2}\right)}{r^2/2}\right)^2$. If we now identify $\mathbb{R}^+ \times S^3$ with $\mathbb{R}^4 - \{\mathbf{0}\}$ via polar coordinates, the map $\wp$ extends as the smooth map

$$\begin{aligned} \wp: \mathbb{C}^2 &\to \mathbb{R}^3 \\ (\mathbf{z}_1, \mathbf{z}_2) &\longmapsto \left(\frac{|\mathbf{z}_1|^2 - |\mathbf{z}_2|^2}{2}, \Re(\mathbf{z}_1\overline{\mathbf{z}}_2), \Im(\mathbf{z}_1\overline{\mathbf{z}}_2)\right), \end{aligned}$$

so that $\wp_* = 0$ at the origin; in particular, $\wp^*\theta$ is smooth on $\mathbb{R}^4$ and vanishes at the origin, while $\wp^*F$ is a smooth function. The above formula for the metric therefore defines a smooth metric on a neighborhood of $\mathbf{0} \in \mathbb{R}^4$ which agrees with the Euclidean metric $dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ at $\mathbf{0}$, since the difference between $\frac{g}{q^2}$ and the Euclidean metric is expressible as a sum of products of the smooth 1-forms

rdr, $r^2\sigma_j$, and $\wp^*\theta$, with smooth coefficients. Since q is smooth at p_j, g is smooth there.

Since the metric is Kähler on the complement of the $\{\hat{p}_j\}$, we may extend it across the $\{\hat{p}_j\}$ by the following procedure: take any path γ within M with final end-point at $\hat{p}_j$, and parallel-transport J along γ to produce a tensor at $\hat{p}_j$; we define this to be the value of J at p_j. This is independent of the choice of path; if it were not, since J is parallel almost everywhere, we would be able to produce one possible value of $J|p_j$ from any other by parallel transport around an arbitrarily small loop based at p_j. Since the metric is smooth at $\{p_j\}$, this would give a contradiction. There is thus a unique extension of J which is invariant under parallel transport, and in particular smooth.

We now examine the behavior of the metric near $z = 0$. Again we have a standard model to work with, this time the flat metric corresponding to $w_0 = \frac{1}{2z}$. Letting $q = \sqrt{2z}$, the metric near $q = 0$ has the form

$$g = (1 + q^2 f)(dx^2 + dy^2 + dq^2) + \frac{q^2(dt + \theta)^2}{1 + q^2 f}$$

where f is a smooth function of $(x, y, q), q \geqslant 0$, and where θ is a smooth 1-form on this (x, y, q) half-space whose dq component may be taken to vanish identically by choosing a suitable gauge. If we now interpret (q, t) as polar coordinates in an $\hat{x}\hat{y}$-plane, this metric becomes a smooth Riemannian metric on a neighborhood of $\mathbb{R}^2 \subset \mathbb{R}^4$. We now introduce an almost complex structure along the xy-plane in $\mathbb{R}^4$ by $dx \mapsto dy, d\hat{x} \mapsto d\hat{y}$, and observe that this is, together with our previous prescription, gives us a continuously differentiable complex structure which is parallel with respect to the metric connection away from the xy-plane, and hence everywhere.

We have therefore constructed a scalar-flat Kähler metric on an abstract complex manifold $\overline{M}$. We will now show that $\overline{M}$ is biholomorphically equivalent to $\mathbb{C}^2$ blown up at the points $\{(a_j + ib_j, 0)\}$.

For simplicity, assume for the moment that the complex numbers $a_j + ib_j$ are distinct. Let $M_0 \subset M \subset \overline{M}$ be the inverse image of $H^3 - \bigcup_{j=1}^{n}(\{(a_j, b_j)\}\times]0, c_j])$. The projection $H^3 \to \mathbb{C} : (x, y, z) \mapsto x + iy$ induces a holomorphic map $\pi : M_0 \to \mathbb{C}$, and the fibers of π are the orbits of a holomorphic $\mathbb{C}_*$-action generated by $\frac{\partial}{\partial t} - iJ(\frac{\partial}{\partial t})$; thus π is a holomorphic principal line bundle. Since $\mathbb{C}$ is Stein, this bundle is trivial. Thus M_0 is biholomorphic to $\mathbb{C} \times (\mathbb{C} - \{0\})$.

Now let $M_1 \supset M_0$ be obtained by adding $\mathbb{R}^2 - \{(a_j, b_j)\}$ to M_0. Since the projection yields a holomorphic map $\mathbf{z}_1 : M_1 \to \mathbb{C}$, and since the constructed second coordinate $\mathbf{z}_2$ is $(\mathbb{C}_*)$-equivariant, the biholomorphism $M_0 \to \mathbb{C} \times (\mathbb{C} - \{0\})$ extends to a continuous map $M_1 \to \mathbb{C}^2$ which is separately complex differentiable, and hence holomorphic. This yields a biholomorphism $M_1 \to \mathbb{C}^2 - \{(a_j + ib_j, 0)\}$.

We now extend this holomorphic map as a continuous map $\pi : M \to \mathbb{C}^2$. Notice that the inverse image of $(a_j + ib_j, 0)$ is homeomorphic to $S^1 \times [0, c_j]/(S^1 \times \{0, c_j\}) \approx S^2$, and is also a smooth complex curve. Since the derivative of π vanishes at this curve, π satisfies the Cauchy-Riemann equations everywhere, and so is a holomorphic map.

The complex curve $\pi^{-1}(a_j + ib_j, 0)$ has self-intersection -1, as can be seen by moving the line segment $\{(a_j, b_j)\} \times [0, c_j]$ to another line segment joining the xy-plane in $\overline{H}^3$ to (a_j, b_j, c_j); the inverse image of this line segment is a 2-sphere in $\overline{M}$ which intersects the original Riemann sphere in exactly one point, and one may deduce that the intersection is transverse, with index -1, by remembering that our standard model of the projection from $\overline{M}$ to $\overline{H}^3$ near (a_j, b_j, c_j) is essentially[1] a cone over the Hopf map $S^3 \to S^2$. Thus $\pi : \overline{M} \to \mathbb{C}^2$ exactly replaces the points $\{(a_j + ib_j, 0)\}$ in $\mathbb{C}^2$ with rational curves of self-intersection -1, and so $\overline{M}$ is the blow-up of $\mathbb{C}^2$ at these points.

(It is relatively easy to check that, should two or more of the complex numbers $a_j + ib_j$ coincide, one obtains an iterated blow-up of $\mathbb{C}^2$, obtained as follows: having blown up a point on the $\mathbf{z}_1$-axis, we may, in turn, blow up the new surface along the resulting exceptional $\mathbb{P}_1$ at the point corresponding to the direction of the $\mathbf{z}_2$-axis. The resulting iterated blow-up may then in turn be blown up at the point corresponding to the direction of the proper transform of $\mathbf{z}_1$ =constant, and the process is then repeated the requisite number of times. Details are left to the interested reader.)

Finally, I claim the metric g is asymptotically flat, and in particular is complete. To be more precise, I claim that there is a compact set $K \subset \overline{M}$ and a diffeomorphism between $M - K$ and the complement of a large closed ball in $\mathbb{R}^4$, such that the induced metric on $\mathbb{R}^4$ has the form

$$g = g_{\text{Euclidean}} + 0\ (\frac{1}{R^2}),$$

where R is the Euclidean radius. Indeed, we have

$$g = (1+f)(dx^2 + dy^2) + (1+f)dq^2 + \tfrac{q^2}{1+f}\ (dt + \theta)^2,$$

where $q = \sqrt{2z}$ and where f falls off like $\frac{1}{q^2+r^2}$ at infinity. Moreover, the components of the 1-form θ can be made to fall off like $\frac{1}{(q^2+r^2)^{3/2}}$ by choosing an appropriate choice of gauge. Thinking of (q, t) as polar coordinates in an $\hat{x}\hat{y}$-plane, the metric is thereby displayed in an asymptotic coordinate system in the desired form.

We have therefore proved the following :

Theorem 1. *The metric of Proposition 1, with $u = \log 2z$ and w given by (3), represents a zero-scalar-curvature, axisymmetric, asymptotically flat Kähler metric on the blow-up of $\mathbb{C}^2$ at n-points situated along a straight complex line.*

[1]The radial map is $r \to r^2/2$ rather than being linear.

Corollary. (cf.[L$_4$]) *The conformal class of this metric represents a self-dual metric on* $\underbrace{\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2}_{n}$.

Proof. Reverse the orientation and add a point at infinity. ∎

Exercize. In fact, one may directly write down a global representative for the constructed conformal metric on $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$, for instance in the form

$$g = (\mathrm{sech}^2\rho)[Vh + V^{-1}\omega^2] \ ,$$

where ρ is the hyperbolic distance from an arbitrary point in $\mathcal{H}^3$. With this choice of conformal factor the scalar curvature becomes

$$R = 12V^{-1},$$

which is a non-negative function on $n\mathbb{CP}_2$. It follows that the metric can be conformally rescaled so as to make the scalar curvature strictly positive. ∎

An interesting 'limit' of our conformal metrics occurs when the chosen centers $p_1, \ldots, p_n \in \mathcal{H}^3$ are allowed to coincide. While this no longer gives a compact self-dual manifold, it is not hard to see that it still defines a compact self-dual *orbifold* with an isolated singular point p of type $\mathbb{Z}_n$; deleting this point and making a suitable conformal change should then give rise to a locally asymptotically Euclidean self-dual manifold. What is this space? To find out, let us identify $\mathcal{H}^3 - \{p\}$ with $\mathbb{R}^+ \times S^2$ via polar coordinates and the exponential map. The hyperbolic metric then becomes

$$h = d\rho^2 + (\sinh^2\rho)g_{S^2} \ ,$$

where ρ is the radial coordinate and g_{S^2} is the standard metric on the unit 2-sphere. We wish to analyze the conformal class of the metric

$$g = Vh + V^{-1}\omega^2 \ ,$$

where

$$V = 1 + \frac{n}{e^{2\rho} - 1}$$

and $d\omega = \star dV$. For Chern-class reasons, our circle bundle may be identified with

$$\mathbb{R}^+ \times (S^3/\mathbb{Z}_n) \to \mathbb{R}^+ \times S^2 \ ,$$

where the projection from the Lens space $S^3/\mathbb{Z}_n$ to S^2 is induced by the Hopf map $\mu : S^3 \to \mathbb{CP}_1$. If $\{\sigma_1, \sigma_2, \sigma_3\}$ is a left-invariant orthonormal coframe for the unit

3-sphere, chosen so that σ_1 and σ_2 vanish on the fibers of the Hopf map, a suitable connection form is given by $\omega = -n\sigma_3$, and we have

$$\begin{aligned} g &= \frac{e^{2\rho}+n-1}{e^{2\rho}-1}[d\rho^2 + 4(\sinh^2\rho)(\sigma_1^2+\sigma_2^2)] + \frac{e^{2\rho}-1}{e^{2\rho}+n-1}n^2\sigma_3^2 \\ &= \frac{e^{2\rho}+n-1}{e^{2\rho}-1}d\rho^2 + \frac{(e^{2\rho}+n-1)(e^{2\rho}-1)}{e^{2\rho}}(\sigma_1^2+\sigma_2^2) + \frac{e^{2\rho}-1}{e^{2\rho}+n-1}n^2\sigma_3^2 \ . \end{aligned}$$

This is then conformally equivalent to

$$\hat{g} = \frac{e^{2\rho}(e^{2\rho}+n-1)}{(e^{2\rho}-1)^2}d\rho^2 + (\frac{e^{2\rho}+n-1}{e^{2\rho}-1})(\sigma_1^2+\sigma_2^2) + \frac{n^2e^{2\rho}}{(e^{2\rho}-1)(e^{2\rho}+n-1)}\sigma_3^2 \ .$$

Setting

$$r := \sqrt{\frac{e^{2\rho}+n-1}{e^{2\rho}-1}}$$

we have

$$\hat{g} = \frac{dr^2}{1+\frac{A}{r^2}+\frac{B}{r^4}} + r^2[\sigma_1^2+\sigma_2^2+(1+\frac{A}{r^2}+\frac{B}{r^4})\sigma_3^2] \ ,$$

with $A = n-2$, $B = 1-n$. But this is precisely the scalar-flat Kähler metric on the Chern-class-(-n) line bundle $\mathcal{O}(-n) \to \mathbb{CP}_1$ found in [L$_5$]. The compact self-dual orbifolds alluded to above are thus just the one-point compactifications of these locally-asymptotically-flat manifolds.

But in what sense are these orbifolds really limits of our metrics on $n\mathbb{CP}_2$? If we allow our n points in $\mathcal{H}^3$ to tend to a single point p in such a way that their angular positions relative to p remain unchanged while their distances to p are rescaled, then, dilating the entire picture, our sequence of conformal metrics may equally well be considered as being given by the generalized Gibbons-Hawking ansatz with fixed centers in a 3-dimensional space whose constant curvature, thought of as an auxiliary parameter, is allowed to tend to zero. Thus, while our conformal metrics certainly converge to the above line bundle metrics on the complement of any fixed neighborhood of p, the limiting conformal structure near p is a Gibbons-Hawking metric[2]. Both of these pictures are, of course, biased by a choice of conformal gauge, and each therefore only tells half the story; the limit of our sequence of self-dual conformal metrics is, correctly conceived, a generalized connected sum of two compact self-dual orbifolds with complementary singularities, namely the respective

[2]For every $a \in \mathbb{R}^+$, the conformal metric $[Vh+V^{-1}\omega^2]$ may also be represented as $[(a^{-1}V)a^2h+(a^{-1}V)^{-1}\omega^2]$, where, with respect to the constant-curvature metric a^2h, we have $d\omega = \star(a^{-1}V)$. The appropriately normalized Green's function of a point p_j with respect to a^2h is given by $\hat{G}_j = a^{-1}G_j$. Thus $V = 1+\sum G_j$ is to replaced with $\hat{V} := a^{-1}V = a^{-1}+\sum \hat{G}_j$ in order to produce the same metric. In the limit $a \to \infty$, (sectional curvature) $\to 0$, V formally becomes just a sum of the Euclidean Green's functions $\frac{1}{2\rho_j}$, and the metric becomes the muti-Eguchi-Hansen metric, rather than the multi-Taub-NUT metric one might have assumed at first glance.

one-point conformal compactifications of the above scalar-flat Kähler metric on $\mathcal{O}(-n)$ and of a Gibbons-Hawking gravitational instanton. As will be explained elsewhere, such non-singular connected sums of complemetary self-dual orbifolds can be constructed in greater generality via the Donaldson-Friedman machine[DF], and the inverse process of "bubbling off" may be hoped to provide the key to the study of the ends of moduli spaces of self-dual metrics.[3]

[3]Here the author is much indebted to Alastair King and Peter Kronheimer, who observed that Poon's metrics on $2\mathbb{CP}_2$ have, as one limit, the generalized connected sum of two Eguchi-Hansen metrics, and suggested that a similar phenomenon should be expected for higher signature.

§6. Twistors, Toda Lattices, and Complex Structures

We now give the twistor-theoretic explanation for the ansatz presented in §1, and observe as an application that the hyperbolic ansatz of §3 is related to the problem of producing Kähler metrics in ways that are quite distinct from those which we have exploited so far.

Associated with any solution u of equation (1) — i.e. of the so-called "Toda-lattice equation" — there is an associated 3-dimensional Einstein-Weyl geometry[4] in the sense of Hitchin [H$_3$] [JT] [W]. Namely, given a function u on a region $\mathcal{V}$ of $\mathbb{R}^3$ such that

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0,$$

we may endow $\mathcal{V}$ with the conformal stucture determined by the Riemannian metric

$$\mathbf{h} := e^u(dx^2 + dy^2) + dz^2,$$

and with the torsion-free connection $\mathbf{D}$ defined by

$$\mathbf{D}_\xi \eta := \nabla_\xi \eta + \nu(\xi)\eta + \nu(\eta)\xi - \mathbf{h}(\xi, \eta)\nu^\#,$$

where $\nu := -u_z dz$, $\nu^\# = -u_z \frac{\partial}{\partial z}$, and ∇ denotes the Levi-Civita connection of $\mathbf{h}$. One immediately notices that

$$\mathbf{D}\mathbf{h} = -2\nu \otimes \mathbf{h},$$

so that $\mathbf{D}$ preserves the conformal class $[\mathbf{h}]$, and is thus a "Weyl connection" for this conformal metric; moreover, the curves (x, y) =constant are geodesic with respect to $\mathbf{D}$. A more involved calculation reveals that the symmetrized Ricci tensor of $\mathbf{D}$ satisfies

$$R_{(ab)} = \mu \mathbf{h}_{ab},$$

where $\mu = u_{zz} + (u_z)^2/2$. Thus $(\mathcal{V}, [\mathbf{h}], \mathbf{D})$ satisfies the so-called Einstein-Weyl equations.

Now assume for simplicity that $(\mathcal{V}, \mathbf{D})$ is geodesically convex, and let $\mathcal{T}$ denote the space of directed geodesics of $\mathbf{D}$, which is then a smooth 4-manifold diffeomorphic to TS^2. The tangent space of $\mathcal{T}$ at a geodesic γ is then just the space of solutions of *Jacobi's equation*

$$\mathbf{D}_\xi \mathbf{D}_\xi \eta = \mathcal{R}_{\xi\eta}\xi \ ,$$

modulo fields tangent to γ; here ξ denotes a tangent field of γ satisfying $\mathbf{D}_\xi \xi = 0$ and $\mathcal{R}^a{}_{bcd}$ is the curvature of the torsion-free connection $\mathbf{D}$. For an Einstein-Weyl 3-manifold, this simplifies to become

$$\mathbf{D}_\xi \mathbf{D}_\xi \eta \equiv -\frac{R}{6}(\xi \cdot \xi)\eta - 5(\xi \cdot v)(\xi \times \eta) \ \mathbf{mod}\ \xi \ ,$$

[4]This sort of geometry was actually first considered by Cartan [Ct].

where $R = h^{ab}R_{ab}$ is the scalar curvature, and $v^a = e^{abc}R_{bc}$ is the Hodge-star of the skew part of the Ricci tensor, and inner and cross-products are with respect to $\mathbf{h}$. The solution space of this equation is then invariant under the 90° rotation $\eta \mapsto \frac{\xi\times\eta}{\|\xi\|}$, and this then gives $\mathcal{T}$ the structure of an almost-complex manifold. Moreover, this almost-complex structure is automatically integrable, as follows seeing this in the real-analytic case is to identify $\mathcal{T}$ with the space of totally geodesic null planes in a small complexification of $(\mathcal{V}, [\mathbf{h}], \mathbf{D})$, which is manifestly a complex surface; more generally, this integrability follows from that of the twistor CR structure on the sphere-bundle of a conformal 3-fold [L1][L2], since the latter induces the above almost-complex structure by projection. The 2-sphere of directed $\mathbf{D}$-geodesics through any point $p \in \mathcal{V}$ now becomes a holomorphic curve $\mathcal{C} \subset \mathcal{T}$ of self-intersection 2, while the map $\sigma : \mathcal{T} \to \mathcal{T}$ obtained by reversing the direction of each geodesic becomes an anti-holomorphic involution of $\mathcal{T}$.

The complex surface $\mathcal{T}$ is called the *minitwistor space* of the given Einstein-Weyl geometry $(\mathcal{V}, [\mathbf{h}], \mathbf{D})$; knowing this space together with the anti-holomorphic involution $\sigma : \mathcal{T} \to \mathcal{T}$ allows us to completely reconstruct the original Einstein-Weyl space. Indeed, $\mathcal{V}$ is precisely the space of smooth, embedded, σ-invariant, compact holomorphic curves in $\mathcal{T}$ with self-intersection 2, and the family of such curves passing through a given point (and hence also passing through its σ-conjugate point) is a geodesic of the connection $\mathbf{D}$. Each point of such a curve $\mathcal{C}_p$ may now be thought of as representing a point in the sphere of directions $(T_p\mathcal{V} - 0)/\mathbb{R}^+$ at the corresponding point $p \in \mathcal{V}$, and the conformal structure of each such $\mathcal{C}_p$ therefore equips $\mathcal{V}$ with a conformal structure, namely $[\mathbf{h}]$. But indeed, we actually could have started with *any* complex surface $\mathcal{T}$ equipped with a fixed-point-free antiholomorphic involution σ and containing a σ-invariant rational curve $\mathcal{C}$ of self-intersection 2; the above prescription would then construct the general Einstein-Weyl 3-fold, starting with this essentially holomorphic data.

Over such a complex surface $\mathcal{T}$, let us suppose we have a holomorphic line bundle L whose Chern class vanishes. Suppose, moreover, that the involution σ lifts to an anti-holomorphic involution $\tilde{\sigma}$ of the total space of L. Via the Penrose transform, we may analyze L in three different ways:

- We may view L as the exponential of an element of $H^1(\mathcal{T}, \mathcal{O})$, at least in a neighborhood of the curve $\mathcal{C}$. Via the Penrose transform, this gives us a function w (secretly a section of the conformal weight -1 line bundle) satisfying

$$d\star(d+\nu)w = 0,$$

where $\star$ denotes the Hodge star operator of $\mathbf{h}$; when our Einstein-Weyl structure arises from a solution u of equation (1), this simplifies to become

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0,$$

and so is precisely equation (2).

• We may instead apply the Hitchin-Ward correspondence [H_4] directly to L to obtain a solution of the "$\mathbf{U(1)}$ Bogomolny equations"

$$d\omega = \star(d+\nu)w$$

on $(\mathcal{V}, [\mathbf{h}], \mathbf{D})$. Here $d\omega$ is the curvature of a complex line bundle over $\mathcal{V}$, while the "Higgs field" w is precisely the same section of the conformal weight -1 line bundle encountered before.
• Finally, we may attempt to treat the complex manifold L as a twistor space, defining M to be the set of $\tilde{\sigma}$-invariant rational complex curves in L with normal bundle $\mathcal{O}(1) + \mathcal{O}(1)$. Any such curve projects to a σ-invariant rational curve in $\mathcal{T}$ of self-intersection 2. Conversely, for each σ-invariant rational curve $\mathcal{C}$ in $\mathcal{T}$ of self-intersection 2, there is a circle's worth of $\overline{\sigma}$-invariant rational complex curves in L obtained as holomorphic sections of $C|_L$; the normal bundle of such a curve is *a priori* an extension

$$0 \to \mathcal{O} \to N \to \mathcal{O}(2) \to 0,$$

and we may identify the splitting obstruction $\in H^1(\mathbb{CP}_1, \mathcal{O}(-2))$ for this sequence with the value of w at the corresponding point of $\mathcal{V}$. If $w \neq 0$ is real-valued, $L - 0$ is therefore the twistor space of a half-conformally-flat 4-manifold. Indeed, this manifold is exactly [JT] the circle bundle with curvature $d\omega = \star(d+\nu)w$, and the conformal metric is just

$$[g] = [w\mathbf{h} + w^{-1}\omega] \; .$$

Conversely, every half-conformally-flat 4-manifold with S^1-action arises in this way.

When an Einstein-Weyl geometry arises from a solution u of equation (1), we have a bit of extra structure. First of all, the system of geodesics $(x, y) =$ constant, equipped with its two possible orientations, becomes a pair of complex curves $\mathcal{D}$ and $\overline{\mathcal{D}} = \sigma(\mathcal{D})$ in $\mathcal{T}$; moreover, $x + iy$ becomes a holomorphic coordinate on $\mathcal{D}$. Secondly, the divisor class $\mathcal{D} + \overline{\mathcal{D}}$ represents the line bundle $K^{-1/2}$ on $\mathcal{T}$, where K is the canonical line bundle, defined by $\mathcal{O}(K) = \Omega^2_{\mathcal{T}}$.

In order to see the latter, let us choose some arbitrary solution w of equation (2), and consider the twistor space Z of the scalar-flat Kähler surface (M, g, J) associated with w via the ansatz of §1. The built-in isometric action of S^1 on M then lifts holomorphically to Z, which is therefore an open set in some holomorphic line bundle over $\mathcal{T}$. Since [AHS] the underlying smooth manifold of Z may be identified with the set of almost-complex structures on M which preserve the metric and the given orientation, the complex structure of J of M defines a section of $Z \to M$, and this section is, moreover, a holomorphic map $M \to Z$; let us call the image Σ. Similarly, we have a conjugate hypersurface $\overline{\Sigma}$, obtained by considering $-J$ as a section of Z. Now the projection $Z \to \mathcal{T}$ is obtained geometrically by applying a given complex structure to the Killing field, projecting to $\mathcal{V}$, and then considering the oriented geodesic in this direction as an element of $\mathcal{T}$. The image of Σ in $\mathcal{T}$ is therefore just $\mathbf{D}$, while the image of $\overline{\Sigma}$ is just $\overline{\mathbf{D}}$. Now the divisor $[\Sigma]+[\overline{\Sigma}]$ is just $K^{-1/2}$ because [Pt] it is precisely the zero-locus of the element of $H^0(Z, K^{-1/2})$ which corresponds to the Kähler form via the Penrose transform; conversely, a real section of $K^{-1/2}$ on any twistor space naturally gives rise to a scalar-flat Kähler metric on the corresponding 4-manifold. Since the vertical tangent bundle of $Z \to \mathcal{T}$ is trivial, it follows that that $[\mathbf{D}]+[\overline{\mathbf{D}}] = K^{-1/2}$ on $\mathcal{T}$.

This this gives the following result:

Proposition. *(1) Solutions of the "Toda-lattice equation"*

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0$$

are locally in one-to-one correspondence with complex surfaces $\mathcal{T}$ equipped with a fixed-point-free anti-holomorphic involution σ, a non-zero σ-invariant holomorphic section of $K^{-1/2}$, and a complex coordinate on the corresponding divisor, such that $\mathcal{T}$ contains a σ-invariant rational complex curve $\mathcal{C}$ of self-intersection 2. (2) Holomorphic line bundles L with $c_1(L) = 0$ over such a surface $\mathcal{T}$ correspond to solutions w of the linearization

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0$$

of the Toda-lattice equation. The case $w \neq 0$ exactly corresponds to the case in which the total space of $L - 0$ is a twistor space, in which case it is the twistor space of the corresponding scalar-flat Kähler surface.

As an example of an Einstein-Weyl geometry, let us consider hyperbolic 3-space. Its space of directed geodesics is just $S^2 \times S^2$ minus the diagonal, since a hyperbolic geodesic is uniquely specified by its two end-points on the ideal conformal sphere at infinity; and our prescription of an almost-complex structure by rotating Jacobi fields through $+90°$ amounts to giving these two factor 2-spheres the complex structures associated with their conformal structures and opposite orientations. Thus, as will be explained in more depth in the next section, the minitwistor space of $\mathcal{H}^3$ is just $\mathbb{CP}_1 \times \mathbb{CP}_1$ minus the graph of an anti-holomorphic map $\mathbb{CP}_1 \to \mathbb{CP}_1$, and the involution or "real structure" $\sigma : \mathbb{CP}_1 \times \mathbb{CP}_1 \to \mathbb{CP}_1 \times \mathbb{CP}_1$ interchanges the factors. But the line bundle $K^{-1/2} = \mathcal{O}(1,1)$ on $\mathbb{CP}_1 \times \mathbb{CP}_1$ has many σ-invariant holomorphic sections; in particular, the divisors of the form $\{point\} \times \mathbb{CP}_1$ plus its conjugate all give examples. By the above yoga, we therefore get a 2-sphere of conformal scalings and associated complex structures, with respect to which the metric becomes Kähler. But in addition to these, there are other σ-invariant elements of $\Gamma(\mathbb{CP}_1 \times \mathbb{CP}_1, K^{-1/2})$, of two kinds, depending on whether the corresponding divisor contains σ-invariant points or not. Let us find the corresponding solutions of the Toda equation (1).

We begin with the case corresponding to a divisor with real points. Rather than choosing a point at infinity of hyperbolic 3-space, this corresponds to instead choosing a conformal circle at infinity. Any such a cicle determines coordinates on $\mathcal{H}^3$ in which the hyperbolic metric is given by

$$\begin{aligned}\mathcal{U} &= \{(x,y,z) |\ y>0,\ 1>z>-1\}\\ &= \mathbb{R}\times\mathbb{R}^+\times\quad]-1,1[\end{aligned}$$

by

$$h = \frac{dx^2+dy^2}{(1-z^2)y^2} + \frac{dz^2}{(1-z^2)^2}.$$

(Indeed, if we set $(\alpha,\beta,\gamma) = (x, yz, y\sqrt{1-z^2})$, this becomes $h = \frac{d\alpha^2+d\beta^2+d\gamma^2}{\gamma^2}$.) The equation $\triangle V = 0$ becomes

$$y^2V_{xx} + y^2V_{yy} + (1-z^2)V_{zz} = 0,$$

so that, setting $w = V/(1-z^2)$ and $u = \log(\frac{1-z^2}{y^2})$, we have

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0$$

and

$$w_{xx} + w_{yy} + (e^u w)_{zz} = 0.$$

Moreover, the metric

$$e^u w(dx^2+dy^2) + w\ dz^2 + w^{-1}\omega^2$$

can now be written as

$$\begin{aligned}(1-z^2)&\left[V\Big(\frac{dx^2+dy^2}{y^2(1-z^2)} + \frac{dz^2}{(1-z^2)^2}\Big) + V^{-1}\omega^2\right]\\ &= (1-z^2)[Vh + V^{-1}\omega^2],\end{aligned}$$

where

$$\begin{aligned}d\omega &= w_x dy\wedge dz + w_y dz\wedge dx + (e^u w)_z dx\wedge dy\\ &= \star dV,\end{aligned}$$

where $\star$ is the Hodge star of h. Thus the conformal metric $[Vh+V^{-1}\omega^2]$ can also be represented by the Kähler metric $(1-z^2)(Vh+V^{-1}\omega^2)$.

These complex structures are quite different from the ones discussed in §3. For example, if we take $V = 1$, the metric of §3 is flat, while the one under discussion here is that of $S^2\times\mathcal{H}^2$ with the product metric. Similarly, $V = 1+\sum_{j=1}^{m} G_j$ gives rise to a metric on an n-fold blow-up of $\mathbb{CP}_1 \times D$, where D is the 2-disk, which is asymptotic to the product metric $S^2 \times \mathcal{H}^2$ at infinity.

Yet another class of complex structures is determined by a σ-invariant section of $K^{-1/2}$ which defines a curve avoiding the fixed-point set of σ. Any such curve is precisely the set of geodesics through some point $p \in \mathcal{H}^3$, and this time our solution of the Toda equation will model $\mathcal{H}^3 - \{p\}$ by $\mathbb{CP}_1 \times \mathbb{R}^+$; $x + iy$ becomes a complex coordinate on $\mathbb{CP}_1$, and z parameterizes the radial direction. Indeed, consider the solution[5]

$$u = \log \frac{4z(1+z)}{(1+x^2+y^2)^2}$$

of equation (1) on the half-space $z > 0$. Setting $V = z(z+1)w$, equation (2) then becomes

$$V_{xx} + V_{yy} + \frac{4z(1+z)}{(1+x^2+y^2)^2} V_{zz} = 0 \ .$$

If we set $\rho = \coth^{-1}(2z+1)$, this can be rewritten as

$$V_{xx} + V_{yy} + \frac{1}{\sinh^2\rho}\left[\frac{4\sinh^2\rho}{(1+x^2+y^2)^2} V_\rho\right]_\rho = 0 \ ,$$

so that

$$\triangle V = \sum_{j,k=1}^{3} |h|^{-\frac{1}{2}} \frac{\partial}{\partial x^j} |h|^{\frac{1}{2}} h^{ij} \frac{\partial}{\partial x^k} V = 0,$$

where $h := d\rho^2 + \frac{4\sinh^2\rho}{(1+x^2+y^2)^2}(dx^2+dy^2) = d\rho^2 + (\sinh^2\rho)g_{S^2}$ is the metric on $\mathcal{H}^3$ written in terms of the exponential map and polar coordinates, combined with stereographic coordinates on the unit 2-sphere (S^2, g_{S^2}). Moreover, the scalar-flat Kähler metric associated by our ansatz with any solution w of (2) is just

$$\begin{aligned} we^u&(dx^2+dy^2) + w\,dz^2 + w^{-1}\omega^2 \\ &= \frac{4V}{(1+x^2+y^2)^2}(dx^2+dy^2) + \frac{V}{z(z+1)}dz^2 + z(z+1)V^{-1}\omega^2 \\ &= z(z+1)[V(d\rho^2 + (\sinh^2\rho)g_{S^2}) + V^{-1}] \\ &= \frac{1}{4\sinh^2\rho}[Vh + V^{-1}\omega^2] \ , \end{aligned}$$

where $d\omega = w_x dy \wedge dz + w_y dz \wedge dx + (e^u w)_z dx \wedge dy = \star dV$ as desired. Thus we see yet another way that hyperbolic monopoles give rise to scalar-flat Kähler surfaces. In addition, this last calculation provides the correct explanation for the complex structures occuring in connection with the self-dual orbifolds discussed at the end of §5.

[5] The author would like to thank Henrik Pedersen for drawing his attention to this solution.

It is now a straightforward exercize to construct locally-asymptotically-flat scalar-flat Kähler metrics on n-fold blow-ups of the Chern-class $-m$ line bundle $\mathcal{O}(-m) \to \mathbb{CP}_1$, where the blow-ups may be taken to occur at any n points of the zero section. This is done by taking the potential to be given by $V = 1 + mG_0 + \sum_{j=1}^{n} G_j$, where the G_j are the Green's functions of $n+1$ distinct points $p_0, \ldots, p_n \in \mathcal{H}^3$, and then using the complex structure determined in the above manner by the point p_0. The blown-up points of the zero section $\mathbb{CP}_1$ will then be given by the points of the 2-sphere at infinity of $\mathcal{H}^3$ at which are the ideal end-points of the geodesic rays from p_0 to p_j, $j = 1, \ldots, n$. In the case of $m = 1$, this allows us to find such metrics on some iterated blow-ups of $\mathbb{C}^2$ which are not covered by Theorem 1.

§7. The Twistor Space is Moishezon

In this section, we will give an explicit description of the twistor spaces of the self-dual 4-manifolds constructed in §5, and see that they are bimeromorphically equivalent to algebraic varieties— that is, they are Moishezon. While this might seem surprising in light of Hitchin's theorem [H_2] to the effect that a self-dual 4-manifold with projective algebraic twistor space is symmetric (i.e. S^4 or $\mathbb{CP}_2$), this phenomenon has been seen before in Poon's work [P_1] on $2\mathbb{CP}_2$. It is worth remarking that the construction presented in this section, inspired by Hitchin's work on the twistor spaces of gravitational instantons [H_1], also stands in contrast to a result of Donaldson and Friedman [DF] to the effect that the twistor space of a *generic* self-dual metric on $n\mathbb{CP}_2$, $n \geq 5$, must have algebraic dimension 0.

Let $\sigma : \mathbb{CP}_1 \times \mathbb{CP}_1 \to \mathbb{CP}_1 \times \mathbb{CP}_1$ denote the anti-holomorphic involution

$$([z_0, z_1], [\zeta_0, \zeta_1]) \mapsto ([\overline{\zeta_0}, \overline{\zeta_1}], [\overline{z_0}, \overline{z_1}]) \ ,$$

and let $S \subset \mathbb{CP}_1 \times \mathbb{CP}_1$ denote the fixed-point set of σ. Then [H_3] [JT] there is a twistor-like correspondence relating $\mathbb{CP}_1 \times \mathbb{CP}_1 - S$ and hyperbolic 3-space; more precisely, the set of σ-invariant, compact holomorphic curves in the generating homology class of $H_2(\mathbb{CP}_1 \times \mathbb{CP}_1 - S, \mathbb{Z}) \cong \mathbb{Z}$ gives a natural model for $\mathcal{H}^3$. Indeed, every holomorphic curve $\mathcal{C}$ in $\mathbb{CP}_1 \times \mathbb{CP}_1$ homologous to the diagonal $\mathbb{CP}_1 \subset \mathbb{CP}_1 \times \mathbb{CP}_1$ is the zero locus of some $P \in \Gamma(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1)) \cong \mathbb{C}^4$, uniquely determined up to a multiplicative constant; here $\mathcal{O}(k,\ell)$ denotes the holomorphic line bundle on $\mathbb{CP}_1 \times \mathbb{CP}_1$ of Chern class $(k,\ell) \in H^2(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. The requirement that C be σ-invariant means that we may take P to to be real with respect to the induced conjugate-linear involution $\mathcal{O}(1,1) \to \mathcal{O}(1,1)$. Now, up to a constant factor, $\Gamma(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1)) \cong \mathbb{C}^4$ carries a natural non-degenerate quadratic form, specified by the requirement that the image of $\mathbb{CP}_1 \times \mathbb{CP}_1$ under the Kodaira embedding map $\mathbb{CP}_1 \times \mathbb{CP}_1 \to \mathbb{P}[\Gamma(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1))]^*$ be its set of null projective covectors; the corresponding real quadratic form on the real subspace $\Gamma(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1))_\sigma \cong \mathbb{R}^4$ thus has the 2-sphere S as its projective null cone, and so (for topological reasons) must be of Minkowski signature $(+ - - -)$. A σ-invariant section P of $\mathcal{O}(1,1)$ now defines a a curve $\mathcal{C}$ disjoint from S iff it has positive Minkowski squarenorm. As such a section may always be multiplied by a non-zero constant without altering the corresponding curve $\mathcal{C} \subset \mathbb{CP}_1 \times \mathbb{CP}_1$, we conclude that the set of σ-invariant, compact holomorphic curves in the generating homology class $1 \in H_2(\mathbb{CP}_1 \times \mathbb{CP}_1 - S, \mathbb{Z}) \cong \mathbb{Z}$ are faithfully parameterized by future-pointing time-like unit vectors in Minkowski space. But the latter hyperboloid of one sheet is, of course, isometric to $\mathcal{H}^3$.

All of the hyperbolic geometry of $\mathcal{H}^3$ is completely encoded by this "minitwistor" model. For example, a hyperbolic geodesic, being a 2-plane section of the above hyperboloid, is exactly the set of curves through a σ-conjugate pair of points in $\mathbb{CP}_1 \times \mathbb{CP}_1 - S$, namely those hyperplane sections of $\mathbb{CP}_1 \times \mathbb{CP}_1 \subset \mathbb{CP}_3$ for which the hyperplane contains the fixed real 2-plane. Concretely, the resulting pair of points $\{[\zeta],[z]),([\overline{z}],[\overline{\zeta}]\}\subset \mathbb{CP}_1\times\mathbb{CP}_1$ is given by the two endpoints $[\zeta],[\overline{z}]\in S^2$ of the geodesic at hyperbolic infinity S, which we identify with $\mathbb{CP}_1$ in the obvious manner. We can therefore identify $\mathbb{CP}_1\times\mathbb{CP}_1-S$ with the space of oriented geodesics in $\mathcal{H}^3$ by thinking of the first and second factors as respectively representing the initial and final endpoints at infinity of hyperbolic geodesics. This has the following consequence, which has a critical rôle in the discussion that follows: *if $\mathcal{C}_1$ and $\mathcal{C}_2$ are any two σ-invariant curves in $\mathbb{CP}_1\times\mathbb{CP}_1-S$ which lie in the discussed homology class, their two intersection points may be systematically labelled so that one "points from $\mathcal{C}_1$ to $\mathcal{C}_2$", while the other "points from $\mathcal{C}_2$ to $\mathcal{C}_1$".*

Let $p_1,\dots,p_n$ be arbitrary, distinct points in $\mathcal{H}^3$, and, for simplicity's sake, assume for the moment that no three of the these n points are collinear[6]. Let $P_1,\dots,P_n\in\Gamma(\mathbb{CP}_1\times\mathbb{CP}_1,\mathcal{O}(1,1))$ be the corresponding polynomials, and let $\mathcal{C}_1,\dots,\mathcal{C}_n\subset\mathbb{CP}_1\times\mathbb{CP}_1-S$ be the n curves they define. We then define an algebraic variety $\tilde{Z}\subset\mathbb{P}(\mathcal{O}(n-1,1)\oplus\mathcal{O}(1,n-1)\oplus\mathcal{O})$ by the equation

$$xy=t^2\prod_{j=1}^{n}P_j\ ,$$

where $x\in\mathcal{O}(n-1,1)$, $y\in\mathcal{O}(1,n-1)$, and $t\in\mathcal{O}:=\mathcal{O}(0,0)$. Our assumption on the $p_1,\dots,p_n$ exactly amounts to requiring that no three of the curves defined by the P_j have a common intersection, so that the locus in $\mathbb{CP}_1\times\mathbb{CP}_1$ defined by $\prod P_j$ therefore has only normal crossing singularities. These intersection points in $\mathbb{CP}_1\times\mathbb{CP}_1\subset\mathbb{P}(\mathcal{O}(n-1,1)\oplus\mathcal{O}(1,n-1)\oplus\mathcal{O})$ are easily seen to be the the only singular points of the hypersurface $\tilde{Z}$.

There is a canonical antiholomorphic identification of $\sigma^*\mathcal{O}(k,\ell)$ with $\mathcal{O}(\ell,k)$, and this induces an anti-holomorphic involution of $\mathbb{P}(\mathcal{O}(n-1,1)\oplus\mathcal{O}(1,n-1)\oplus\mathcal{O})$, and hence of $\tilde{Z}$. Let us denote the latter involution by $\tilde{\sigma}:\tilde{Z}\to\tilde{Z}$.

We will now construct a complex 3-manifold Z bimeromorphic to $\tilde{Z}$. First of all, notice that the two surfaces $x=t=0$ and $y=t=0$ are contained in $\tilde{Z}$, and that, identifying them with the base $\mathbb{CP}_1\times\mathbb{CP}_1$ via the canonical projection, their normal bundles relative to $\tilde{Z}$ are respectively $\mathcal{O}(-1,1-n)$ and $\mathcal{O}(1-n,-1)$. Each can therefore be blown down to yield a rational curve $\mathbb{CP}_1$, both of which will have normal bundle $\mathcal{O}(1-n)\oplus\mathcal{O}(1-n)$. This blowing down constitutes the first step in the construction of Z from $\tilde{Z}$.

We complete the construction by making "small resolutions" of the the singular points of $\tilde{Z}$. To do this, notice that the singular points of $\tilde{Z}$ occur in pairs, which are "conjugate" via σ. Each such singularity may be put in the canonical form $xy=zw$ by choosing local coordinates z and w on $\mathbb{CP}_1\times\mathbb{CP}_1$ centered at the given crossing point of $\mathcal{C}_j$ and $\mathcal{C}_k$, and a "small resolution" of this singularity can therefore be obtained by using the map

$$\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{C}^4$$
$$(u,v)|_{[\zeta^0,\zeta^1]}\mapsto(x,y,z,w):=(u\zeta^0,v\zeta^1,u\zeta^1,v\zeta^0)$$

[6]i.e. on a common hyperbolic geodesic.

as a local model; here $\mathcal{O}(-1)$ denotes the "tautological" line bundle over $\mathbb{CP}_1$. (The singular point is thus replaced with a rational curve $\mathbb{CP}_1$.) Inspection of this resolution reveals that it just amounts to blowing up the 4-fold at a point at which the hypersurface $\tilde{Z}$ has a quadratic singularity, taking the proper transform of the hypersurface, and then collapsing a chosen factor of the resulting exceptional 2-quadric $\mathbf{Q}_2 \cong \mathbb{CP}_1 \times \mathbb{CP}_1$ so as to produce a rational curve. As this depends on the choice of a factor at each singularity, we will insist on the following rule: at the intersection point of a pair of real curves $\mathcal{C}_j$ and $\mathcal{C}_k$ which "points from $\mathcal{C}_j$ to $\mathcal{C}_k$," the resolutions should be obtained from the above prescription by letting z define $\mathcal{C}_j$ and letting w define $\mathcal{C}_k$, while x and y are again coordinates on $\mathcal{O}(n-1,1)$ and $\mathcal{O}(1,n-1)$, respectively. Notice that, with this choice of conventions, the involution $\tilde{\sigma}$ lifts to an involution $\hat{\sigma} : Z \to Z$ of the constructed complex 3-fold.

Theorem 2. *The complex 3-fold Z constructed above is the twistor space of the self-dual metric with centers $p_1, \ldots, p_n \in \mathcal{H}^3$ constructed in §5. Moreover, even if the general position hypothesis on $p_1, \ldots, p_n \in \mathcal{H}^3$ is dropped, the corresponding twistor space is still bimeromorphically equivalent to the variety $\tilde{Z}$ constructed by the given prescription, starting with this system of centers.*

Proof. There is a 4-parameter family of $\hat{\sigma}$-invariant rational curves in Z constructed as follows: for each real curve $\mathcal{C} \subset \mathbb{CP}_1 \times \mathbb{CP}_1 - S$ in the appropriate homology class, $\mathcal{C} \neq \mathcal{C}_1, \ldots, \mathcal{C}_n$, let $\ell_1, \ldots, \ell_n$ be the intersection points which "point from $\mathcal{C}$ to $\mathcal{C}_1, \ldots, \mathcal{C}_n$," and let $r_1, \ldots, r_n$ denote the intersection points which "point from $\mathcal{C}_1, \ldots, \mathcal{C}_n$ to $\mathcal{C}$." The restrictions of $\mathcal{O}(n-1,1)$ and $\mathcal{O}(1,n-1)$ to $\mathcal{C}$ are both isomorphic to $\mathcal{O}(n)$, and we may choose the isomorphisms so that σ just acts by the usual involution of $\mathcal{O}(n)$ covering the antipodal map $\mathbb{CP}_1 \to \mathbb{CP}_1$. Let $f \in \Gamma(C, \mathcal{O}(n))$ be a polynomial of degree n with zeroes $\ell_1, \ldots, \ell_n$, and notice that there is then a circle's worth of complex constants λ such that $|\lambda|^2 f \cdot \sigma^*(f) = \prod P_j|_C$. Then $x = \lambda f$, $y = \sigma^*(\lambda f)$ is a σ-invariant smooth rational curve in $\tilde{Z}$, and therefore certainly lifts as such to Z provided that $\mathcal{C}$ avoids the intersection points of the $\mathcal{C}_1, \ldots, \mathcal{C}_n$. In the exceptional case that $\mathcal{C}$ does meet some of the $\mathcal{C}_j \cap \mathcal{C}_k$, our recipe for resolving the singularities has the effect that we still have unique liftings, and these are, by virtue of our construction, limits of the previously described curves. We will now see that these curves are all twistor lines of a self-dual metric.

The constructed manifold Z comes equipped with a $(\mathbb{C} - 0)$-action, given on $\tilde{Z}$ by

$$[x, y, t] \mapsto [\zeta x, \zeta^{-1} y, t] \ .$$

While this does not quite make $(x, y) \neq 0$, $t \neq 0$ into a holomorphic principal line bundle— after all, the inverse image of each point in $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n \subset \mathbb{CP}_1 \times \mathbb{CP}_1$ consists of two orbits, not one— we can still analyze Z via the techniques of §6 by using the following trick. Choose any σ-invariant curve $\mathcal{C} \neq \mathcal{C}_1, \ldots, \mathcal{C}_n$ in the fixed homology class, and let p denote the corresponding point of $\mathcal{H}^3$. Let $\mathcal{V}$ be a geodesically convex neighborhood of p in $\mathcal{H}^3 - \{p_1, \ldots, p_n\}$, and let $\mathcal{T} \subset \mathbb{CP}_1 \times \mathbb{CP}_1$ denote the set of directed geodesics meeting $\mathcal{V}$. Then $\mathcal{T} \cap \mathcal{C}_j$ consists of two connected components D_j and $\overline{D}_j$, where D_j contains ℓ_j and $\overline{D}_j$ contains r_j. Let $\mathcal{I}$ denote the line bundle given by the ideal sheaf of $\bigcup_{j=1}^{n}[D_j]$, and let $L = \mathcal{I} \otimes \mathcal{O}(n-1,1)$. Let s denote the canonical section of $\mathcal{I}^*$. There is then a canonical open embedding $(L-0) \hookrightarrow \tilde{Z}$ given by

$$\alpha \mapsto [s\alpha, \frac{P_1 \dots P_n}{s\alpha}, 1] \ .$$

Because the restriction of L to $\mathcal{C}$ is trivial, it can be analyzed by the Hitchin-Ward correspondence discussed in §6. Since L can be rewritten as $(1-n)\mathcal{O}(-1,1) + \sum_{j=1}^{n}(\mathcal{O}(0,1) - [D_j])$, where sums in the Picard group are used to represent tensor products of line bundles, the linearity of the Penrose transform on $H^1(\mathcal{T}, \mathcal{O})$ allows us to find the harmonic function V on $\mathcal{V}$ by separately analyzing the bundles $\mathcal{O}(-1,1)$ and $\mathcal{O}(0,1) - [D_j] = [\overline{D_j}] - \mathcal{O}(1,0)$. I claim that these correspond to $V = 1$ and $V = 1 + G_j$, respectively.

When $V = 1$, the metric $Vh + V^{-1}\omega^2$ just becomes the product metric on $\mathcal{H}^3 \times S^1$, and so is conformally equivalent[7] to the standard metric on $S^4 - S^2$ and so has an open set in $\mathbb{CP}_3$ as its twistor space. In fact, this open set is just the complement of the union of the real twistor-lines joining the skew pair of projective lines gotten by taking the two holomorphic lifts of $S^2 \subset S^4$. But the complement of two skew lines in $\mathbb{CP}_3$ is the exactly $\mathcal{O}(-1,1) \to \mathbb{CP}_1 \times \mathbb{CP}_1$ minus the zero section, proving that $\mathcal{O}(-1,1)$ does indeed correspond to $V = 1$.

When $V = 1 + G_j$, we saw in §4 that the resulting metric is just the Burns metric on the blow-up of the origin in $\mathbb{C}^2$. Since the Burns metric is conformally diffeomorphic (in an orientation reversing manner) to $\mathbb{CP}_2$ minus a point equipped with the Fubini-Study metric [L4], the twistor space corresponding to $V = 1 + G_j$ is therefore[AHS] an open subset of the flag manifold

$$F_{1,2,3} := \{([Z^0, Z^1, Z^2], [W_0, W_1, W_2]) \in \mathbb{CP}_2 \times \mathbb{CP}_2 \mid \sum_{k=0}^{2} Z^k W_k = 0\} \ .$$

One may also observe that any (conformally) isometric circle-action on $\mathbb{CP}_2$ which fixes a 2-surface must be rotation around a projective line, and so the corresponding action on the twistor space may be taken to be

$$([Z^0, Z^1, Z^2], [W_0, W_1, W_2]) \to ([Z^0, Z^1, \zeta^{-1}Z^2], [W_0, W_1, \zeta W_2]) \ ;$$

the fixed center p_j, corresponding as it does to the unique fixed point of the relevant circle-action, must by symmetry therefore correspond to the curve $P = 0$, which we

[7]The Riemannian product $S^n \times \mathcal{H}^m$ is a hypersurface of the light-cone in Minkowski $n+m+2$-space $\mathbb{R}^{n+m+1,1}$, and so is conformally mapped to an open set of the (n+m)-sphere by the central projection; inspection shows that this set is precisely $S^{n+m} - S^{m-1}$. An more pedestrian proof is to identify $\mathbb{R}^{n+m} - \mathbb{R}^{m-1}$ with $S^n \times H^m$ via cylindrical coordinates, and then observe that the Euclidean metric divided by r^2 is just the product of the spherical metric and the hyperbolic metric on the upper half-space H^m.

may thus identify with the previously considered curve $P_j = 0$. We may now define a $\mathbb{C}_*$-equivariant open embedding by

$$\begin{aligned}(\mathcal{O}(0,1) - [D_j]) - 0 &\hookrightarrow F_{1,2,3}\\ \alpha|_{([z_0,z_1],[\zeta_0,\zeta_1])} &\mapsto ([z_0, z_1, \frac{P_j}{s_j\alpha}], [\zeta_0, \zeta_1, s_j\alpha]) ,\end{aligned}$$

where s_j is the canonical section of the divisor line bundle $[D_j]$. Under this embedding, the curves in $\mathcal{O}(0,1) - [D_j]$ obtained as sections of the bundle over minitwistor lines are sent to embedded rational curves on which $c_1(F_{1,2,3}) = 4$, and so have bidegree $(1,1)$ in $\mathbb{CP}_2 \times \mathbb{CP}_2$; but the only such curves in $F_{1,2,3}$ are the twistor lines. We conclude that $\mathcal{O}(0,1) - [D_j]$ corresponds to the potential $V = 1 + G_j$.

Covering $\mathcal{H}^3 - \{p_1, \dots, p_n\}$ with geodesically convex sets, we conclude that Z contains an open dense subset which is the twistor space of $Vh + V^{-1}\omega^2$, where

$$V = (1-n)1 + \sum_{j=1}^{n}(1 + G_j) = 1 + \sum_{j=1}^{n} G_j .$$

We now will show that the compactification of this subset exactly corresponds to the process used to conformally compactify this self-dual manifold in §5.

Indeed, an additional set of real curves is provided by the inverse images of the points of $S \subset \mathbb{CP}_1 \times \mathbb{CP}_1$; and while these inverse images of course have trivial normal bundle in $\tilde{Z}$, our blowing down of $x = t = 0$ and $y = t = 0$ alters this to yield a normal bundle of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ in Z. In addition, there is a finite collection of real curves given by the strict transforms of the curves $\mathcal{C}_j$, and the normal bundle of these curves in Z is again is again $\mathcal{O}(1) \oplus \mathcal{O}(1)$ as a direct consequence of our procedure for resolving the singularities. Together with the previously described real twistor lines, these yield a connected compact family of real twistor lines filling all of Z, from whence it follows that Z is the twistor space of a compact self-dual 4-manifold M. Moreover, M is obtained from a monopole metric over $\mathcal{H}^3 - \{p_j\}$ with potential $V = 1 + \sum G_j$ by adding a single point for each p_j and for each point of the 2-sphere S. It follows that M is exactly the self-dual manifold described in §5.

While the Hitchin-Ward correspondence part of the above argument goes through even if the points $p_1, \dots, p_n$ are not in general position, the detailed procedure we have described for resolving the sigularities of $\tilde{Z}$ is no longer adequate. Nonetheless, we have a natural identification of open dense sets of $\tilde{Z}$ and the twistor space Z of the coresponding compact self-dual 4-manifold, and this identification still induces a bimeromorphic map between the two spaces because, after blowing down of $x = t = 0$ and $y = t = 0$, this identification is a biholomorphism on the complement of sets of complex codimension two.[8] ∎

[8]In fact, it is even not hard to see that Z is obtained from $\tilde{Z}$ by using the A_k resolutions discussed in [Fr] as local models.

Example. Consider the self-dual metric on $\mathbb{CP}_2\#\mathbb{CP}_2$ produced by the monopole ansatz, starting with a pair of points $p_1, p_2 \in \mathcal{H}^3$ separated by hyperbolic distance ρ. Let us realize $\mathbb{CP}_1 \times \mathbb{CP}_1$ as the quadric $x_o^2 - x_1^2 - x_2^2 - x_3^2 = 0$ in $\mathbb{CP}_3$; we may take our two points to be given by the points $(\cosh \frac{\rho}{2}, \pm\sinh \frac{\rho}{2}, 0, 0) \in \mathcal{H}^3 \subset \mathbb{R}^{1,3}$, so that the corresponding sections of $\mathcal{O}(1,1)$ are just the pull-backs of the sections of $\mathcal{O}(1) \to \mathbb{CP}_3$ given by

$$P_1 = (\cosh \frac{\rho}{2})x_0 - (\sinh \frac{\rho}{2})x_1$$
$$P_2 = (\cosh \frac{\rho}{2})x_0 + (\sinh \frac{\rho}{2})x_1 \ .$$

By Theorem 2, a bimeromorphic model for the twistor space is given by $xy = t^2 P_1 P - 2$ in $\mathbb{P}(\mathcal{O}(1,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O})$, or equivalently by the intersection of the hypersurfaces

$$xy - t^2(\cosh^2 \frac{\rho}{2})x_0^2 + t^2(\sinh^2 \frac{\rho}{2})x_1^2 = 0$$
$$x_o^2 - x_1^2 - x_2^2 - x_3^2 = 0$$

in $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{CP}_3$. Since $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{CP}_3$ is just the blow-up of $\mathbb{CP}_5$ along $\mathbb{CP}_1$, as is seen by setting $x = tx_4, y = tx_5$, it follows that the twistor space must be bimeromorphic to the intersection of the two quadrics

$$x_4x_5 - \frac{1}{2}(1 + \cosh\ \rho)x_0^2 + \frac{1}{2}(-1 + \cosh\ \rho)x_1^2 = 0$$
$$x_o^2 - x_1^2 - x_2^2 - x_3^2 = 0$$

in $\mathbb{CP}_5$. Introducing new coordinates by

$$z_0 = ix_2,\ z_1 = ix_3,\ z_2 = \frac{ix_1}{\sqrt{(1 + \mathrm{sech}\ \rho)/2}},\ z_3 = \sqrt{2}x_0,$$

$$z_4 = \frac{i(x_4 + x_5)}{\sqrt{2(1 + \cosh\ \rho)}},\ z_5 = \frac{x_4 - x_5}{\sqrt{2(1 + \cosh\ \rho)}}$$

puts this locus in the standard form [P1]

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0$$
$$2z_0^2 + 2z_1^2 + \lambda z_2^2 + \frac{3}{2}z_3^2 + z_4^2 + z_5^2 = 0\ ,$$

where the value of Poon's parameter $\lambda \in]\frac{3}{2}, 2[$ is given by

$$\lambda = \frac{3}{2} + \frac{1}{2}\mathrm{sech}\ \rho.$$

We thus obtain *every* positive-scalar-curvature self-dual metric on $\mathbb{CP}_2\#\mathbb{CP}_2$ by varying the distance between our two points in $\mathcal{H}^3$. A clear picture of the two ends of this 1-dimensional moduli space of self-dual metrics now emerges: $\lambda \to \frac{3}{2}$ corresponds to $\rho \to \infty$, and therefore to degeneration into two copies of the Fubini-Study conformal structure; $\lambda \to 2$ corresponds to $\rho \to 0$, and thus to degeneration into two copies of the Eguchi-Hansen conformal structure; cf. §5. ∎

Since the vector bundles $\mathcal{O}(k,k) \otimes [\mathcal{O}(n-1,1) \oplus \mathcal{O}(1,n-1) \oplus \mathcal{O}]$ are very ample for k large, the total space of the bundle $\mathbb{P}(\mathcal{O}(n-1,1) \oplus \mathcal{O}(1,n-1) \oplus \mathcal{O}) \to \mathbb{CP}_1 \times \mathbb{CP}_1$ supports ample line bundles, namely those given by negative powers of the tautological line bundle twisted by pull-backs of $\mathcal{O}(k,k)$; all these complex 4-manifolds, and hence their hypersurfaces $\tilde{Z}$, are therefore projective algebraic. Theorem 2 thus has the following immediate consequence:

Corollary. *All of the constructed self-dual metrics on $n\mathbb{CP}_2$ have Moishezon twistor spaces.*

On the other hand, Campana [Cp] has recently proved that a compact self-dual 4-manifold with Moishezon twistor space must be homeomorphic to $n\mathbb{CP}_2$ for some $n \geq 0$; our construction shows that this is essentially sharp.[9] The above corollary also allows the reader uninterested in solving the (pure thought) scalar curvature exercize of §5 to instead prove the positivity of the Yamabe constant by citing a result of Poon [P_2], which asserts that this holds whenever the twistor space is Moishezon.

[9] It remains to be seen, of course, whether a self-dual 4-manifold with Moishezon twistor space is necessarily *diffeomorphic* to $n\mathbb{CP}_2$.

Bibliography

[AHS] M. Atiyah, N. Hitchin and I. Singer, "Self-Duality in Four Dimensional Riemannian Geometry," **Proc. R. Soc. Lond. A 362** (1978) 425-461.

[BF] C. Boyer and J. Finley, "Killing Vectors in Self-Dual Euclidean Einstein Spaces," **J. Math. Phys. 23** (1982) 1126-1130.

[Bu] D. Burns, "Twistors and Harmonic Maps," Amer. Math. Soc. Conference talk, Charlotte, N.C., October 1986.

[Cp] F. Campana, "On Twistor Spaces of the Class $\mathcal{C}$," *Forschungsschwerpunkt Komplexe Mannigfaltigkeiten 46*, *preprint*, Bayreuth, 1989; (*to appear in* **J. Differential Geometry**)

[Ct] E. Cartan, "Sur une Classe d'Espaces de Weyl," **Ann. Scient. Ec. Norm. Sup. 60** (1943) 1-16

[D] A. Derdzinski, "Self-Dual Kähler Manifolds and Einstein Manifolds of Dimension Four," **Comp. Math. 49** (1983) 405-433.

[DF] S.K. Donaldson and R. Friedman, "Connected Sums of Self-Dual Manifolds and Deformations of Singular Spaces," **Nonlinearity 2,** (1989) 197-239.

[Fl] A. Floer, "Self-Dual Conformal Structures on $\ell\mathbb{CP}^2$," **J. Diff. Geometry**, *to appear.*

[Fr] R. Friedman, "Simultaneous Resolution of Threefold Double Points," **Math. Ann. 274** (1986) 671-689.

[GH] G.W. Gibbons and S. W. Hawking, "Gravitational Multi-Instantons," **Phys. Lett.** 78B (1978) 430-432.

[H_1] N.J. Hitchin, "Polygons and Gravitons," **Math. Proc. Cambridge Philos. Soc. 83** (1979) 465-476.

[H_2] N.J. Hitchin, "Kählerian Twistor Spaces," **Proc. Lond. Math. Soc. 43** (1981) 133-150.

[H_3] N.J. Hitchin, "Complex Manifolds and Einstein's Equations," *Lecture Notes in Mathematics* **970** (1982) 73-99.

[H_4] N.J. Hitchin, "Monopoles and Geodesics," **Comm. Math. Phys. 83** (1982) 579-602.

[JT] P. E. Jones and K. P. Tod, "Minitwistor Spaces and Einstein-Weyl Spaces," **Class. Quantum Grav. 2** (1985) 565-577.

[L_1] C. LeBrun, "Twistor CR Manifolds and Three-Dimensional Conformal Geometry," **Trans. Am. Math. Soc. 284** (1984) 601-616.

[L_2] C. LeBrun, "Foliated CR Manifolds, **J. Differential Geometry 22** (1985) 81-96.

[L_3] C. LeBrun, "On the Topology of Self-Dual Manifolds," **Proc. AMS 98** (1986) 637-640.

[L_4] C. LeBrun, "Poon's Self-Dual Metrics and Kähler Geometry," **J. Differential Geometry 28** (1988) 341-343.

[L_5] C. LeBrun, "Counter-Examples to the Generalized Positive Action Conjecture," **Comm. Math. Phys. 118** (1988) 591-596.

[Pa] Q.H. Park, "Extended Conformal Symmetries in Real Heavens," *preprint* 90-037, University of Maryland, 1989.

[Pd] H. Pedersen, "Einstein Metrics, Spinning-Top Motions, and Monopoles," **Math. Ann. 274** (1986) 35-59.

[PT] H. Pedersen and K.P. Tod, "Three Dimensional Einstein-Weyl Geometry," *preprint*, Odense Universitet, 1988; *to appear in* **Adv. Math.**

[Pt] M. Pontecorvo, "On Twistor Spaces of Anti-Self-Dual Hermitian Surfaces," *preprint*, SISSA, 1990.

[P_1] Y.S. Poon, "Compact Self-dual Manifolds with Positive Scalar Curvature," **J. Differential Geometry 24** (1986) 97-132.

[P_2] Y.S. Poon, "Algebraic Dimension of Twistor Spaces," **Math. Ann. 282** (1988) 621-627.

[W] R.S. Ward, "Einstein-Weyl Spaces and $\mathbf{SU(\infty)}$ Toda Fields," **Class. Quantum Grav. 7** (1990) L95-L98.