Yamabe Invariants and Spin^c Structures

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Abstract

The *Yamabe invariant* of a smooth compact manifold is by definition the supremum of the scalar curvatures of unit-volume Yamabe metrics on the manifold. For an explicit infinite class of 4-manifolds, we show that this invariant is positive but strictly less than that of the 4-sphere. This is done by using spin^c Dirac operators to control the lowest eigenvalue of a perturbation of the Yamabe Laplacian. These results dovetail perfectly with those derived from the perturbed Seiberg-Witten equations [14], but the present method is much more elementary in spirit.

1 Introduction

There is a natural diffeomorphism invariant [10, 21] which arises from a variational problem for the total scalar curvature of Riemannian metrics on any given compact smooth n-manifold M. Observe that the group of smooth positive functions $u : M \to \mathbf{R}^+$ acts on the space of smooth Riemannian metrics g by conformal rescaling $g \mapsto u^2 g$. The *conformal class* of a Riemannian metric g is by definition the orbit $\gamma = [g]$ of this action which contains g. Let $\mathcal{C}(M) = \{\gamma\}$ denote the set of conformal classes of metrics on M. We may then define an invariant of the smooth compact manifold M by setting

$$Y(M) := \sup_{\gamma \in \mathcal{C}(M)} \inf_{g \in \gamma} \frac{\int_M s_g \, d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}},$$

where s_g and $d\mu_g$ respectively denote the scalar curvature and volume measure of the Riemannian metric g. We will call this the *Yamabe invariant* of M.

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To put this definition in context, recall [4] that, for $n > 2$, a Riemannian metric is Einstein iff it is a critical point of the functional

$$g \mapsto \mathcal{S}(g) := \frac{\int_M s_g \, d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}}.$$

The functional $\mathcal{S}$, however, is neither bounded above nor below, so one cannot hope to find a critical point by either minimizing or maximizing it. However, the restriction of $\mathcal{S}$ to any conformal class *is* bounded below, and a remarkable theorem [2, 16, 20] of Yamabe, Trudinger, Aubin, and Schoen asserts that each conformal class γ contains metrics g, called *Yamabe metrics*, which attain the minimum value

$$Y_\gamma = \inf_{g \in \gamma} \mathcal{S}(g).$$

This number is called the *Yamabe constant* of the conformal class γ. A simple and beautiful argument of Aubin [2] shows that $Y_\gamma \leq Y(S^n) = n(n-1)V_n^{2/n}$ for any conformal class on any n-manifold, where V_n is the volume of the standard metric on S^n. Thus the scalar curvatures of unit-volume Yamabe metrics on M are bounded above, and their least upper bound is a real number $Y(M) \leq Y(S^n)$. Of course, this by no means guarantees[1] that $Y(M)$ is a critical value of $\mathcal{S}$. Indeed, there are many low-dimensional examples [4] of manifolds which do not admit any Einstein metrics.

A conformal class γ contains a metric $g \in \gamma$ with positive scalar curvature iff $Y_\gamma > 0$, so the Yamabe invariant $Y(M)$ is positive iff M carries a metric of positive scalar curvature. Now there is a substantial body of results [18, 9, 8, 23, 11, 24] concerning manifolds which admit metrics of positive scalar curvature, and these results may be understood as simply providing one kind of estimate for Yamabe invariants. The bulk of this literature consists of variations on a theme of Lichnerowicz: on a spin manifold of positive scalar curvature, the Dirac operator must have index zero. In the present article, we will show that essentially the same method, applied to the *twisted* Dirac operators associated with spin^c structures, can be used to calculate the Yamabe invariant for many 4-manifolds.

For rather mysterious reasons, the Yamabe invariant seems to be most easily computed in low dimensions. In dimension 2, for example, the Gauss-Bonnet theorem asserts that $Y(M)$ is a multiple of the Euler characteristic. In dimension 3, Anderson [1] has announced a computation of the Yamabe invariants of all irreducible 3-manifolds with infinite π_1. And in dimension 4, which will be our field of concern, the advent of Seiberg-Witten theory [25] has made it

[1] A plausible folk tradition maintains that it was Yamabe himself who first considered the question of whether $Y(M)$ can be realized as the scalar curvature of a unit-volume Yamabe metric. However, $Y(M)$ seems to make its first published appearance in an important paper of O. Kobayashi [10], who called it the mu invariant. Other authors [1, 21] have elsewhere called it the sigma constant.

possible [13, 15] to compute the Yamabe invariants of most complex algebraic surfaces. One remarkable feature that emerges is that $Y(M)$ often distinguishes between different smooth structures on the same topological 4-manifold.

The Seiberg-Witten method, however, is most finely tuned to manifolds with $Y(M) \leq 0$, although a perturbed version can be used [14] to show, for example, that $Y(\mathbf{CP}_2) = 12\sqrt{2}\pi < 8\sqrt{6}\pi = Y(S^4)$. In this article, the last result will be reproved by a much simpler method, while at the same time proving the following substantial generalization:

Theorem A *The Yamabe invariant of* $\mathbf{CP}_2$ *is unaltered by* 0*-surgeries:*

$$Y(\mathbf{CP}_2 \# m(S^1 \times S^3)) = Y(\mathbf{CP}_2) = 12\sqrt{2}\pi$$

for all $m \geq 0$*. In particular, these projective planes with handles all have Yamabe invariant strictly less than* $Y(S^4) = 8\sqrt{6}\pi$*.*

The same reasoning also proves the following:

Theorem B *Let* $k \in \{1, 2, 3\}$*, and let* m *be any natural number. Then*

$$12\sqrt{2}\pi \leq Y(k\mathbf{CP}_2 \# m(S^1 \times S^3)) \leq 4\pi\sqrt{2k + 16}.$$

In particular, these connected sums of $\mathbf{CP}_2$*'s and* $S^1 \times S^3$*'s all have Yamabe invariant strictly less than* $Y(S^4)$*.*

For Kähler-type complex surfaces of Kodaira dimension ≥ 0, Seiberg-Witten theory allows one to show [15] that $Y(M)$ is a bimeromorphic invariant — i.e. it is unchanged by blowing up and down. One might therefore blithely suppose that the same is true of *all* complex surfaces. However, the present methods show that this supposition simply does not hold water:

Theorem C *The Hopf surface and its one-point blow-up have different Yamabe invariants. Thus the Yamabe invariant is* not *a bimeromorphic invariant for complex surfaces of class* VII*.*

The key inequality for the Yamabe invariant developed here is sensitive only to homotopy type rather than to diffeomorphism type. The present methods are thus oblivious to the deeper aspects of 4-manifold topology detected by Seiberg-Witten invariants. Nonetheless, several peculiarities of dimension 4 — e.g. the conformal invariance of harmonic 2-forms — will play a a crucial rôle. It thus remains to be seen whether the ideas developed in this article have any ramifications in higher dimensions.

2 Perturbed Yamabe Laplacians

Let (M^4, g) be a smooth 4-dimensional Riemannian manifold. The *Yamabe Laplacian* of g will mean the elliptic operator

$$\Box_g = 6\Delta_g + s_g \tag{1}$$

acting on functions on M. Here s_g denotes the scalar curvature of g, and $\Delta = d^*d = -\mathrm{div}\,\mathrm{grad}$ is the (positive) Laplace-Beltrami operator of g. Considered as a map between spaces of functions (or densities) of appropriate conformal weight, this operator is *conformally invariant*; namely, if $\tilde{g} = u^2 g$ for some positive C^2 function u, then

$$\Box_{\tilde{g}}\varphi = u^{-3}\Box_g(u\varphi) \tag{2}$$

for any function φ. The geometric essence of this statement is the fact that the scalar curvature transforms under conformal rescalings according to the rule

$$s_{\tilde{g}} = u^{-3}\Box_g u.$$

Let $E \subset \otimes^2 T^*M$ be the real line bundle spanned by the metric g. Evidently, this depends only on the conformal class γ, and conversely the conformal class is uniquely determined by E. A section f of E may simply be thought of as a real valued function on M which transforms according to the rule

$$f \mapsto \tilde{f} = u^{-2}f$$

when the metric g is replaced by $\tilde{g} = u^2 g$, since this transformation rule ensures that $\tilde{f}\tilde{g} = fg$. Sections of E will therefore be called *functions of conformal weight* -2.

Example. Let ω be a smooth 2-form. The function

$$f = |\omega|_g$$

then transforms according to the rule

$$f \mapsto \tilde{f} = u^{-2}f$$

when $g \mapsto \tilde{g} = u^2 g$. Thus f is a function of conformal weight -2. Notice that while f^2 is smooth, f will typically merely be Lipschitz if the locus where ω vanishes is non-empty.

Lemma 1 *Let γ be a smooth conformal class on a 4-manifold M, and let f be a function of conformal weight -2 on M. Then the operator $\Diamond_g = \Box_g - f$ transforms according to the rule*

$$\Diamond_{\tilde{g}}\varphi = u^{-3}\Diamond_g(u\varphi)$$

when g is replaced by $\tilde{g} = u^2 g$.

Proof. We have

$$\begin{aligned}\diamondsuit_{\tilde g}\varphi &= \Box_{\tilde g}\varphi - \tilde f\varphi \\ &= u^{-3}\Box_g(u\varphi) - u^{-2}f\varphi \\ &= u^{-3}[\Box_g(u\varphi) - uf\varphi] \\ &= u^{-3}\diamondsuit_g(u\varphi)\end{aligned}$$

by the conformal invariance of the Yamabe Laplacian. ∎

Definition 1 *Let g be a metric on M, and let f be a function of conformal weight -2. The* modified scalar curvature *of the pair (g, f) will mean the function $\sigma = \sigma_{(g,f)} = s - f$, where $s = s_g$ is the usual scalar curvature of g.*

Lemma 2 *Under conformal changes $g \mapsto \tilde g = u^2 g$, the modified scalar curvature transforms according to the rule $\sigma \mapsto \tilde\sigma = u^{-3}\diamondsuit_g u$.*

Proof. Indeed, $\sigma_{(g,f)} = \diamondsuit_g(1)$. By the previous lemma, we therefore have $\sigma_{(\tilde g,\tilde f)} = \diamondsuit_{\tilde g}(1) = u^{-3}\diamondsuit_g u$. ∎

Proposition 3 *Let g be a smooth Riemannian metric on a compact smooth 4-manifold M, and let $f \in C^{0,\alpha}(M, E)$, $\alpha \in (0, 1)$, be a Hölder continuous function of conformal weight -2. Then there is a conformally related metric $\tilde g = u^2 g$ of class $C^{2,\alpha}$ whose modified scalar curvature satisfies $\tilde\sigma > 0$, $\tilde\sigma < 0$, or $\tilde\sigma \equiv 0$. Moreover, these three possibilities are mutually exclusive.*

Proof. Let λ_g be the lowest eigenvalue of $\diamondsuit_g$:

$$\lambda_g = \inf_{\substack{u \in L^2_1 \\ \|u\|_{L^2} = 1}} \langle \diamondsuit_g u, u\rangle_{L^2(g)}.$$

Let u be a non-zero eigenfunction of $\diamondsuit_g$ corresponding to this eigenvalue:

$$\diamondsuit_g u = \lambda_g u.$$

By the interior Schauder estimates [7, p.109], u is of class $C^{2,\alpha}$. By the minimum principle [7, p.35], $u \neq 0$, so $\tilde g = u^2 g$ is a $C^{2,\alpha}$ metric conformal to g. Its modified scalar curvature is

$$\tilde\sigma = u^{-3}\diamondsuit_g u = u^{-2}\lambda_g,$$

and so is strictly positive, strictly negative, or identically zero, exactly as promised.

Now notice that $\diamondsuit_{\tilde g} = 6\Delta_{\tilde g} + \tilde\sigma$. Thus, for any positive C^2 function v, the modified scalar curvature of $v^2\tilde g$ is at most $v^{-2}\tilde\sigma$ at the minima of v, and at least

$v^{-2}\tilde{\sigma}$ at the maxima of v. The three possibilities under discussion are therefore mutually exclusive. ∎

Notice that the L^2 norm

$$\|\omega\|_2 = \left(\int_M |\omega|_g^2 d\mu_g\right)^{1/2}$$

of a 2-form ω on any compact 4-manifold M is *conformally invariant*; that is, it depends only on the conformal class $\gamma = [g]$ of the metric.

Corollary 4 *Let γ be a smooth conformal class on a smooth compact 4-manifold M, and let ω be a differentiable 2-form on M. Then one of the following must hold:*

- *there is a C^∞ metric $g \in \gamma$ with scalar curvature $s > |\omega|_g$; or*
- *$Y_\gamma < \|\omega\|_2$; or*
- *$Y_\gamma = \|\omega\|_2$, and there is a (C^∞) Yamabe metric $g \in \gamma$ with $s = |\omega| \equiv$ const. In particular, this happens only if ω is nowhere zero or vanishes identically.*

Proof. Let $f = |\omega|$. The corresponding modified scalar curvature $\sigma = s - |\omega|$ then defines a continuous map from the Banach space of C^2 metrics in γ to the Banach space of C^0 functions. Thus the set of C^2 metrics in γ with $\sigma > 0$ is therefore C^2 open. However, the smooth metrics in γ are dense in the C^2 metrics. Thus, if there is no smooth metric in γ with $s > |\omega|$, there cannot be a $C^{2,\alpha}$ metric with $s > |\omega|$ either. But by Proposition 3, this happens precisely if there is instead a $C^{2,\alpha}$ metric $g \in \gamma$ with $s \leq |\omega|$.

If the latter happens, we then have a metric $g \in \gamma$ for which

$$\frac{\int s\, d\mu}{\sqrt{\int d\mu}} \leq \frac{\int |\omega| d\mu}{\sqrt{\int d\mu}} \leq \sqrt{\int |\omega|^2 d\mu} = \|\omega\|_2,$$

so that the definition of the Yamabe constant yields

$$Y_\gamma \leq \|\omega\|_2.$$

If equality holds, moreover, the metric g is a Yamabe metric, and satisfies $s \equiv |\omega|$. Since g is a Yamabe metric, it is smooth and has constant scalar curvature. In particular, $|\omega| \equiv s$ must be constant. ∎

3 Polarizations

Let M be a compact oriented 4-manifold, and let γ be a conformal class on M. Then the orientation and conformal structure induce a Hodge star operator

$$\star : \Lambda^2 \to \Lambda^2$$

on the bundle of 2-forms. That is to say, the Hodge star operator on middle-dimensional forms determined by any metric $g \in \gamma$ actually depends only on the conformal class γ. This linear endomorphism of Λ^2 satisfies $\star^2 = 1$, so that we have an eigenspace decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

depending only on γ and the orientation. The factors $\Lambda^\pm$, corresponding to the eigenvalues ± 1, are vector bundles of rank 3, and reversing the orientation of M just interchanges them. Sections of Λ^+ are called self-dual 2-forms, whereas sections of Λ^- are called anti-self-dual 2-forms.

Now the Hodge theorem tells us that

$$H^2(M, \mathbf{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \ d \star \varphi = 0\}.$$

Since $\star$ defines an involution of the right-hand side, however, we therefore have a direct sum decomposition

$$H^2(M, \mathbf{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where

$$\mathcal{H}^\pm = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

are the spaces of self-dual and anti-self-dual harmonic forms. Given any cohomology class $\zeta \in H^2(M, \mathbf{R})$, we thus have a γ-induced decomposition

$$\zeta = \zeta^+ + \zeta^-,$$

where $\zeta^\pm \in \mathcal{H}^\pm$.

The subspace $\mathcal{H}^+ \subset H^2(M, \mathbf{R})$ is called the *polarization* determined by γ. The intersection form $\cup : H^2 \times H^2 \to H^4 = \mathbf{R}$ becomes positive-definite when restricted to $\mathcal{H}^+$, and $\mathcal{H}^+$ is a maximal subspace with this property. Indeed, $\mathcal{H}^-$ is the orthogonal complement of $\mathcal{H}^+$ with respect to $\cup$, and the restriction of $\cup$ to $\mathcal{H}^-$ is *negative*-definite. The dimension of $\mathcal{H}^\pm$ is therefore a homotopy invariant $b^\pm$ of M, and the difference $\tau = b^+ - b^-$ is called the *signature* of M. It is important to point out that the polarization $\mathcal{H}^+ \subset H^2$ really does [6] depend on the conformal class γ unless $b^- = 0$.

If ω is a self-dual harmonic 2-form with respect to γ, we have

$$\|\omega\|_2^2 = [\omega] \cup [\omega] = [\omega]^2,$$

since $|\omega|^2 d\mu = \omega \wedge \star\omega = \omega \wedge \omega$. Thus Corollary 4 immediately implies

Proposition 5 *Let $\zeta \in H^2(M, \mathbf{R})$ be a fixed cohomology class on a smooth compact 4-manifold M, and let γ be any conformal class on M. Let ϕ denote the unique γ-harmonic 2-form with $[\phi] = \zeta$. Then one of the following must hold:*

- *there is a C^∞ metric $g \in \gamma$ with scalar curvature $s > |\phi^+|_g$; or*
- *$Y_\gamma < \sqrt{(\zeta^+)^2}$; or*
- *$Y_\gamma = \sqrt{(\zeta^+)^2}$, and there is a Yamabe metric $g \in \gamma$ with $s = |\zeta^+| \equiv \mathrm{const}$.*

Proof. Set $\omega = \phi^+$ and apply Corollary 4. ∎

4 Dirac Operators and Spin^c Structures

Let M be a compact oriented 4-manifold. A cohomology class $\eta \in H^2(M, \mathbf{Z})$ is then called *characteristic* if $\eta \equiv w_2(M) \bmod 2$; by a theorem of Wu, such elements always exist. Given such a class η, let L be the Hermitian complex line bundle with $c_1(L) = \eta$. This L is unique up to isomorphism. Moreover, given a conformal class γ on M, the obstruction to the existence of a square-root $L^{1/2}$ of L precisely coincides with the obstruction to defining the spin bundles $\mathbf{S}_\pm$ of (M, γ). Thus one may define two rank-2 Hermitian vector bundles $V_\pm$ on M such that

$$V_\pm = \mathbf{S}_\pm \otimes L^{1/2},$$

in the formal sense that on an any spin open set of M, $\mathbf{S}_\pm$ and $L^{1/2}$ may be defined, and there is a canonical (but sign-ambiguous) isomorphism $V_\pm \to \mathbf{S}_\pm \otimes L^{1/2}$. A choice of such bundles $V_\pm$ is called a spin^c structure. If $H_1(M, \mathbf{Z})$ contains no elements of order 2, the spin^c structures on M are in one-to-one correspondence with the set of characteristic elements $\eta \in H^2(M, \mathbf{Z})$.

Now fix a spin^c structure on M, and choose some Hermitian connection θ on the associated line bundle $L \to M$. If g is any metric on M, its Levi-Cività connection and θ together induce a connection

$$\nabla^\theta : \Gamma(V_+) \to \Gamma(V_+ \otimes T^*M)$$

via the local identifications

$$V_+ = \mathbf{S}_+ \otimes L^{1/2}.$$

On the other hand, Clifford multiplication induces a bundle homomorphism

$$V_+ \otimes T^*M \xrightarrow{\cdot} V_-.$$

Composing these maps gives us a (twisted) Dirac operator

$$D^\theta : \Gamma(V_+) \to \Gamma(V_-).$$

The latter is an elliptic operator whose index is given by

$$\text{ind }(D^\theta) = \frac{c_1^2(L) - \tau(M)}{8}.$$

When this index is positive, we get an estimate for the Yamabe constant of any conformal class:

Theorem 6 *Let M be a smooth compact oriented 4-manifold, and let $\eta \in H^2(M, \mathbf{Z})$ be a non-torsion, characteristic element such that $\eta^2 > \tau(M)$. Let γ be any smooth conformal class on M. Then*

$$Y_\gamma \leq 4\pi\sqrt{2(\eta^+)^2}.$$

Moreover, equality occurs iff M is diffeomorphic to a rational complex surface, in such a manner that η becomes the first Chern class $c_1(M)$, and some Yamabe metric representing γ becomes a Kähler metric of constant, non-negative scalar curvature.

Proof. Let φ denote the unique γ-harmonic 2-form such that the de Rham class $[\varphi]$ coincides with the image of η in real cohomology. If we had $Y_\gamma \geq 4\pi\sqrt{2(\eta^+)^2}$, Proposition 5, applied to $\zeta = 4\pi\sqrt{2}\eta$, would assert the existence of a smooth metric $g \in \gamma$ with $s_g \geq 4\pi\sqrt{2}|\varphi^+|_g$; and if equality holds, moreover, g may be further assumed to be a Yamabe metric. We claim, however, that this leads to a contradiction unless equality holds and the geometry is of the special kind detailed above.

Indeed, set $F = -2\pi i\varphi$, and let $L \to M$ be the unique Hermitian line bundle with $c_1(L) = \eta$. Since $\frac{i}{2\pi}F$ then represents the image of $c_1(L)$ in real cohomology, the Chern-Weil theorem tells us there is a $U(1)$ connection θ on L whose curvature is F. Choose a spin^c structure with associated line bundle L, and let $D^\theta : \Gamma(V_+) \to \Gamma(V_-)$ be the corresponding Dirac operator. By construction, the index

$$\text{ind }(D^\theta) = \frac{\eta^2 - \tau(M)}{8}$$

of this operator is positive. Thus there exists a smooth section $\psi \not\equiv 0$ of V_+ with $D^\theta\psi = 0$. But, by the Weitzenböck formula [9, 11]

$$D^{\theta *}D^\theta = \nabla^{\theta *}\nabla^\theta + \frac{s}{4} + \frac{1}{2}F^+,$$

and we therefore have

$$0 = (\psi, \nabla^*\nabla\psi) + \frac{s}{4}|\psi|^2 + \frac{1}{2}(\psi, F^+ \cdot \psi),$$

where the self-dual 2-form F^+ acts on V_+ by Clifford multiplication. The latter action is diagonalizable, with eigenvalues $\pm\sqrt{2}|F^+| = \pm 2\pi\sqrt{2}|\varphi^+|$. Thus

$$0 \geq (\psi, \nabla^*\nabla\psi) + \frac{s - 4\pi\sqrt{2}|\varphi^+|}{4}|\psi|^2.$$

Integrating over M, we thus have

$$0 \geq \int_M [|\nabla\psi|^2 + \frac{s - 4\pi\sqrt{2}|\varphi^+|}{4}|\psi|^2]d\mu.$$

But, by assumption, our metric g satisfies $s \geq 4\pi\sqrt{2}|\varphi^+|$. It follows that $s = 4\pi\sqrt{2}|\varphi^+|$, g is a Yamabe metric, and $\nabla\psi = 0$. The non-zero self-dual 2-form $\psi \odot \bar{\psi}$ is therefore parallel, so g is a Kähler metric. Moreover, L is now the anti-canonical line bundle of the associated complex structure on M, so $\eta = c_1(M)$. But since η is not a torsion class by assumption, and since M admits a Kähler metric of non-negative scalar curvature, our complex surface is rational or ruled [26]. Moreover, its Todd genus $(c_1^2 - \tau)/8$ is also positive, so it follows [3] that M is rational — i.e. obtained from $\mathbf{CP}_2$ by blowing up and down. ∎

5 The Main Theorems

We now restrict the last result to 4-manifolds with positive-definite intersection forms.

Theorem 7 *Let M be a smooth compact oriented 4-manifold with $b_-(M) = 0$, and suppose that $\eta \in H^2(M, \mathbf{Z})$ is a characteristic element such that $\eta^2 > b_2(M)$. Then*

$$Y(M) \leq 4\pi\sqrt{2\eta^2}.$$

Moreover, if there is a a conformal class γ on M such that $Y_\gamma = 4\pi\sqrt{2\eta^2}$, then $\eta^2 = 9$ and M is diffeomorphic to $\mathbf{CP}_2$ in such a manner that γ becomes the conformal class of the Fubini-Study metric.

Proof. Because $b_-(M) = 0$, $\eta^+ = \eta$ for any conformal class γ, and Theorem 6 therefore asserts that $Y_\gamma \leq 4\pi\sqrt{2\eta^2}$ for any conformal class γ. Taking the supremum over all γ then yields the first part of the result.

The second part of the result similarly follows from Theorem 6 because $\mathbf{CP}_2$ is the only rational surface with $b_- = 0$, and because [17] the isometry group of any constant-scalar-curvature Kähler metric on $\mathbf{CP}_2$ must be a maximal compact subgroup of the complex automorphism group $PGL(3, \mathbf{C})$. ∎

Corollary 8 *Let M be a smooth compact 4-manifold with non-trivial, positive-definite intersection form. Then $Y(M) \leq 4\pi\sqrt{2b_2(M)+16}$.*

Proof. We may assume that $b_2 < 4$, since the the upper bound in question is otherwise a trivial consequence of Aubin's estimate. Thus the intersection form of M is automatically diagonalizable [19, p.19]. Choose a basis for the free part of H^2 relative to which the intersection form is represented by the identity matrix, and let η be a characteristic element whose free part is a truncation of $(3,1,1)$ in this basis. Then $\eta^2 = 8 + b_2(M) > b_2(M)$. Now apply the previous theorem. ∎

Theorem 9 *Let $X_1, X_2, \ldots, X_\ell$ be 3-dimensional spherical space-forms, and let*

$$M = k\mathbf{CP}_2\#(S^1 \times X_1)\# \cdots \#(S^1 \times X_\ell)$$

for some $k \geq 1$. Then

$$12\sqrt{2}\pi \leq Y(M) \leq 4\pi\sqrt{2k+16}.$$

Proof. The upper bound in question is precisely that provided by the previous corollary.

To obtain the lower bound, first let g denote the Fubini-Study metric on $\mathbf{CP}_2$. This is an Einstein metric, and hence a Yamabe minimizer by the 4-dimensional Gauss-Bonnet theorem. Thus

$$Y(\mathbf{CP}_2) \geq Y_{[g]} = \mathcal{S}(g) = 12\sqrt{2}\pi.$$

Next, recall [10, 22] that $Y(S^1 \times X_j) = Y(S^4)$. Now a fundamental result of O. Kobayashi [10] asserts that

$$Y(M_j) \geq 0 \ \forall j \implies Y(M_1\# \cdots \#M_n) \geq \min_j Y(M_j),$$

so we therefore have

$$Y(k\mathbf{CP}_2\#(S^1 \times X_1)\# \cdots \#(S^1 \times X_\ell)) \geq Y(\mathbf{CP}_2) \geq 12\sqrt{2}\pi,$$

which is precisely the promised lower bound. ∎

Theorems A and B are simply interesting special cases of this result.

Theorem C also follows quite easily. Indeed, a primary Hopf surface is diffeomorphic to $S^1 \times S^3$, whereas its one-point blow-up is diffeomorphic to $\mathbf{CP}_2\#(S^1 \times S^3)$, albeit in an orientation-reversing manner. Thus a primary Hopf surface has Yamabe invariant equal to $Y(S^4) = 8\sqrt{6}\pi$ by [10, 22], whereas its blow-up has Yamabe invariant equal to $Y(\mathbf{CP}_2) = 12\sqrt{2}\pi$ by the above result. Incidentally, the same argument also works for secondary Hopf surfaces (finite quotients of primary Hopf surfaces).

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