*Trends in Mathematical Physics, V. Alexiades and G. Siopsis, eds.*


# EINSTEIN METRICS AND THE YAMABE PROBLEM

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**Abstract**

Which smooth compact $n$-manifolds admit Riemannian metrics of constant Ricci curvature? A direct variational approach sheds some interesting light on this problem, but by no means answers it. This article surveys some recent results concerning both Einstein metrics and the associated variational problem, with the particular aim of highlighting the striking manner in which the 4-dimensional case differs from the case of dimensions $\geq 5$.

## 1 The Total Scalar Curvature

For the purposes of this article, the term *Einstein metric* will mean a Riemannian metric of constant Ricci curvature. In physics jargon, that's to say that an Einstein metric $g$ is a Euclidean-signature solution of the Einstein vacuum equations

$$r = \lambda g,$$

for some (unspecified) value of the cosmological constant $\lambda \in \mathbb{R}$. Here $r$ denotes the Ricci tensor

$$r_{ab} = R^c{}_{acb}$$

of the (positive definite) Riemannian metric $g$. Notice that the scalar curvature $s = r_a{}^a = R^{ab}{}_{ab}$ of $g$ is related to the the constant $\lambda$ by

$$s = n\lambda,$$

where $n$ is the dimension of the manifold on which $g$ is defined.

Einstein metrics are the solutions of a natural variational problem which can be traced back to Hilbert [1]. Choose a smooth compact oriented manifold $M$ of dimension $n > 2$, and let $\mathcal{M} = \mathcal{M}_M$ denote the space of $C^\infty$ Riemannian metrics on $M$. We may then consider the *total scalar curvature* functional, or *Einstein-Hilbert action*

$$\mathcal{S} : \mathcal{M} \to \mathbb{R},$$

given by

$$\mathcal{S}(g) = \int_M s_g \mu_g ,$$

where $s_g$ and $\mu_g = \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$ are respectively the scalar curvature and volume $n$-form of the metric $g$. If $h_{ab}$ is any symmetric tensor field, one then has

$$\frac{d}{dt}\mathcal{S}(g+th)|_{t=0} = -\int_M \langle h, r - \frac{s}{2}g\rangle \mu_g \ ,$$

so that, as discovered by Hilbert, the critical points of $\mathcal{S}$ are then just the *Ricci-flat* metrics. If we restrict the functional to the submanifold $\mathcal{M}_1 \subset \mathcal{M}$ of metrics of total volume 1, however, the Lagrange-multiplier method immediately predicts that the critical points of $\mathcal{S}|_{\mathcal{M}_1}$ are precisely the unit-volume Einstein metrics. Without singling out the metrics of unit volume, we can accomplish much the same thing by instead considering the 'renormalized' functional

$$\mathfrak{S} = V^{(2-n)/n}\mathcal{S},$$

where $V(g) = \int_M \mu_g$ is the total volume of $M$ with respect to $g$; the power of $V$ has been chosen precisely so as to make $\mathfrak{S}$ invariant under rescalings $g \mapsto cg$, for $c$ any positive constant. One then has

$$\frac{d}{dt}\mathfrak{S}(g+th)|_{t=0} = -V^{(2-n)/n}\int_M \left\langle h, r - \left(\frac{s_0}{n} + \frac{s-s_0}{2}\right) g \right\rangle \mu_g,$$

where $s_0 = \mathcal{S}(g)/V(g)$ is the average value of the scalar curvature, and, since $n > 2$, this expression vanishes for all $h$ iff $s \equiv s_0$ and $r \equiv \frac{s_0}{n}g$. The critical points of $\mathfrak{S}$ thus coincide with the Einstein metrics on $M$.

Naively, one might therefore be tempted to try to construct Einstein metrics on $M$ by minimizing or maximizing $\mathfrak{S}$. Unfortunately, however, this functional is neither bounded above nor below! Nonetheless, there is a natural way of circumventing this difficulty, discovered by Hidehiko Yamabe [2] shortly before his premature death in 1960. Yamabe realized that the restriction of $\mathfrak{S}$ to any *conformal class*

$$\gamma = [g] = \{ug \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}$$

of metrics *is* always bounded below. For each conformal class, one can thus define an associated number $Y(\gamma)$, called the *Yamabe constant* of the class, by

$$Y(\gamma) = \inf_{g\in\gamma} \mathfrak{S}(g).$$

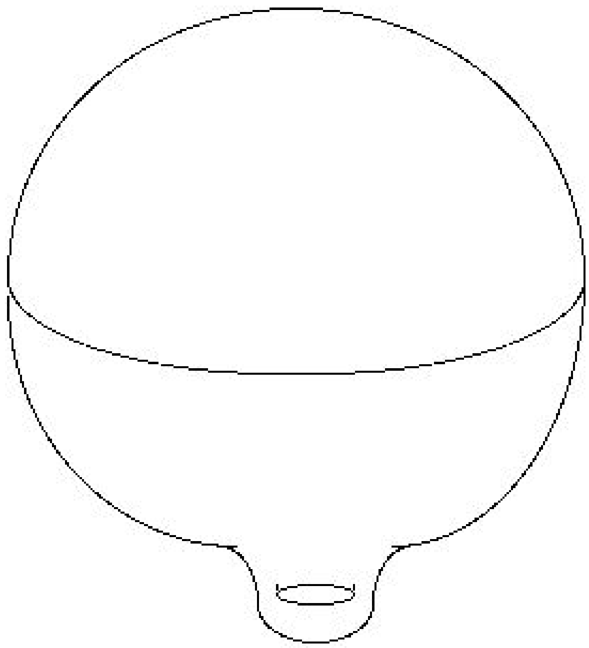
Yamabe went on to give a flawed but highly original argument purporting to show that the infimum $Y(\gamma)$ is always achieved. While this assertion and Yamabe's strategy for proving it are quite correct, his proof overlooks a fundamental analytic difficulty concerning Sobolev inequalities of critical exponent. This issue is a profound one, and Yamabe's proof was ultimately only corrected in stages, first by Trudinger [3], then by Aubin [4], and finally by Schoen [5]. For a clear, self-contained account of the complete proof, see [6].

Thus, any conformal class $\gamma$ on our manifold $M$ contains a metric $g$ which minimizes $\mathfrak{S}$. Such metrics are called *Yamabe metrics.* Yamabe metrics always have constant scalar curvature; and conversely, metrics with $s \equiv$ const $\leq 0$ are always Yamabe metrics. An important result of Obata [7] guarantees that Einstein metrics are always Yamabe metrics, even when $s > 0$.

Aubin's contributions to the subject center on the observation that the Yamabe constant of any conformal class $\gamma$ on any compact $n$-manifold $M$ automatically satisfies

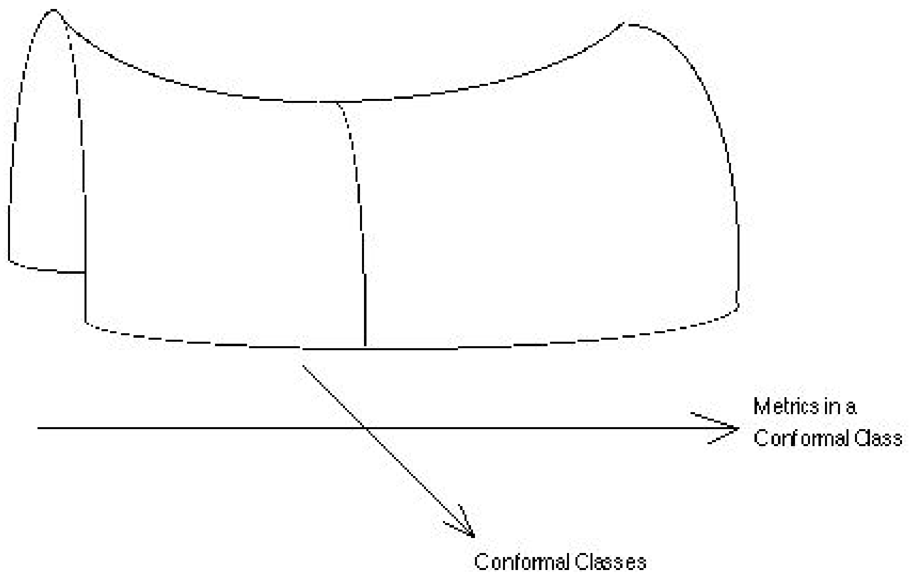
$$Y(\gamma) \leq \mathfrak{S}(S^n, g_0) = \begin{cases} 2\pi n(n-1)\left[\frac{2}{(n-1)(n-3)\cdots 1}\right]^{2/n}, & n \text{ even}, \\ 2\pi n(n-1)\left[\frac{\sqrt{2\pi}}{(n-1)(n-3)\cdots 2}\right]^{2/n}, & n \text{ odd}, \end{cases}$$

where $g_0$ is the usual 'round' metric induced on the $n$-sphere by the standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. To see this, remember that any Riemannian manifold looks like Euclidean space on a microscopic scale, and that $(S^n - \{\text{point}\}, g_0)$ is conformally related to Euclidean space via stereographic projection. Any metric on any manifold may therefore be conformally rescaled so as to make the manifold look like a nearly round, spherical balloon with a tiny, topology-laden gondola attached:



For any $\varepsilon > 0$, this trick constructs metrics $g$ in any conformal class $\gamma$ with $\mathfrak{S}(g) \leq \mathfrak{S}(g_0) + \varepsilon$. The desired inequality follows.

Thus, while $\mathfrak{S}$ is neither bounded above nor below, one might try to find a critical point of the functional by a 'mountain pass' trick. Let $\mathcal{C}_M = \mathcal{M}_M / C^\infty(M, \mathbb{R}^+)$ denote the set of conformal classes of Riemannian metrics on $M$. We have just seen that $\mathfrak{S}$ can be minimized in each conformal class $\gamma \in \mathcal{C}_M$. We now attempt to maximize the restriction of $\mathfrak{S}$ to these Yamabe metrics.




In particular, this idea leads [8,9] to the definition of a real-valued diffeomorphism invariant

$$\mathfrak{Y}(M) = \sup_{\gamma \in \mathcal{C}_M} Y(\gamma) = \sup_{\gamma \in \mathcal{C}_M} \inf_{g \in \gamma} \mathfrak{S}(g),$$

which I will call the *Yamabe invariant* of the smooth compact manifold $M$. Notice that, while there are always sequences of unit-volume Yamabe metrics $g_j$ with $s_{g_j} \nearrow \mathfrak{Y}(M)$, we have no reason to expect that the minimax is actually achieved. Nonetheless, it is worth introducing some terminology to describe the most optimistic situation:

**Definition 1.1** *Let $M$ be a smooth compact manifold, and suppose that $g$ is an Einstein metric on $M$. We will say that $g$ is a* supreme Einstein metric *if*

$$\mathfrak{S}(g) = \mathfrak{Y}(M).$$

For example, Aubin's estimate asserts that the standard Einstein metric $g_0$ on $S^n$ is indeed supreme.

One of the main mathematical reasons for studying Einstein metrics is the desire to find a higher-dimensional analog of the uniformization theorem for Riemann surfaces. That is, given a compact manifold, one would like find a Riemannian metric on it which is fairly unique. Now Einstein metrics are just the critical points of the functional $\mathfrak{S}$, and it might very well be that there are many different critical values. In this case, it is natural to choose the *highest* critical value since this definitely gives one the right answer for the sphere, and, moreover, can be seen [10] to give one the standard metric on compact quotients of locally symmetric spaces of rank 1. The supreme condition, of course, is *even stronger* than this, for it is defined in terms of a comparison involving non-Einstein metrics. The purpose of this article is to explore what this condition means in practice. Here are some of the questions I will try to explore:

- Which compact manifolds admit supreme Einstein metrics?

- Are there manifolds which admit Einstein metrics, but do not admit *supreme* Einstein metrics?
- Is the Yamabe invariant $\mathfrak{Y}(M)$ really computable in practice?

Of course, it should be clear from the outset that it is impossible to find supreme Einstein metrics on many manifolds. For example, in all dimensions $\geq 3$ there are manifolds like $S^{n-1} \times S^1$ which admit metrics of positive scalar curvature, but which have infinite fundamental group. A supreme Einstein metric on such a manifold would have to have positive Ricci curvature — which is of course ruled out by Myers' Theorem. Thus, a direct minimax construction of supreme Einstein metrics could generally hope only to produce, at best, Einstein metrics on related spaces obtained by 'bubbling off' some of the topology. But in any case it seems worthwhile to attempt to catalog the known topological obstructions to the existence of supreme Einstein metrics. In the process, it will become clear that dimension 4 is utterly anomalous for problems involving the scalar curvature.

## 2 Dimension Three

In dimension 3, the Ricci curvature algebraically determines the entire curvature tensor via the formula

$$\star R \star = \frac{s}{2} g - r,$$

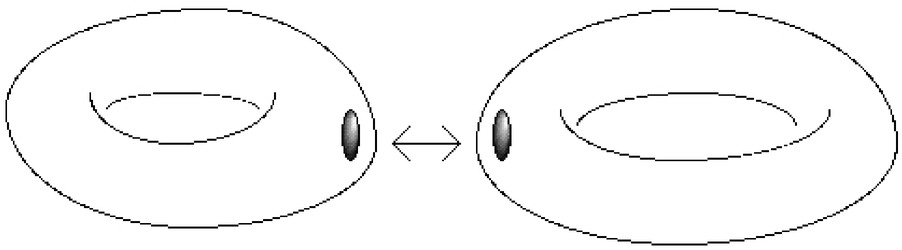
where the Hodge star operator $\star$ has been used to identify $\odot^2 \Lambda^2$ with $\odot^2 \Lambda^1$. Consequently, a Riemannian 3-manifold is Einstein iff it has constant sectional curvature. While this makes 3-dimensional Einstein manifolds 'locally trivial,' in the sense that they are necessarily locally isometric to standard models depending only on the Einstein constant $\lambda$, there are still some global issues worthy of discussion.

### *2.1 Obstructions*

Not every 3-manifold admits an Einstein metric. Indeed, since the a 3-dimensional Einstein manifold $M$ must have constant sectional curvature, its universal cover $\tilde{M}$ must be diffeomorphic to $S^3$ or $\mathbb{R}^3$, which implies that

$$\pi_2(M) = \pi_2(\tilde{M}) = 0.$$

This can be used to rule out not only $S^1 \times S^2$, but also any connected sum $M_1 \# M_2$ where neither $M_1$ nor $M_2$ is a homotopy 3-sphere. Here the connected sum operation $\#$, which is defined for any pair of manifolds of the same dimension, is performed



by deleting a standard ball from each manifold, and then identifying the resulting spherical boundaries via a reflection.

### 2.2 Uniqueness

As we have already observed, the problem of classifying 3-dimensional Einstein manifold reduces to that of classifying Riemannian 3-manifolds of constant sectional curvature $M$. The *sign* of this sectional curvature, and hence of the Einstein constant $\lambda$, is determined by the structure of the fundamental group $\pi_1(M)$. Namely,

$$\begin{aligned}
&\lambda > 0 \Longleftrightarrow |\pi_1(M)| < \infty;\\
&\lambda = 0 \Longleftrightarrow \mathbb{Z}\oplus\mathbb{Z} \subset \pi_1(M);\\
&\lambda < 0 \Longleftrightarrow |\pi_1(M)| = \infty, \text{ but } \mathbb{Z}\oplus\mathbb{Z} \not\subset \pi_1(M).
\end{aligned}$$

The moduli space of Einstein metrics (modulo rescalings and diffeomorphisms) is at most a point in the $\lambda < 0$ case. This is the prototypical case of *Mostow rigidity* [11], which tells us that there is, up to global isometry, at most one hyperbolic metric on any compact manifold of dimension $> 2$. When $\lambda > 0$, the same phenomenon occurs, albeit for somewhat different reasons [12]. When $\lambda = 0$, however, the situation is different, as the $n$-dimensional torus carries a non-trivial moduli space $SL(n,\mathbb{Z})\setminus SL(n,\mathbb{R})/SO(n)$ of geometrically distinct flat metrics of unit volume.

### 2.3 Supreme Einstein Metrics

Some two decades ago, Schoen and Yau showed [13] that the 3-torus $T^3$ cannot admit metrics of positive scalar curvature. (Using a complete different set of ideas, this result was then generalized to arbitrary dimension by Gromov and Lawson [14].) If a compact 3-manifold admits a flat metric, it thus follows that this metric is a supreme Einstein metric, since any flat manifold is finite covered by a torus [15]. It is natural to hope that similar statements can be made about 3-dimensional Einstein manifolds for other values of $\lambda$.

In this direction, Michael Anderson [16,17] is attempting to show that the hyperbolic metric on any compact hyperbolic 3-manifold $M = \mathcal{H}^3/\Gamma$ is in fact supreme — meaning, once again, that it realizes the Yamabe invariant $\mathfrak{Y}(M)$. This would follow from an ambitious program he is now pursuing in an attempt to prove Thurston's geometrization conjectures. His strategy is to first demonstrate the existence of minimizers for the functionals

$$g \mapsto V^{-1/3}\int_M (s_g^2 + \varepsilon|R|^2)\mu_g,$$

and then prove Gromov-Hausdorff convergence of these solutions as $\varepsilon \to 0$. Surprisingly, the most delicate technical issues hinge on uniqueness questions for static black holes! Up-to-date e-prints detailing describing Anderson's current progress can be obtained from his website

http://www.math.sunysb.edu/˜anderson

### 2.4 Yamabe Invariants

As we have just seen, Anderson hopes to prove, in particular, that hyperbolic 3-manifolds always have negative Yamabe invariant. Hyperbolic manifolds, of course, necessarily have large fundamental groups; and one might instead ask about the behavior of the Yamabe invariant for *simply connected manifolds.* But in dimension

3, the celebrated Poincaré conjecture speculates that the only simply connected 3-manifold is the 3-sphere $S^3$, and this most certainly has positive Yamabe invariant! That is, the usual Poincaré conjecture would immediately imply

**Conjecture 2.1 (Weak Poincaré Conjecture)** *Any simply connected compact 3-manifold $M^3$ has positive Yamabe invariant $\mathfrak{Y}(M)$.*

This weak assertion is of independent interest, as will become clear in §4.1.

Of course, the full Poincaré conjecture is equivalent to the assertion that every simply connected 3-manifold admits an Einstein metric. Richard Hamilton's work on heat flow for the Ricci curvature [18] provides a powerful tool for constructing Einstein metrics with $\lambda > 0$ on suitable 3-manifolds, and may well yet offer the best strategy for proving the conjecture.

## 3 Dimension Four

Dimension 4 is the transitional dimension for Einstein's equations. In dimension $n \geq 4$, the curvature tensor of $g$ is no longer determined by the Einstein condition in a point-wise manner, and Einstein metrics are no longer describable in terms of universal local models. On the other hand, a constellation of low-dimensional accidents gives one strong topological information about Einstein 4-manifolds for which there is no apparent high-dimensional analog.

### *3.1 Obstructions*

Suppose that $(M, g)$ is a compact oriented Riemannian 4-manifold. Then the rank-6 bundle of 2-forms on $M$ invariantly decomposes into two rank-3 bundles

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \tag{1}$$

defined as the $(\pm 1)$-eigenspaces of the Hodge star operator

$$\star : \Lambda^2 \to \Lambda^2.$$

Sections of $\Lambda^+$ are called *self-dual 2-forms*, whereas sections of $\Lambda^-$ are called *anti-self-dual 2-forms*. The numbers

$$b_\pm(M) = \dim\{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

are independent of the metric, and depends only on the oriented homeomorphism type of $M^4$. The sum

$$b_2(M) = b_+(M) + b_-(M)$$

is called the *second Betti number* of the manifold, while the difference

$$\tau(M) = b_+(M) - b_-(M)$$

is called the signature. The *Euler characteristic* of $M$ can then be defined as

$$\chi(M) = 2 - 2b_1(M) + b_2(M),$$

where the *first Betti number*

$$b_1(M) = \dim Hom(\pi_1(M), \mathbb{R})$$

counts the number of independent group homomorphisms from the fundamental group to the real numbers.

**Example.** Let $\mathbb{CP}_2$ denote the complex projective plane, with its *standard* orientation, and let $\overline{\mathbb{CP}}_2$ be the same smooth 4-manifold, but equipped with the *opposite* orientation. Then one can construct new manifolds from these by the connected sum operation $\#$, where one removes small standard balls from a pair of manifolds and glues the resulting $S^3$ boundaries together *in a manner consistent with the orientations*, much as a surface of genus 2 can be constructed by gluing together two 2-dimensional tori. Because the invariants $b_\pm$ and $b_1$ behave additively with respect to $\#$, the iterated connected sum

$$k\mathbb{CP}_2\#\ell\overline{\mathbb{CP}}_2 = \underbrace{\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2}_{k}\#\underbrace{\overline{\mathbb{CP}}_2\#\cdots\#\overline{\mathbb{CP}}_2}_{\ell},$$

satisfies $b_+ = k$, $b_- = \ell$, and $b_1 = 0$. In particular, this manifold has $\tau = k - \ell$, and $\chi = 2 + k + \ell$. $\diamondsuit$

Now, for an arbitrary Riemannian metric $g$ on $M$, let us think of the curvature tensor as a linear map $R : \Lambda^2 \to \Lambda^2$. Decomposing the 2-forms as in (1), we have

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W^- + \frac{s}{12} \end{array}\right). \tag{2}$$

Here $W_\pm$ are the trace-free pieces of the appropriate blocks, and are called the self-dual and anti-self-dual Weyl curvatures, respectively. The scalar curvature $s$ is understood to act by scalar multiplication, whereas the trace-free Ricci curvature

$$\overset{\circ}{r} = r - \frac{s}{4}g$$

acts on 2-forms by

$$\varphi_{ab} \mapsto \overset{\circ}{r}_{ac}\,\varphi^c{}_b - \overset{\circ}{r}_{bc}\,\varphi^c{}_a.$$

Now both the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ are actually indices of elliptic operators on $M$, and the index theorem therefore predicts that they can be expressed as curvature integrals. The formula

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left[|W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\overset{\circ}{r}|^2}{2}\right]\mu$$

is usually called the 4-dimensional *Gauss-Bonnet formula*, while its analog

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left[|W^+|^2 - |W^-|^2\right]\mu$$

is closely related to the Hirzebruch signature theorem. Here the curvatures, norms $|\cdot|$, and volume form $\mu$ are all, of course, those of the same arbitrary Riemannian metric $g$ on $M$. In particular,

$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left[2|W_\pm|^2 + \frac{s^2}{24} - \frac{|\overset{\circ}{r}|^2}{2}\right]\mu. \tag{3}$$

Since the above integrand is non-negative for any Einstein metric, we therefore have the following celebrated result of Thorpe [19] and Hitchin [20]:

**Theorem 3.1 (Hitchin-Thorpe Inequality)** *If the smooth compact oriented 4-manifold $M$ admits an Einstein metric $g$, then*

$$2\chi(M) \geq 3|\tau(M)|,$$

*with equality iff the $g$-induced connection on one of the bundles $\Lambda^\pm$ is flat.*

In particular, there are [20] *simply connected* 4-manifolds which do not admit Einstein metrics.

**Example.** Since the 4-manifold $k\mathbb{CP}_2\#\ell\overline{\mathbb{CP}}_2$ has $\chi = 2+k+\ell$ and $\tau = k-\ell$, it cannot admit an Einstein metric if $\ell > 4+5k$ or $k > 4+5\ell$. ♢

Now let us define a diffeomorphism invariant of $M$ by

$$\mathcal{I}(M) = \inf_g \int_M s_g^2 \mu_g,$$

where the infimum is taken over *all Riemannian metrics* on $M$. Referring back to (3), it is then immediate that the existence of an Einstein metric on $M$ would force

$$(2\chi \pm 3\tau)(M) \geq \frac{1}{96\pi^2}\mathcal{I}(M).$$

Thus, we have an apparent improvement on the Hitchin-Thorpe inequality — but devoid of content until something can be said about the invariant $\mathcal{I}$!

Intriguingly, $\mathcal{I}$ is determined by the Yamabe invariant of $M$, according to the formula [21,22,23]

$$\mathcal{I}(M) = \begin{cases} 0, & \text{if } \mathfrak{Y}(M) \geq 0 \\ |\mathfrak{Y}(M)|^2, & \text{if } \mathfrak{Y}(M) \leq 0. \end{cases}$$

This is easily proved by considering one conformal class at a time. The good news is thus that the invariant $\mathcal{I}$ is related to another invariant we've already agreed plays a natural rôle in the theory of Einstein metrics. The bad news is that this still doesn't help calculate anything!

Seiberg-Witten theory, however, provides us with just the sort of information we need. This theory captures certain properties of a 4-manifold $M$ which are invariant under diffeomorphisms, but not generally under homeomorphisms. The relevant invariants are calculated by counting solutions to a non-linear version of the Dirac equation.

While it is not possible to define spinors on an arbitrary 4-manifold, it *is* always possible to define rank-2 complex vector bundles $V_\pm \to M$ which formally satisfy

$$V_\pm = \mathbb{S}_\pm \otimes L^{1/2},$$

where $\mathbb{S}_\pm$ are the left- and right-handed spin bundles, and where $L = \wedge^2 V_\pm$ is some Hermitian complex line bundle on $M$. After possibly passing to a finite cover of $M$, the possible choices of such $V_\pm$ become determined by knowing the de Rham class $c_1(L) \in H^2_{DR}(M)$, represented by $\frac{i}{2\pi}$ times the curvature 2-form $F$ of any Hermitian connection on $L$. One then says that $c_1(L)$ is a *spin$^c$ structure* on $M$. If $c_1(L)$ and $c_1(L')$ are two different spin$^c$ structures, their difference is a cohomology class whose integral on every compact surface is an even integer, and this criterion completely characterizes the set of spin$^c$ structures.

Now given a metric $g$ and a spin$^c$ structure $c_1(L)$ on an oriented 4-manifold $M$, the locally defined Dirac operators

$$D : \Gamma(\mathbb{S}_+) \to \Gamma(\mathbb{S}_-)$$

do not quite suffice to give one a globally defined Dirac operator. However, the choice of any Hermitian connection $A$ has been chosen on $L \to M$, there *is* a natural Dirac-type operator

$$D_A : \Gamma(V_+) \to \Gamma(V_-).$$

This allows us to consider the *Seiberg-Witten* equations

$$D_A \Phi = 0 \tag{4}$$

$$F_A^+ = i\sigma(\Phi) \tag{5}$$

for a pair $(\Phi, A)$, consisting of a twisted spinor $\Phi \in \Gamma(V_+)$ and a Hermitian connection $A$ on $L \to M$. Here $F_A^+$ denotes the self-dual part of the curvature 2-form of $A$, and $\sigma : V_+ \to \Lambda^+$ is a natural real-quadratic map arising from the identity $\Lambda^+ \otimes \mathbb{C} = \odot^2 \mathbb{S}_+$.

The solution space of these equations tends to be very large, because the infinite-dimensional *gauge group* of smooth functions $M \to S^1$ acts on the solution. After modding out by this action, however, we obtain a *moduli space* which is compact and, for a generic metric on a manifold with $b_+(M) \neq 0$, is actually a manifold, the dimension of which is given by $[c_1(L)^2 - (2\chi + 3\tau)(M)]/4$. If $b_+(M) > 1$, the moduli spaces corresponding to different generic metrics are cobordant, and this allows one to define invariants which force the existence of solutions of (4-5) for *any* metric, given only suitable information about just one *particular* metric on $M$. For example, if $c_1(L)^2 = (2\chi + 3\tau)(M)$, the moduli space is just a set of points. These can be given orientations in a canonical manner, and the number of points, counted with signs, is then independent of the metric; if this number is non-zero, we then deduce that the Seiberg-Witten equations will have solutions, no matter *which* metric we choose on $M$. It is this property alone which will interest us here.

**Definition 3.1** *If $c_1(L)$ is a spin$^c$ structure on the smooth oriented 4-manifold $M$, we will say that $c_1(L)$ is a* monopole class *if the Seiberg-Witten equations (4-5) have at least one solution $(\Phi, A)$ for every smooth Riemannian metric $g$ on $M$.*

As was realized by Witten [24], the existence of a monopole class can sometimes be used to show that $\mathcal{I}(M) \neq 0$. The reason is that (4-5) imply the Weitzenböck formula

$$0 = 4\nabla_A^* \nabla_A \Phi + s\Phi + |\Phi|^2\Phi. \tag{6}$$

From this one may then obtain the inequality

$$\frac{1}{4\pi^2}\int_M \frac{s_g^2}{24}\mu_g \geq \frac{1}{3}(c_1^+)^2,$$

where $c_1^+$ denotes the orthogonal projection of $c_1(L)$ into the cohomology classes represented by closed self-dual 2-forms. Now a remarkable result of Taubes shows that symplectic 4-manifolds with $b_+ > 1$ always have a monopole class, and one therefore gets *many* monopole classes on the blow-ups of symplectic 4-manifolds. this then leads [23] to results of the following type:

**Theorem 3.2 (LeBrun)** *Let $X$ be a symplectic 4-manifold with $b_+ > 1$. Then $M = X\#k\overline{\mathbb{CP}}_2$ does not admit Einstein metrics if $k \geq \frac{2}{3}(2\chi + 3\tau)(X)$.*

In particular, this gives one many simple examples [23] of simply connected 4-manifolds which do not admit Einstein metrics, even though the Hitchin-Thorpe inequality makes no such prediction.

However, Theorem 3.2 can be significantly improved upon. The reason is that so far we have only considered the scalar-curvature term in (3). In fact, a more subtle Weitzenböck argument [25] shows that the existence of a solution of (4-5) also implies that

$$\frac{1}{4\pi^2}\int_M \left(2|W_+|^2 + \frac{s_g^2}{24}\right)\mu_g \geq \frac{5}{9}(c_1^+)^2.$$

Using this, together with a recent result of Ozsváth and Szabó [26] concerning monopole classes of non-simple type, one immediately obtains

**Theorem 3.3** *Let $X$ be a symplectic 4-manifold with $b_+ > 1$. Then*

$$M = X\#k\overline{\mathbb{CP}}_2\#\ell(S^1 \times S^3)$$

*does not admit Einstein metrics if $k + 4\ell \geq \frac{4}{9}(2\chi + 3\tau)(X)$.*

While the Hitchin-Thorpe inequality only involves homotopy invariants of the 4-manifold in question, the newer obstructions discussed here strongly depend on the smooth structure of the manifold. Indeed, these results allow one [27,25] to find examples of homeomorphic pairs of 4-manifolds where one member does not admit Einstein metric, but where the homeomorphic partner is known to admit an Einstein metric by Yau's solution of the Calabi conjecture [28].

### *3.2 Uniqueness*

Theorem 3.1 gives uniqueness results as well as obstructions. For example, any Einstein metric on a 4-torus must be flat, since $\chi = \tau = 0$ for this manifold; thus the picture is similar to that encountered in dimension 3, albeit for highly non-trivial reasons. It is therefore natural to ask whether the analogy extends to 4-manifolds which admit metrics of constant curvature. When the curvature is negative, the affirmative answer is supplied [10] by the following remarkable result:

**Theorem 3.4 (Besson-Courtois-Gallot)** *Let $M^4$ be a smooth compact quotient of hyperbolic 4-space $\mathcal{H}^4 = SO(4,1)/SO(4)$, and let $g_0$ be its standard metric of constant sectional curvature. Then every Einstein metric $g$ on $M$ is of the form $g = \lambda\varphi^* g_0$, where $\varphi : M \to M$ is a diffeomorphism and $\lambda > 0$ is a constant.*

If one knew that the standard metric minimized $\int s^2\mu$, this would follow from (3). Unfortunately, however, this $L^2$ estimate is only a conjecture [22] at present. However, an entropy argument can be used to prove [10] that the standard metric minimizes $(\inf r)^2 V$, and when restricted to the set of Einstein metrics this amounts to the same inequality anyway. This entropy-comparison technique can also be used to prove the non-existence of Einstein metrics on certain manifolds of large fundamental group [29].

Instead of considering manifolds of constant sectional curvature, one might consider manifolds for which the curvature tensor is covariantly constant: $\nabla R = 0$. Spaces of this type are called *locally symmetric*. In dimension 4, a compact irreducible locally symmetric space with non-constant sectional curvature is either $\mathbb{CP}_2$ with a multiple of its standard metric, or else is covered by the so-called complex hyperbolic plane $\mathbb{C}\mathcal{H}_2$ with a multiple of its standard metric. Concretely, $\mathbb{C}\mathcal{H}_2$ may be realized as the unit ball in $\mathbb{C}^2$ with the so-called *Bergmann metric.*

One then has the following result [30]:

**Theorem 3.5 (LeBrun)** *Let $M = \mathbb{C}\mathcal{H}_2/\Gamma$ be a compact complex-hyperbolic 4-manifold, and let $g_0$ be its tautological metric. Then every Einstein metric $g$ on $M$ is of the form $g = \varphi^* c g_0$, where $\varphi : M \to M$ is a diffeomorphism and $c > 0$ is a constant.*

In this case, Seiberg-Witten theory tells us that the standard metric minimizes $\int s^2\mu$, and the result then follows from (3).

One might naively hope that it would be as easy to classify solutions of Einstein's equations on $S^4$ and $\mathbb{CP}_2$ as on their negative-curvature analogs. But the simplicity of an Einstein manifold and the simplicity of its Einstein moduli space are largely unrelated. In fact, there is essentially only one other 4-manifold for which the Einstein moduli space is completely known, namely $K3$. The Hitchin-Thorpe inequality tells us that every Einstein metric on $K3$ is one of the hyper-Kähler metrics constructed by Yau [28]. While these metrics cannot themselves be written down in closed form, the algebraic geometry allows us [31] to canonically identify their moduli space with the locally symmetric space $[O(3,19) \cap GL(22,\mathbb{Z})] \setminus O(3,19)/[O(3) \times O(19)]$.

### *3.3 Supreme Einstein Metrics*

We have seen that Seiberg-Witten theory gives rise to certain estimates concerning the scalar curvature. These estimates are actually *sharp*, and are precisely saturated by the Kähler-Einstein metrics:

**Theorem 3.6** *Let $M$ be a smooth 4-manifold which admits a Kähler-Einstein metric $g$ of scalar curvature $\leq 0$. Then $g$ is a supreme Einstein metric. Conversely, every other supreme Einstein metric on $M$ is Kähler-Einstein.*

For a large class of 4-manifolds, this and Yau's solution [28] of the Calabi conjecture allow one to identify the moduli space of *supreme* Einstein metrics with a finite union of connected components of the moduli space of non-singular complex algebraic surfaces in $\mathbb{CP}_5$.

What about Kähler-Einstein manifolds with $s > 0$? The standard Seiberg-Witten invariants vanish for such spaces, and thus shed no light at all on the story. However, a *perturbed* version of the Seiberg-Witten equations can be used [32] to prove

**Theorem 3.7** *The Fubini-Study metric on* $\mathbb{CP}_2$ *is a supreme Einstein metric. In particular,* $\mathfrak{Y}(\mathbb{CP}_2) = 12\sqrt{2}\pi$.

It might thus seem that the story is really independent of the sign of the Einstein constant $\lambda$. However, Tian's Kähler-Einstein metrics [33] on $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$, $3 \leq k \leq 8$, turn out *not* to be supreme!

**Proposition 3.8** *While the 4-manifolds* $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$*,* $k = 1,3,4,5,6,7,8$ *admit Einstein metrics, they do not admit* supreme *Einstein metrics.*

**Proof.** By a pioneering result of Osamu Kobayashi [8],

$$\mathfrak{Y}(M_1), \mathfrak{Y}(M_2) > 0 \Longrightarrow \mathfrak{Y}(M_1\#M_2) \geq \min[\mathfrak{Y}(M_1), \mathfrak{Y}(M_2)].$$

Since the Yamabe invariant is independent of orientation, this tells us that

$$\mathfrak{Y}(\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2) \geq \mathfrak{Y}(\mathbb{CP}_2) = 12\sqrt{2}\pi.$$

On the other hand, a beautiful Weitzenböck argument of Matthew Gursky [34] shows that any positive-scalar-curvature metric on a 4-manifold $M$ with $b_+ \neq 0$ must satisfy

$$\int_M |W_+|^2\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

If the metric is Einstein, (3) allows us to rewrite this as

$$\int_M s^2\mu \leq 32\pi^2(2\chi + 3\tau)(M),$$

so that

$$\mathfrak{S}(g) \leq 4\pi\sqrt{2(2\chi + 3\tau)(M)}.$$

For $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$, we have $2\chi + 3\tau = 9 - k$, so this assertion would become

$$\mathfrak{S}(g) \leq 4\pi\sqrt{2(9-k} < 12\sqrt{2}\pi \leq \mathfrak{Y}(\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2),$$

showing that the Einstein metric is not supreme.

On the other hand, these spaces actually do admit Einstein metrics by the work of Page [35] and Tian [33], so the result follows. ■

While this argument shows, in particular, that Tian's metrics are not supreme, it also shows, incidentally, that these metrics *do* correspond to the highest critical value of $\mathfrak{S}$. This weaker conclusion also holds for the standard product metric on $S^2 \times S^2$, and inspection of the Hessian of $\mathfrak{S}$ at this metric indicates that $S^2 \times S^2$ does not admit a supreme Einstein metric, either.

### *3.4 Yamabe Invariants*

The Seiberg-Witten estimates can also be used, in conjunction with appropriate direct geometric arguments, to calculate the Yamabe invariants of many other 4-manifolds. For example, one has [25]

**Theorem 3.9 (LeBrun)** *Let $M$ be a complex surface of general type, and let $X$ be its minimal model. Then*

$$\mathfrak{Y}(M) = \mathfrak{Y}(X) = -4\pi\sqrt{2(2\chi + 3\tau)(X)}.$$

*Moreover, $M$ does not admit a supreme Einstein metric if $M \neq X$.*

Here one says that $X$ is a minimal model of $M$ if $M$ can be obtained by blowing up $X$ at some finite number of points, but $X$ is not itself the blow-up of another complex surface [31]. A surface $M$ is said to be of general type if its *Kodaira dimension* $\mathrm{Kod}(M)$ equals 2. The invariant $\mathrm{Kod}(M)$ of a complex surface takes values in $\{-\infty, 0, 1, 2\}$, and is a rough measure of the number of holomorphic sections of powers of the canonical line bundle. This ostensibly depends on the particular complex structure $J$, but actually turns out to instead reflect the differential topology of the underlying 4-manifold. Indeed, it is related to the Yamabe invariant, in the following manner:

**Theorem 3.10** *Let $M$ be the underlying 4-manifold of a complex surface $(M, J)$ of Kähler type. Then*

$$\begin{aligned}
\mathfrak{Y}(M) > 0 &\Longleftrightarrow \mathit{Kod}(M, J) = -\infty \\
\mathfrak{Y}(M) = 0 &\Longleftrightarrow \mathit{Kod}(M, J) = 0 \ \mathit{or}\ 1 \\
\mathfrak{Y}(M) < 0 &\Longleftrightarrow \mathit{Kod}(M, J) = 2.
\end{aligned}$$

Notice that these results give the exact value of the Yamabe invariant for any complex algebraic surface of Kodaira dimension $\geq 0$.

Also notice that we now know that the Yamabe invariant is negative for infinitely many simply connected 4-manifolds. Indeed, any complete intersection of high-degree hypersurfaces in a complex projective space will be both simply connected and of general type, and so has this property.

Leaving the arena of simply connected manifolds, there is a beautiful and simple way of altering a 4-manifold without changing its Yamabe invariant [36]:

**Theorem 3.11 (Petean)** *If $M$ is any 4-manifold with $\mathfrak{Y}(M) \leq 0$, then*

$$\mathfrak{Y}(M\#[S^1 \times S^3]) = \mathfrak{Y}(M).$$

The proof stems from surgical ideas which will be described in the next section.

**Corollary 3.12** *Let $X$ be any minimal complex surface of general type, and, for integers $k, \ell \geq 0$, let $M = X\#k\overline{\mathbb{CP}}_2\#\ell(S^1 \times S^3)$. Then*

$$\mathfrak{Y}(M) = \mathfrak{Y}(X) = -4\pi\sqrt{2(2\chi + 3\tau)(X)}.$$

This corollary can also be proved directly from Seiberg-Witten ideas, using the results of Ozsváth and Szabó [26].

Finally, let us consider 4-manifolds with positive Yamabe invariant. To prove that $\mathfrak{Y}(M) > 0$, all one has to is produce a metric of positive scalar curvature on $M$, and examples can therefore easily be produced in great abundance. On the other hand, Aubin's argument says that the Yamabe invariant can never be bigger than that of the sphere. Showing that $0 < \mathfrak{Y}(M^4) < \mathfrak{Y}(S^4)$ for some particular $M$ remains, however, is a non-trivial problem. So far, we have seen that this does, however, actually happen in the case of $M = \mathbb{CP}_2$. There are [37] some other known examples of this type:

**Theorem 3.13** *For $k = 1, 2, 3$, and any integer $\ell \geq 0$,*

$$12\sqrt{2}\pi \leq \mathfrak{Y}(k\mathbb{CP}_2\#[S^1 \times S^3]) \leq 4\pi\sqrt{16 + 2k}.$$

*In particular, these manifolds all have Yamabe invariant $< \mathfrak{Y}(S^4)$.*

Presumably there are many more 4-manifolds with this property. Whether there are high-dimensional analogs of these examples remains completely open.

## 4 High Dimensions

While the work of many people [28,33,38,39,40] has given us a fascinating menagerie of high-dimensional Einstein manifolds, the high-dimensional arena is notably devoid of any results regarding the non-existence or uniqueness of Einstein metrics. As I shall point out here, however, the situation is quite different for *supreme* Einstein metrics. My remarks in this direction are inspired, in part, by a paper of Futaki [41] and some spectacular recent results of my student Jimmy Petean [42,36,43].

### *4.1 Yamabe Invariants*

The Yamabe invariant $\mathfrak{Y}(M)$ turns out to be relatively well behaved with respect to surgeries in high codimension [42]. Recall [44] that if $M$ is any smooth compact $n$-manifold, and if $S^q \subset M$ is a smoothly embedded $q$-sphere with trivial normal bundle, we may construct a new $n$-manifold $N$ by replacing a tubular neighborhood $S^q \times \mathbb{R}^{n-q}$ of $S^q$ with $S^{n-q-1} \times \mathbb{R}^{q+1}$. One then says $N$ is obtained from $M$ by performing a *surgery* in codimension $n-q$ (or dimension $q$). This operation precisely describes the way that level sets of a Morse function change as one passes a critical point of index $q+1$, and two manifolds are therefore cobordant iff one can be obtained from the other by such a sequence of surgeries.

**Proposition 4.1 (Petean-Yun)** *Let $M$ be any smooth compact $n$-manifold, and let $N$ be obtained from $M$ by performing a surgery in codimension $\geq 3$. If $\mathfrak{Y}(M) \leq 0$, then*

$$\mathfrak{Y}(N) \geq \mathfrak{Y}(M).$$

**Proof.** Let us consider the invariant $\mathcal{I}_-(M) = \inf_g \int_M |s_{-g}|^{n/2}\mu_g$, where

$$s_- = \begin{cases} 0, & s \geq 0 \\ s, & s \leq 0. \end{cases}$$

Assume that $n \geq 3$, as otherwise there is nothing to prove. With this assumption, $\mathcal{I}_-$ can be rewritten as

$$\mathcal{I}_-(M) = \begin{cases} 0, & \mathfrak{Y}(M) \geq 0 \\ |\mathfrak{Y}(M)|^{n/2}, & \mathfrak{Y}(M) \leq 0. \end{cases}$$

Indeed, if $M$ admits a metric of positive scalar curvature, it also [45] admits a metric with $s \equiv 0$, so both sides vanish. If, on the other hand, $M$ does *not* admit a metric of positive scalar curvature, both functionals [10] are minimized in each conformal class by a metric of constant scalar curvature $\leq 0$, and the claim is then an immediate consequence.

Now let $g$ be a metric on $M$ such that

$$\int_M |s_{-g}|^{n/2}\mu_g < \mathcal{I}_-(M) + \frac{\epsilon}{2},$$

and suppose that $S^q \subset M$ is any embedded sphere of codimension $n - q \geq 3$. By making a conformal change which is trivial outside a small tubular neighborhood of the sphere, one may produce a conformally related metric $\tilde{g} = ug$ which has positive scalar curvature along $S^q$, but still satisfies

$$\int_M |s_{-\tilde{g}}|^{n/2}\mu_{\tilde{g}} < \mathcal{I}_-(M) + \epsilon.$$

But on the manifold $N$ obtained by surgery on $S^q$, a celebrated local construction of Gromov-Lawson [46] then gives us a metric $\hat{g}$ which has positive scalar curvature in the surgered region, and agrees with $\tilde{g}$ on the set where $s_{\tilde{g}} \leq 0$. Thus

$$\int_N |s_{-\hat{g}}|^{n/2}\mu_{\hat{g}} = \int_M |s_{-\tilde{g}}|^{n/2}\mu_{\tilde{g}} < \mathcal{I}_-(M) + \epsilon,$$

so that

$$\mathcal{I}_-(N) = \inf_{\hat{g}} \int_N |s_{-\hat{g}}|^{n/2}\mu_{\hat{g}} \leq \mathcal{I}_-(M),$$

and the claim follows. ∎

Even in dimension 4, this implies Theorem 3.11, since $M^4$ and $M^4\#[S^1 \times S^3]$ are obtainable from each other in codimension $\geq 3$. But in dimension $\geq 5$, the consequences are much more startling. Indeed, it implies [43]

**Theorem 4.2 (Petean)** *Let $M$ be any smooth, simply connected compact $n$-manifold, $n \geq 5$. Then $\mathfrak{Y}(M) \geq 0$.*

That is, modulo Conjecture 2.1, the only simply connected manifolds with $\mathfrak{Y}(M) < 0$ live in dimension 4 — where they exist in profusion!

To prove the above result, it suffices to show that any simply connected manifold can be obtained by performing a sequence of codimension $\geq 3$ surgeries on some manifold with $\mathfrak{Y} \geq 0$. Using related results of Gromov-Lawson [46] and Stolz [47], this can be reduced to the problem of generating the spin-cobordism ring with manifolds for which $\mathfrak{Y} \geq 0$. Fortunately, most of the needed generators have been known for some time, and the last piece of the puzzle is supplied by Joyce's construction [38] of 8-manifolds of holonomy $Spin(7)$.

### *4.2 Supreme Einstein Metrics*

It is relatively easy to produce high dimensional examples of supreme Einstein manifolds. For example, most Calabi-Yau manifolds give us examples:

**Proposition 4.3** *Let $(M,g)$ be a simply connected Ricci-flat Kähler manifold. Assume that the complex manifold $M$ cannot be expressed as a Cartesian product of other complex manifolds. Then $g$ is a supreme Einstein metric iff $\dim_{\mathbb{C}} M \not\equiv 3 \bmod 4$.*

**Proof.** The canonical line bundle of $M$ is trivial, since it is flat and $M$ is simply connected. It follows [20] that $M$ is spin. For the Kähler metric $g$, moreover, the spin bundles are given by

$$\mathbb{S}_+ = \bigoplus_{p \text{ even}} \Lambda^{0,p},$$

$$\mathbb{S}_- = \bigoplus_{p \text{ odd}} \Lambda^{0,p},$$

and the Dirac operator $D : \Gamma(\mathbb{S}_+) \to \Gamma(\mathbb{S}_-)$ is given by $D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$. In particular, the index of the Dirac operator is just the Todd genus

$$\hat{A}(M) = \chi(M, \mathcal{O}) = \sum_p (-1)^p h^0(M, \Omega^p),$$

and

$$\dim Ker(D) = \sum_{p \text{ even}} h^0(M, \Omega^p).$$

Now let $m$ denote the complex dimension of $M$, so that $n = 2m$ is the real dimension. The index of $D$ is independent of the Riemannian metric, and does not vanish *a priori* if $n \equiv 0 \bmod 4$. If $n \equiv 2 \bmod 8$, the Hitchin invariant

$$\hat{\mathfrak{a}}(M) \equiv \dim Ker(D) \bmod 2$$

is similarly metric-independent. If either of these invariants were non-zero, the Lichnerowicz Weitzenböck formula [48,49,50]

$$D^*D = \nabla^*\nabla + \frac{s}{4}$$

would preclude the existence of a metric of positive scalar curvature on $M$, and the Ricci-flat metric $g$ would be supreme.

Now Bochner observed that a holomorphic form on a Ricci-flat Kähler manifold must be parallel, and our hypothesis forces $M$ to be holonomy irreducible [40]. If $m$ is odd, Berger's holonomy classification predicts that $M$ must have holonomy $SU(m)$, so the only parallel holomorphic forms are in dimensions 0 and $m$, so that

$$\hat{\mathfrak{a}}(M) \equiv \dim Ker(D) \equiv 1 \bmod 2$$

if $\dim_{\mathbb{C}} M \equiv 1 \bmod 4$. If, on the other hand, $m = 2k$ is even, the holonomy could be either $SU(m)$ or $Sp(k)$, but in either case every parallel form would be of even degree, and we would have $\chi(M) > 0$. Thus $g$ is a supreme Einstein metric if $m \not\equiv 3 \bmod 4$.

On the other hand, any simply connected compact manifold with dimension $\equiv 6 \bmod 8$ admits metrics of positive scalar curvature by a cobordism argument of Gromov-Lawson [46] and Stolz [47]. A supreme Einstein metric would thus have to have $s > 0$, and $g$ is therefore *not* supreme in this case. ■

If $M$ is an arbitrary simply connected Calabi-Yau manifold, it can be written as a product

$$M = M_1 \times \cdots \times M_\ell$$

of simply connected irreducible factors. A Ricci-flat Kähler metric on $M$ will thus be supreme iff none of the factors has complex dimension $\equiv 3 \bmod 4$.

Similar reasoning can also be applied to the Ricci-flat manifolds of exotic holonomy constructed by Joyce [38]. Thus any compact Einstein 8-manifold of holonomy $Spin(7)$ is supreme. On the other hand, a compact 7-manifolds with holonomy $G_2$ is *never* supreme.

We now come to the problem of constructing manifolds which do not admit supreme Einstein metrics. These are particularly easy to construct when the dimension is $\equiv 1 \bmod 8$.

**Proposition 4.4** *Let $M$ be any smooth, simply connected, compact spin manifold of dimension $n = 9, 17, 25, 33, \ldots$ Then $M$ is homeomorphic to a smooth manifold $N$ which does not admit a supreme Einstein metric.*

**Proof.** If the Hitchin invariant $\hat{\mathfrak{a}}(M) \in \mathbb{Z}_2$ is non-zero, set $N = M$. Otherwise, let $\Sigma$ be an exotic $n$-sphere with $\hat{\mathfrak{a}}(\Sigma) \neq 0$, and let $N$ be the connected sum $M\#\Sigma$. In either case, $N$ is then homeomorphic to $M$ and has non-trivial $\hat{\mathfrak{a}}$ invariant. Since $N$ is simply connected, Petean's result then tells us that $\mathfrak{Y}(N) = 0$.

Now suppose that there were a supreme Einstein metric $g$ on $N$. Since $\mathfrak{Y}(N) = 0$, $g$ would then have to be Ricci-flat. On the other hand, since $N$ is simply connected, the de Rham lemma tells us that we can express it as a Riemannian product $N = N_1 \times \cdots N_k$, where each of the factors is holonomy irreducible, and since $(N, g)$ is Ricci-flat, simply connected, and has non-trivial $\hat{\mathfrak{a}}$ invariant, the same is true of each factor. But because $N$ has odd dimension, so must at least one of the factors, say $N_1$. Since $N_1$ is simply connected, it cannot have dimension 1, and its holonomy irreducibility and Ricci-flatness would therefore force it, by Berger's classification, to be a 7-manifold with holonomy $G_2$. But since the $\hat{\mathfrak{a}}$ invariant is trivial in dimension 7, this is a contradiction. Thus $N$ does not admit a supreme Einstein metric, and the claim follows. ■

The situation is similar when the dimension is $\equiv 2 \bmod 8$.

**Proposition 4.5** *Let $M$ be any smooth, compact, 2-connected manifold of dimension $n = 10, 18, 26, 34, \ldots$ Then $M$ is homeomorphic to a smooth manifold $N$ which does not admit a supreme Einstein metric.*

**Proof.** Any such manifold has $H^2(M, \mathbb{Z}_2) = 0$, and so is spin. By again taking the connected sum of $M$ and a suitable exotic sphere $\Sigma$, we can again construct a manifold $N = M \# \Sigma$ which is homeomorphic to $M$ and has non-trivial $\hat{\mathfrak{a}}$ invariant. As before, a supreme Einstein metric $g$ on $N$ would therefore have to be Ricci-flat. Now assume that such a metric exists, and consider the holonomy decomposition $N = N_1 \times \cdots N_k$ of $(N, g)$. Each factor $N_j$ must be 2-connected, so none can be Kähler. Also, each has non-zero $\hat{\mathfrak{a}}$ invariant, and so has special holonomy. This eliminates all possible holonomy groups for the factors except $Spin(7)$, and $N$ is therefore a product of 8-manifolds. But since $\dim N \equiv 2 \bmod 8$, this is a contradiction, and it follows that $N$ cannot carry a supreme Einstein metric. ■

Similar reasoning also shows

**Proposition 4.6** *There are simply connected 8-manifolds which admit Kähler-Einstein metrics with $\lambda < 0$, but which nonetheless do not admit supreme Einstein metrics.*

**Proof.** Let $m$ be a positive integer, and let $M$ be the non-singular complex hypersurface of degree $2m + 6$ in $\mathbb{CP}_5$; for concreteness, we could thus take $M$ to be defined by the homogeneous equation

$$z_1^{2m+6} + z_2^{2m+6} + \cdots + z_6^{2m+6} = 0.$$

The adjunction formula then tells us that the canonical line bundle of $M$ is then given by $K = \mathcal{O}(2m)|_M$, and it follows that $M$ is spin. For any Kähler metric $g$ on $M$, the kernel and cokernel of the Dirac operator can be identified with the Dolbeault cohomology spaces

$$\bigoplus_{p \text{ even}} H^p(M, \mathcal{O}(m)) \text{ and } \bigoplus_{p \text{ odd}} H^p(M, \mathcal{O}(m)),$$

respectively, and the fact that

$$\hat{A}(M) = 2h^0(\mathbb{CP}_5, \mathcal{O}(m)) = 2\binom{m+5}{5}$$

can therefore be read off directly from the algebraic geometry without needlessly invoking the index theorem. In particular, the underlying simply connected 8-dimensional spin manifold has $\hat{A}(M) \neq 0$, so that $\mathfrak{Y}(M) = 0$ by Petean's theorem.

It follows that $M$ cannot admit a supreme Einstein metric. Indeed, we have just seen that $\mathfrak{Y}(M) = 0$, and a supreme Einstein metric $g$ on $M$ would therefore have to be Ricci-flat. Since $\hat{A}(M) = 2\binom{m+5}{5} \geq 12$, there would also have to be at least 12 linearly independent harmonic sections of the spinor bundle $\mathbb{S}_+$ over $M$, and each of these would have to be parallel by the Weitzenböck formula. But the maximal dimension of the space of parallel sections of $\mathbb{S}_+$ is the rank of this bundle, which is $8 < 12$. This contradictions shows that $M$ does not admit a supreme Einstein metric.

On the other hand, $M$ certainly *does* admit non-supreme Einstein metrics. Indeed, the complex manifold $M$ has ample canonical line bundle $K = \mathcal{O}(2m)|_M$, so that $M$ therefore admits a Kähler-Einstein metric of negative scalar curvature compatible with the given complex structure. Of course, $M$ admits many geometrically distinct Einstein metrics by the same argument, since for free we get a Kähler-Einstein metric for each of the different complex structures obtained by varying the homogeneous polynomial which defines the hypersurface $M \subset \mathbb{CP}_5$. ∎

Thus Theorem 3.6 no longer applies when the real dimension is greater than 4. A more extreme form of this failure is illustrated by [51]

**Theorem 4.7 (Catanese-LeBrun)** *There is a smooth 8-manifold $M$ which admits a pair of Einstein metrics for which the Einstein constants $\lambda$ have opposite signs. Moreover, one may arrange for both of these Einstein metrics to be Kähler, albeit with respect to wildly unrelated complex structures.*

Indeed, one may take the 8-manifold $M$ to be $X \times X$, where $X$ is the 4-manifold $\mathbb{CP}_2 \# 8\overline{\mathbb{CP}}_2$, which admits the Kähler-Einstein metrics with $\lambda > 0$ constructed by Tian [33]. However, one can show that this $M$ is diffeomorphic to $Z \times Z$, where $Z$ is the Barlow surface [52]. By a deformation argument, one can construct [51] complex structures on $Z$ with $c_1 < 0$, and Yau's proof [28] of the Calabi conjecture thus makes $Z$ into a Kähler-Einstein manifold with $\lambda < 0$. Taking product metrics on $X \times X$ and $Z \times Z$ proves the claim.

Thus the geometry of general Einstein metrics seems rather loosely tied to topology in high dimensions. On the other hand, restricting one's attention to *supreme* Einstein metrics would seem to offer a better chance of geometrizing manifolds in a meaningful way. However, as we have seen in this article, one cannot expect most manifolds to admit such metrics. The most interesting avenue would thus perhaps be to try, in the spirit of Anderson's 3-dimensional program, to break a general manifold into supreme Einstein pieces and collapsed pieces. The technical obstacles to doing this are formidable even in low dimensions, however, so the prospects for such a general scheme remain extremely murky.

**Acknowledgment**

This research was funded in part by NSF grant DMS-9802722.

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