On Einstein Manifolds of Positive Sectional Curvature

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Abstract

Let (M,g) be a compact oriented 4-dimensional Einstein manifold. If M has positive intersection form and g has non-negative sectional curvature, we show that, up to rescaling and isometry, (M,g) is $\mathbb{CP}_2$, with its standard Fubini-Study metric.

1 Introduction

A Riemannian manifold (M,g) is said to be *Einstein* if it has constant Ricci curvature — i.e. if its Ricci tensor r is a constant multiple of the metric:

$$r = \lambda g. \tag{1}$$

If g is complete and $\lambda > 0$, Myers' theorem [18] then tells us that M is compact, and has finite fundamental group.

The simplest examples of compact Einstein manifolds with positive Ricci curvature ($\lambda > 0$) are provided by the irreducible symmetric spaces of compact type. In dimension 4, this observation yields exactly two orientable examples: $S^4 = SO(5)/SO(4)$ and $\mathbb{CP}_2 = SU(3)/U(2)$, both of which actually have positive *sectional* curvature. A slight generalization would be to allow for *reducible* symmetric spaces; in dimension 4, this gives us the additional oriented examples of $S^2 \times S^2 = SO(4)/[SO(2) \times SO(2)]$ and its quotient by the simultaneous antipodal map on both factors. The latter examples have non-negative sectional curvature, although some of their sectional curvatures are actually zero.

While [24] there certainly are other compact 4-dimensional Einstein manifolds with $\lambda > 0$, none are known which have non-negative sectional curvature.

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One might hope that this is not merely accidental. In this direction, we are able to offer the following partial result:

Theorem A *Let M be a smooth compact oriented 4-manifold with (strictly) positive intersection form, and suppose that g is an Einstein metric on M which has non-negative sectional curvature. Then (M,g) is homothetically isometric to $\mathbb{CP}_2$, equipped with its standard Fubini-Study metric.*

To clarify the statement, let us recall that one can always find bases for the de Rham cohomology $H^2(M,\mathbb{R})$ in which the intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R})\times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) & \mapsto & \int_M \varphi\wedge\psi \end{array}$$

is represented by a diagonal matrix

$$\left[\begin{array}{cc} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{b_+(M)} & \\ & b_-(M)\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{array}\right]$$

and, in these terms, our first hypothesis stipulates that $b_-(M)=0$ and that $b_+(M)\neq 0$. The *Fubini-Study metric* is the unique $U(2)$-invariant metric on $\mathbb{CP}_2 = SU(3)/U(2)$ with total volume $\pi^2/2$; it is Einstein, and has sectional curvatures $K(P)\in[1,4]$. By *homothetically isometric*, we mean isometric after rescaling; in other words, the theorem concludes by asserting the existence of a diffeomorphism $\Phi: M\to\mathbb{CP}_2$ such that $g=\Phi^* c g_0$ for some some positive constant c.

Theorem A is actually a consequence of the following, more general result:

Theorem B *Let (M,g) be a smooth compact oriented Einstein 4-manifold with non-negative sectional curvature. Assume, moreover, that g is neither self-dual nor anti-self-dual. Then the Euler characteristic χ and the signature τ of M satisfy*

$$9\geq\chi>\frac{15}{4}|\tau|.$$

Here $\tau(M):=b_+(M)-b_-(M)$, whereas $\chi(M)=2+b_+(M)+b_-(M)$ if M has finite fundamental group. Thus, for example, while Tian [24] has shown that the manifolds $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$, $3\leq k\leq 8$, admit Einstein metrics with positive Ricci curvature, these spaces definitely *do not* admit Einstein metrics of positive *sectional* curvature.

Theorem B was directly inspired by the following result of Hitchin [14]: if a compact oriented 4-manifold M admits an Einstein metric g of positive sectional curvature, then the Euler characteristic χ and signature τ of M must satisfy

$$\chi \geq \left(\frac{3}{2}\right)^{3/2} |\tau|.$$

By freely quoting the subsequent topological results of Freedman [9], this actually contains enough information to conclude that an Einstein 4-manifold of positive intersection form and positive sectional curvature must be *homeomorphic* to either $\mathbb{CP}_2$ or $\mathbb{CP}_2 \# \mathbb{CP}_2$. Even on the crude level of homeomorphism type, however, Theorem A represents a 2-fold improvement over Hitchin's result. This reflects the fact that Hitchin's coefficient of $\left(\frac{3}{2}\right)^{3/2} = 1.837\ldots$ is less than half the coefficient of $\frac{15}{4} = 3.75$ appearing in Theorem B.

Let us conclude this introduction with some general remarks regarding the class of Riemannian manifolds under consideration. First of all, a celebrated result of Synge [22] asserts that any compact, orientable, even-dimensional Riemannian manifold of positive sectional curvature is necessarily simply connected. On the other hand, we have already seen, by the example of $(S^2 \times S^2)/\mathbb{Z}_2$, that no such result holds when the sectional curvature is merely assumed to be non-negative, even if the Ricci curvature is positive. Nonetheless, Theorem B *does* tell us that a compact orientable Einstein 4-manifold of non-negative sectional curvature and *non-zero signature* must be simply connected. Indeed, if $|\tau| \geq 1$, the inequality $\chi > \frac{15}{4}|\tau|$ and the observation that $\chi \equiv \tau \bmod 2$ together then guarantee that $\chi \geq 5$. But if such a manifold were not simply connected, its universal cover would then violate the inequality $9 \geq \chi$.

Finally, let us observe that there are, up to diffeomorphism, only finitely many compact 4-manifolds with Einstein metrics of non-negative Ricci curvature. The flat 4-manifolds, of course, nominally form a subclass of the the manifolds under discussion, but Bieberbach's theorem [3] in any case tells us that there are finitely many diffeomorphism types of these. For the others, which are our real concern here, the Ricci curvature must be positive, and we may thus rescale the metric so that, for example, $r = 3g$. The definition of the Ricci curvature then tells us that the sectional curvatures all satisfy $0 \leq K(P) \leq 3$. On the other hand, Myers' theorem [18] predicts that the diameter is $\leq \pi$. Moreover, the 4-dimensional Gauss-Bonnet theorem [2, 14] and our upper bound on curvature imply the volume is $\geq 8\pi^2/15$. With such bounds, Cheeger's finiteness theorem [6] then predicts that the given class of manifolds is precompact in the C^α topology[1], and therefore consists of finitely many diffeomorphism classes. Unfortunately, however, such arguments by no means predict the actual number of diffeotypes. By contrast, Theorem B and Freedman's classification [9] tell us that there are at most twelve *homeotypes* of simply connected compact Einstein 4-manifolds with non-negative sectional curvature.

[1]Indeed, the ellipticity of the Einstein equations guarantees [1] that the class of manifolds in question is actually compact in the C^∞ topology.

2 The Curvature of 4-Manifolds

We begin by recalling that the rank-6 bundle of 2-forms Λ^2 on an oriented Riemannian 4-manifold (M^4, g) has an invariant decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{2}$$

as the sum of two rank-3 vector bundles. Here $\Lambda^\pm$ are by definition the eigenspaces of the Hodge duality operator

$$\star : \Lambda^2 \to \Lambda^2,$$

corresponding respectively to the eigenvalue ± 1. Sections of Λ^+ are called *self-dual* 2-forms, whereas sections of Λ^- are called *anti-self-dual* 2-forms. But since the curvature tensor of g may be thought of as a map $\mathcal{R} : \Lambda^2 \to \Lambda^2$, (2) gives us a decomposition [21] of the curvature into primitive pieces

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W^- + \frac{s}{12} \end{array} \right), \tag{3}$$

where the *self-dual* and *anti-self-dual Weyl curvatures* $W^\pm$ are trace-free as endomorphisms of $\Lambda^\pm$. The scalar curvature s is understood here to act by scalar multiplication. On the other hand, $\overset{\circ}{r}$ represents the trace-free Ricci curvature $r - \frac{s}{4}g$, and so vanishes iff g is Einstein.

This last fact has a simple but crucial consequence.

Lemma 1 *Let (M, g) be an oriented Einstein 4-manifold. If the sectional curvature of g is non-negative, then*

$$\frac{s}{\sqrt{6}} \geq |W^+| + |W^-| \tag{4}$$

at each point of M.

Proof. Every 2-form φ on M can be uniquely written as $\varphi = \varphi^+ + \varphi^-$, where $\varphi^\pm \in \Lambda^\pm$. Now a 2-form is expressible as a simple wedge product of 1-forms iff $\varphi \wedge \varphi = 0$. But this condition can be rewritten as $|\varphi^+|^2 - |\varphi^-|^2 = 0$. Thus the sectional curvature of g is non-negative iff the curvature operator $\mathcal{R} : \Lambda^2 \to \Lambda^2$ satisfies

$$\langle \varphi^+ + \varphi^-, \mathcal{R}(\varphi^+ + \varphi^-) \rangle \geq 0$$

for all *unit-length* self-dual 2-forms φ^+ and all *unit-length* anti-self-dual 2-forms φ^-. But for an Einstein manifold, (3) tells us that this can be rewritten as

$$\frac{s}{6} + \lambda_+ + \lambda_- \geq 0 \tag{5}$$

where, for each $x \in M$, $\lambda_\pm(x) \leq 0$ is by definition the smallest eigenvalue of the trace-free endomorphism $W^\pm_x : \Lambda^\pm_x \to \Lambda^\pm_x$.

The claim will thus follow immediately from (5) if we can show that

$$|\lambda_\pm| \geq \frac{1}{\sqrt{6}}|W^\pm|.$$

To see, this, let $\lambda_+ \leq \mu_+ \leq \nu_+$ be the eigenvalues of W^+. Thus

$$|W^+|^2 = \lambda_+^2 + \mu_+^2 + \nu_+^2.$$

But since W^+ is trace-free, $\lambda_+ + \mu_+ + \nu_+ = 0$, and hence

$$\begin{aligned} |W^+|^2 &= \lambda_+^2 + \mu_+^2 + \nu_+^2 + (\lambda_+ - \mu_+ - \nu_+)(\lambda_+ + \mu_+ + \nu_+) \\ &= 2\left[\lambda_+^2 - \mu_+\nu_+\right]. \end{aligned}$$

If $\mu_+ \geq 0$, this last expression is less than $2|\lambda_+|^2$. Otherwise, $\lambda_+ \leq \mu_+ < 0$, $0 < \nu_+ \leq 2|\lambda_+|$, and hence $|W^+|^2 \leq 6|\lambda_+|^2$. Thus $|\lambda_+| \geq \frac{1}{\sqrt{6}}|W^+|$. Since $|\lambda_-| \geq \frac{1}{\sqrt{6}}|W^-|$ by the same argument, we are done. ∎

The curvatures $W^\pm$, $\mathring{r}$, and s correspond to different irreducible representation of $SO(4)$, so the only invariant quadratic polynomials in the curvature of an oriented 4-manifold are linear combinations of s^2, $|\mathring{r}|^2$, $|W_+|^2$ and $|W_-|^2$. This observation can be applied, in particular, to simplify the integrands [2, 14] of the 4-dimensional Chern-Gauss-Bonnet

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left[|W_+|^2 + |W_-|^2 + \frac{s^2}{24} - \frac{|\mathring{r}|^2}{2}\right] d\mu \tag{6}$$

and Hirzebruch signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left[|W_+|^2 - |W_-|^2\right] d\mu \tag{7}$$

formulæ. Here the curvatures, norms $|\cdot|$, and volume form $d\mu$ are, of course, those of any given Riemannian metric g on M.

Applying Lemma 1 now gives us an elementary but useful result:

Lemma 2 *Let (M, g) be a compact 4-dimensional Einstein manifold of non-negative sectional curvature. If g is not flat, then*

$$\chi(M) < \frac{5}{8\pi^2}\int_M \frac{s_g^2}{24} d\mu_g. \tag{8}$$

Proof. Because g is not flat, and the sectional curvature is non-negative, our Einstein metric g must have positive scalar curvature, and hence positive Ricci curvature. Myers' Theorem [18] thus forces M to have finite fundamental group, so that, in particular, $b_1(M) = 0$.

By passing to a double cover if necessary, we may assume that M is orientable. Let us choose to orient M so that $\tau(M) \geq 0$.

Lemma 1 now tells us that

$$|W_g^+|^2 + |W_g^-|^2 \leq \left(|W_g^+| + |W_g^-|\right)^2 \leq \frac{s_g^2}{6},$$

so that

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left[|W_g^+|^2 + |W_g^-|^2 + \frac{s_g^2}{24}\right] d\mu_g \leq 5 \cdot \frac{1}{8\pi^2}\int_M \frac{s_g^2}{24}.$$

If equality were to hold, we would have $|W_g^+|\,|W_g^-| \equiv 0$ and $|W_g^+| + |W_g^-| \equiv \frac{s}{\sqrt{6}}$. But any Einstein metric g is real-analytic [8] in harmonic coordinates. With our orientation conventions, we must therefore have $|W_g^-| \equiv 0$. Our Gauss-Bonnet formulæ (6–7) then tell us that $\chi(M) = \frac{15}{8}\tau(M)$. However, the Weitzenböck formula for harmonic 2-forms implies [5] that a self-dual 4-manifold with $s > 0$ has $b_- = 0$. Since we also have $b_1(M) = 0$, it follows that $\chi(M) = 2 + \tau(M)$. Solving for the signature, we find that $\tau(M) = \frac{16}{7}$. As this is of course a contradiction, it follows that the inequality is always strict. ∎

This has an important consequence:

Lemma 3 *Let (M, g) be a compact 4-dimensional Einstein manifold of non-negative sectional curvature. Then $\chi(M) \leq 9$.*

Proof. We may assume that g has positive Ricci curvature, since otherwise the Euler characteristic would vanish. By rescaling, we can thus arrange for our Einstein metric to have Ricci tensor $r = 3g$. Bishop's inequality [4] then asserts that the total volume of (M, g) is less than or equal to that of the 4-sphere with its standard metric g_1. Since both g and g_1 have $s = 12$, Lemma 2 now asserts that

$$\chi(M) < \frac{5}{8\pi^2}\int_M \frac{s_g^2}{24} d\mu_g \leq \frac{5}{8\pi^2}\int_{S^4} \frac{s_{g_1}^2}{24} d\mu_{g_1} = 5\chi(S^4) = 10.$$

Since the Euler characteristic is an integer, it follows that $\chi(M) \leq 9$. ∎

3 L^2 Curvature Estimates

The key observations of §2 were basically point-wise in character. We now turn to some results of a fundamentally global nature, beginning with a simplified proof of a surprising fact discovered in [12].

Lemma 4 *Suppose (M,g) is a compact oriented Einstein 4-manifold of positive scalar curvature. Then either $W^+ \equiv 0$, or else there is a smooth, conformally related metric $\hat{g} = u^2 g$ such that*

$$\int_M \left[s_{\hat{g}} - 2\sqrt{6}|W^+_{\hat{g}}|_{\hat{g}}\right] d\mu_{\hat{g}} \leq 0.$$

Moreover, one can either arrange for the inequality to be strict, or for the metric $\hat{g}$ to be locally Kähler.

Proof. For each metric $\hat{g}$ on our oriented 4-manifold M, let us consider the quantity $\mathfrak{S}_{\hat{g}}$ defined by

$$\mathfrak{S}_{\hat{g}} = s_{\hat{g}} - 2\sqrt{6}|W^+_{\hat{g}}|_{\hat{g}}.$$

Under conformal rescaling, this curvature function behaves very much like the usual scalar curvature s. Indeed, if $\hat{g} = u^2 g$, where u is a smooth positive function, then

$$\mathfrak{S}_{\hat{g}} = u^{-3} \Diamond u, \tag{9}$$

where, in terms of the (positive) Laplace-Beltrami operator $\Delta = d^*d = -\text{div grad}$, the linear elliptic operator $\Diamond = \Diamond_g$ is defined by

$$\Diamond = 6\Delta_g + \mathfrak{S}_g.$$

Since $d\mu_{\hat{g}} = u^4 d\mu_g$, we thus have

$$\int_M \left[s_{\hat{g}} - 2\sqrt{6}|W^+_{\hat{g}}|_{\hat{g}}\right] d\mu_{\hat{g}} = \int_M \mathfrak{S}_{\hat{g}} d\mu_{\hat{g}} = \int_M (u\Diamond u) d\mu_g.$$

The above generalities apply to any conformally related pair of metrics. But in the present case, the given metric g is assumed to be Einstein. The second Bianchi identity therefore tells us that its self-dual Weyl curvature is harmonic, in the sense that

$$\nabla^a W^+_{abcd} = 0. \tag{10}$$

In spinor terms, this says that $\nabla W^+ \in \mathbb{S}_- \otimes \bigodot^5 \mathbb{S}_+$. Now suppose $U \in \bigodot^4 \mathbb{S}_+$ and $v \in \mathbb{S}_- \otimes \mathbb{S}_+ = \mathbb{C} \otimes TM$ are real elements, and let $(v \otimes U)^{\|}$ denote the orthogonal projection of $v \otimes U$ to $\mathbb{S}_- \otimes \bigodot^5 \mathbb{S}_+$. Using the notational conventions of [20], we then have

$$\begin{aligned}\left[(v \otimes U)^{\|}\right]_{A'ABCDE} &= v_{A'(A}U_{BCDE)} \\ &= \frac{1}{5}\left[v_{A'A}U_{BCDE} + v_{A'B}U_{ACDE}\right. \\ &\qquad \left. + v_{A'C}U_{ABDE} + v_{A'D}U_{ABCE} + v_{A'E}U_{ABCD}\right],\end{aligned}$$

so that

$$\begin{aligned}
|(v\otimes U)^{\parallel}|^2 &= v^{A'A}U^{BCDE}v_{A'(A}U_{BCDE)}\\
&= \frac{1}{5}v^{A'A}U^{BCDE}\left[v_{A'A}U_{BCDE}+v_{A'B}U_{ACDE}+v_{A'C}U_{ABDE}\right.\\
&\qquad\left.+v_{A'D}U_{ABCE}+v_{A'E}U_{ABCD}\right]\\
&= \frac{|v|^2}{5}U^{BCDE}\left[U_{BCDE}+\frac{1}{2}\varepsilon_B{}^AU_{ACDE}+\frac{1}{2}\varepsilon_C{}^AU_{ABDE}+\right.\\
&\qquad\left.+\frac{1}{2}\varepsilon_D{}^AU_{ABCE}+\frac{1}{2}\varepsilon_E{}^AU_{ABCD}\right]\\
&= \frac{3}{5}|v|^2|U|^2.
\end{aligned}$$

The Cauchy-Schwarz inequality therefore predicts that

$$\langle v\otimes U,\nabla W^+\rangle \leq \sqrt{\frac{3}{5}}|v|\;|U|\;|\nabla W^+|.$$

Away from the zeroes of W^+, setting $U=W^+$ thus yields

$$|W^+|\;\nabla_v|W^+| = \langle v\otimes W^+,\nabla W^+\rangle \leq \sqrt{\frac{3}{5}}\;|v|\;|W^+|\;|\nabla W^+|,$$

giving us the Kato inequality

$$|\nabla W^+| \geq \sqrt{\frac{5}{3}}\;\big|\nabla|W^+|\big|. \tag{11}$$

On the other hand, Derdziński [7] observed that equation (10) also implies the Weitzenböck formula

$$0=\frac{1}{2}\Delta|W^+|^2+|\nabla W^+|^2+\frac{s}{2}|W^+|^2-18\det W^+ \tag{12}$$

where Δ is again the positive Laplacian and $\det W^+$ is the determinant of the bundle endomorphism $W^+:\Lambda^+\to\Lambda^+$; cf. [20, equation (6.8.40)]. In conjunction with the (sharp) algebraic inequality $3\sqrt{6}\det W^+\leq|W^+|^3$, equations (11–12) imply that the non-negative function $u_0=|W^+|^{1/3}$ satisfies

$$0\geq \Diamond u_0 \tag{13}$$

in the classical sense, except at the locus $u_0=0$, where it presumably fails to be smooth. Now, for each $\epsilon>0$, let $f_\epsilon:[0,\infty)\to(0,\infty)$ be a smooth positive function which is constant on $[0,\epsilon/2]$, satisfies $f_\epsilon(x)=x$ for $x>\epsilon$, and has non-negative second derivative everywhere. We may then consider the smooth positive function $u_\epsilon=f_\epsilon\circ u_0$, and the metric $g_\epsilon=u_\epsilon^2g$. Let M_ϵ be the set where $u_0<\epsilon$. Then

$$\begin{aligned}
\int_M \mathfrak{S}_{g_\epsilon}d\mu_{g_\epsilon} &= \int_M(u_\epsilon\Diamond u_\epsilon)d\mu_g\\
&\leq C\epsilon^2\mathrm{Vol}(M_\epsilon)+\int_{M-M_\epsilon}(u_0\Diamond u_0)d\mu_g,
\end{aligned}$$

where C is any positive upper bound for $\mathfrak{S}_g$.

Now assume that $W^+ \not\equiv 0$. Since g is real-analytic [8] in harmonic coordinates, so is $u_0^6 = |W^+|^2$, and hence $\mathrm{Vol}(M_\epsilon) \longrightarrow 0$ as $\epsilon \to 0$. Thus $\int_M \mathfrak{S}_{g_\epsilon} d\mu_{g_\epsilon}$ is negative for small ϵ unless equality holds in (13).

In the latter case, however, we have $|W^+|^3 \equiv 3\sqrt{6} \det W^+$, and hence W^+ has at most 2 distinct eigenvalues at each point. Derdziński's theorem [7] thus asserts that $W^+ \neq 0$, and that $\hat{g} = u_0^2 g$ is locally Kähler — i.e. becomes Kähler after possibly pulling back to a double cover of M. ∎

This implies [12] a remarkable "gap theorem" for W^+:

Theorem 1 *Let (M, g) be a compact oriented Einstein 4-manifold with $s > 0$ and $W^+ \not\equiv 0$. Then*

$$\int_M |W_g^+|_g^2 d\mu_g \geq \int_M \frac{s_g^2}{24} d\mu_g,$$

with equality iff $\nabla W^+ \equiv 0$.

Proof. A fundamental result of Obata [19] implies that any Einstein metric is a Yamabe minimizer; moreover, such a metric is always the *unique* Yamabe minimizer, modulo homotheties and — on the round sphere — global conformal transformations. Thus, if $\hat{g} = u^2 g$ is any conformal rescaling of our Einstein metric g, we have

$$\frac{\int_M s_g d\mu_g}{\sqrt{\int_M d\mu_g}} \leq \frac{\int_M s_{\hat{g}} d\mu_{\hat{g}}}{\sqrt{\int_M d\mu_{\hat{g}}}}.$$

However, assuming that $W^+ \not\equiv 0$, Lemma 4 tells us that u can be chosen so that

$$\begin{aligned} \int s_{\hat{g}} d\mu_{\hat{g}} &\leq 2\sqrt{6} \int |W_{\hat{g}}^+| d\mu_{\hat{g}} \\ &\leq \left(24 \int |W_{\hat{g}}^+|^2 d\mu_{\hat{g}}\right)^{\frac{1}{2}} \left(\int d\mu_{\hat{g}}\right)^{\frac{1}{2}}. \end{aligned}$$

Since s_g is constant, and because the L^2 norm of W^+ is conformally invariant, it therefore follows that

$$\begin{aligned} \left(\int_M s_g^2 d\mu_g\right)^{1/2} &= \frac{\int_M s_g d\mu_g}{\sqrt{\int_M d\mu_g}} \\ &\leq \frac{\int_M s_{\hat{g}} d\mu_{\hat{g}}}{\sqrt{\int_M d\mu_{\hat{g}}}} \\ &\leq \left(24 \int |W_{\hat{g}}^+|^2 d\mu_{\hat{g}}\right)^{\frac{1}{2}} \\ &= \left(24 \int |W_g^+|^2 d\mu_g\right)^{\frac{1}{2}}. \end{aligned}$$

Moreover, equality can occur only if $\hat{g}$ is both locally Kähler and isometric to a constant times g. The latter, of course, happen iff g is itself locally Kähler. But since $s \neq 0$ is constant and $W^+ \not\equiv 0$, the latter is equivalent to requiring that $\nabla W^+ \equiv 0$. ∎

Reading this in the mirror, we have:

Corollary 1 *Let (M,g) be an oriented compact Einstein 4-manifold with $s > 0$ and $W^- \not\equiv 0$. Then*

$$\begin{aligned} \int_M |W_g^-|^2 d\mu_g &\geq \int \frac{s_g^2}{24} d\mu_g \ , \textit{ and} \qquad (14) \\ \frac{2\chi - 3\tau}{3}(M) &\geq \frac{1}{4\pi^2}\int_M \frac{s_g^2}{24} d\mu_g \ . \qquad (15) \end{aligned}$$

Moreover, both these inequalities are strict unless $\nabla W^- \equiv 0$.

Proof. Reversing the orientation of M interchanges W^+ and W^-. Applying this observation to Theorem 1 immediately yields (14). But this and the Gauss-Bonnet-type formulæ (6–7) then tell us that

$$\begin{aligned} (2\chi - 3\tau)(M) &= \frac{1}{4\pi^2}\int_M \left[2|W_g^-|^2 + \frac{s^2}{24}\right] d\mu_g \\ &\geq \frac{3}{4\pi^2}\int_M \frac{s_g^2}{24} d\mu_g \ , \end{aligned}$$

thus proving (15). ∎

4 The Main Theorems

Combining the estimates of §§2–3 to now allows us to prove our main inequality:

Theorem B *Let (M,g) be a smooth compact oriented Einstein 4-manifold with non-negative sectional curvature. Assume, moreover, that g is neither self-dual nor anti-self-dual. Then the Euler characteristic χ and the signature τ of M satisfy*

$$9 \geq \chi > \frac{15}{4}|\tau|.$$

Proof. Combining (8) and (15), we have

$$\frac{2}{3}\chi - \tau \geq \frac{1}{4\pi^2}\int \frac{s^2}{24} d\mu > \frac{2}{5}\chi,$$

or in other words $\chi > \frac{15}{4}\tau$. Reversing the orientation of M, we also have $\chi > -\frac{15}{4}\tau$. Since Lemma 3 tells us that $\chi \leq 9$, we are therefore done. ∎

Our other main result now follows:

Theorem A *Let M be a smooth compact oriented 4-manifold with (strictly) positive intersection form, and suppose that g is an Einstein metric on M which has non-negative sectional curvature. Then (M, g) is homothetically isometric to $\mathbb{CP}_2$, equipped with its standard Fubini-Study metric.*

Proof. By assumption, $b_+ > 0$ and $b_- = 0$, so $\chi(M) = 2 + b_+$ and $\tau(M) = b_+ > 0$. Hence

$$\frac{15}{4}\tau = \frac{15}{4}b_+ \geq \frac{9}{4} + b_+ > 2 + b_+ = \chi.$$

Theorem B therefore insists that our Einstein metric g of non-negative sectional curvature must satisfy either $W^+ \equiv 0$ or $W^- \equiv 0$. But since $\tau > 0$, the signature formula (7) thus forces $W^- \equiv 0$ and $W^+ \not\equiv 0$. In particular, g is not flat, and, since it has non-negative sectional curvature, its scalar curvature must somewhere be positive. Thus (M, g) is a non-conformally-flat, self-dual Einstein 4-manifold of positive scalar curvature. A celebrated result of Hitchin [2, Theorem 13.30], originally discovered via twistor methods [14, 10], therefore tells us that (M, g) must, up to isometry, be $\mathbb{CP}_2$, equipped with a constant multiple of the Fubini-Study metric. ∎

5 Einstein Constants

Given a smooth compact n-manifold M, for what values of λ do Einstein's equations (1) have a unit-volume solution? The collection of all such λ is called [2] the set of *Einstein constants* for M, and constitutes an interesting smooth invariant of the manifold. Assuming that $n > 2$, this set is just the collection of critical values of the Riemannian functional $\mathcal{S}/n$, where

$$\mathcal{S}(g) = \frac{\int_M s_g d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}},$$

since a metric is a critical point of $\mathcal{S}$ iff it is Einstein.

For $\mathbb{CP}_2$, it is a relatively recent result [17, 13] that the set of Einstein constants has a maximal element, represented by, and only by, the Fubini-Study metric. This provides one new explanation for the rigidity [16] of the Fubini-Study metric. Theorem A of course provides an ostensibly different explanation of this phenomenon, since the positivity of sectional curvatures is an open condition in the C^2 topology. However, the proof of Theorem A tells us more. While it does not rule out the existence of an Einstein metric on $\mathbb{CP}_2$ with sectional curvatures of varying sign, it does assert that the maximal element in the set of Einstein constants for $\mathbb{CP}_2$ is *isolated.* Moreover, the form of this assertion is actually quantitative.

Theorem 2 *Suppose that $\mathbb{CP}_2$ admits an Einstein metric g which is not isometric to any multiple of the Fubini-Study metric g_0. Then*

$$\mathcal{S}(g) < \frac{1}{\sqrt{3}}\mathcal{S}(g_0),$$

where $\mathcal{S}(g_0) = 12\pi\sqrt{2}$.

Proof. We may assume that the scalar curvature of g is positive, since the result is otherwise trivial. By Hitchin's theorem [14, 17], we therefore know that $W_g^- \not\equiv 0$. For $\mathbb{CP}_2, \chi = 3$ and $\tau = 1$. Thus (15) tells us that

$$\frac{1}{4\pi^2}\int_{\mathbb{CP}_2} \frac{s_g^2}{24} d\mu_g \leq \frac{2\chi - 3\tau}{3}(\mathbb{CP}_2) = 1.$$

Moreover, the inequality is strict, since Corollary 1 would otherwise predict that the universal cover of $(\mathbb{CP}_2, g)$ is *reverse-oriented* Kähler, contradicting the fact that $\mathbb{CP}_2$ is simply connected and has $b_- = 0$. Thus

$$\mathcal{S}(g) = \left(\int_{\mathbb{CP}_2} s_g^2 d\mu_g\right)^{1/2} < \sqrt{4\pi^2 \cdot 24} = 4\pi\sqrt{6}.$$

Since $\mathcal{S}(g_0) = 12\pi\sqrt{2}$, the result follows. ∎

While Bishop's inequality immediately implies that the set of Einstein constants for S^4 has a maximal element, represented by, and only by, the "round" metric, the S^4-analog of Theorem 2 was only recently proved [12]. Unfortunately, however, an S^4-analog of Theorem A remains out of our reach. The best our present techniques can offer in this direction is

Theorem 3 *Let g be an Einstein metric of non-negative sectional curvature on S^4. If g is not isometric to some multiple of the standard metric g_1, then*

$$\frac{1}{\sqrt{5}}\mathcal{S}(g_1) < \mathcal{S}(g) < \frac{1}{\sqrt{3}}\mathcal{S}(g_1),$$

where $\mathcal{S}(g_1) = 8\pi\sqrt{6}$.

Proof. If g is Einstein but has non-constant curvature, it cannot be conformally flat. By (15), we thus have

$$\int_{S^4} s_g^2 d\mu_g \leq 4\pi^2 \cdot 24\frac{2\chi - 3\tau}{3}(S^4) = 2^7\pi^2$$

and the inequality is in fact strict, since S^4 cannot admit a (reverse-oriented) Kähler metric. Thus

$$\mathcal{S}(g) = \left(\int_{S^4} s_g^2 d\mu_g\right)^{1/2} < 8\pi\sqrt{2}.$$

On the other hand, if g has non-negative sectional curvature, (8) tells us that

$$\frac{5}{8\pi^2}\int \frac{s_g^2}{24}d\mu_g > \chi(S^4) = 2,$$

so that

$$\mathcal{S}(g) = \left(\int_{S^4} s_g^2 d\mu_g\right)^{1/2} > \sqrt{\frac{2^7 \cdot 3\pi^2}{5}} = 8\pi\sqrt{\frac{6}{5}}.$$

Since $\mathcal{S}(g_1) = 8\pi\sqrt{6}$, the assertion follows. ∎

References

[1] M. Anderson, *The L^2 Structure of Moduli Spaces of Einstein Metrics on 4-Manifolds*, **Geom. Func. An. 2** (1992) 29–89.

[2] A. Besse, **Einstein Manifolds**, Springer-Verlag, 1987.

[3] L. Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume I*, **Math. Ann. 70** (1911) 297–336; *II*, **72** (1912) 400–412.

[4] R.L. Bishop and R.J. Crittenden, **Geometry of Manifolds**, Academic Press, 1964.

[5] J.P. Bourgignon, *Les Variétés de Dimension 4 à Signature Non Nulle dont la Courbure Est Harmonique Sont d'Einstein*, **Inv. Math. 63** (1981) 263–286.

[6] J. Cheeger, *Finiteness Theorems for Riemannian Manifolds*, **Amer. J. Math. 92** (1970) 61–74.

[7] A. Derdziński, *Self-dual Kähler Manifolds and Einstein Manifolds of Dimension Four*, **Comp. Math. 49** (1983) 405–433.

[8] D. DeTurck and J. Kazdan, *Some Regularity Theorems in Riemannian Geometry*, **Ann. Sci. Ec. Norm. Sup. 14** (1981) 249–260.

[9] M. Freedman, *On the Topology of 4-Manifolds*, **J. Diff. Geom. 17** (1982) 357–454.

[10] T. Friedrich and H. Kurke, *Compact Four-Dimensional Self-Dual Einstein Manifolds with Positive Scalar Curvature*, **Math. Nachr. 106** (1982) 271–299.

[11] M. Gursky, *The Weyl Functional, DeRham Cohomology, and Kähler-Einstein Metrics*, preprint, 1996.

[12] M. Gursky, *Four-Manifolds with $\delta W^+ = 0$ and Einstein Constants on the Sphere*, preprint, 1997.

[13] M. Gursky and C. LeBrun, *Yamabe Invariants and Spin^c Structures*, **Geom. Func. An.** *to appear*.

[14] N.J. Hitchin, *On Compact Four-Dimensional Einstein Manifolds*, **J. Diff. Geom. 9** (1974) 435–442.

[15] N.J. Hitchin, *Kählerian Twistor Spaces*, **Proc. Lond. Math. Soc. 43** (1981) 133–150.

[16] N. Koiso, *Rigidity and Stability of Einstein Metrics. The Case of Compact Symmetric Spaces*, **Osaka J. Math. 17** (1980) 51–73.

[17] C. LeBrun, *Yamabe Constants and the Perturbed Seiberg-Witten Equations*, **Comm. An. Geom. 5** (1997) 535–553.

[18] S.B. Myers, *Riemannian Manifolds with Positive Mean Curvature*, **Duke Math. J. 8** (1941) 401–404.

[19] M. Obata, *The Conjectures on Conformal Transformations of Riemannian Manifolds*, **J. Diff. Geom. 6** (1971) 247–258.

[20] R. Penrose and W. Rindler, **Spinors and Space-Time. vol. 2: Spinor and Twistor Methods in Space-Time Geometry**, Cambridge University Press, 1986.

[21] I. M. Singer and J. A. Thorpe, *The Curvature of* 4*-dimensional Einstein Spaces*, **Global Analysis (Papers in Honor of K. Kodaira)**, pp. 355–365, Univ. Tokyo Press, Tokyo, 1969.

[22] J.L. Synge, *On the Connectivity of Spaces of Positive Curvature*, **Quart. J. Math. 7** (1936) 316–320.

[23] J.A. Thorpe, *Some Remarks on the Gauss-Bonnet Formula*, **J. Math. Mech. 18** (1969) 779–786.

[24] G. Tian, *On Calabi's Conjecture for Complex Surfaces with Positive First Chern Class,* **Inv. Math. 101** (1990) 101-172.