On the Scalar Curvature of Einstein Manifolds

Fabrizio Catanese
Georg-August-Universität Göttingen
and
Claude LeBrun*
SUNY at Stony Brook

May 12, 1997

Abstract

We show that there are high-dimensional smooth compact manifolds which admit pairs of Einstein metrics for which the scalar curvatures have opposite signs. These are counter-examples to a conjecture considered by Besse [6, p. 19]. The proof hinges on showing that the Barlow surface has small deformations with ample canonical line bundle.

*Supported in part by NSF grant DMS-9505744.

1 Introduction

Let M be a smooth compact n-manifold. An *Einstein metric* on M is a smooth Riemannian metric g on M such that

$$r = \lambda g,$$

where r is the Ricci curvature of g and λ is any real constant. Such a metric has scalar curvature $s = n\lambda$, so the so-called Einstein constant λ and the scalar curvature of g have the same sign.

When $n < 4$, any Einstein metric has constant sectional curvature $\lambda/(n-1)$. In dimensions 2 and 3, the sign of λ is therefore determined by the size of the fundamental group, and so is really a topological invariant of M. This motivated Besse [6, p.19] to consider the conjecture that no smooth compact n-manifold can ever admit Einstein metrics with different signs of λ.

The Kähler-Einstein case superficially seems to support such a conjecture. Indeed, if J_1 and J_2 are deformation-equivalent complex structures on an even dimension smooth manifold M, then there cannot be a Kähler-Einstein metric with $\lambda > 0$ compatible with J_1 and a Kähler-Einstein metric with $\lambda < 0$ compatible with J_2. When $n = 4$, Seiberg-Witten theory [26, 18] and the Hitchin-Thorpe inequality [15] allow one to make an even more encouraging statement: if a smooth compact 4-manifold admits a Kähler-Einstein metric with $\lambda \leq 0$, then it cannot admit *any* Einstein metric for which λ has a different sign.

Unfortunately, these pieces of evidence are just red herrings. Not only is the conjecture false, but counter-examples can be constructed as Kähler-Einstein metrics on products of complex surfaces!

Theorem 1 *For any $k \geq 2$, there is a smooth compact $4k$-manifolds M which admits both an Einstein metric g_1 with $\lambda > 0$ and an Einstein metric g_2 with $\lambda < 0$. Moreover, M may be taken to be the k-fold product of $\mathbb{CP}_2\#8\overline{\mathbb{CP}}_2$ with itself, and g_1 and g_2 may both be taken to be Kähler-Einstein metrics.*

It follows, of course, that the relevant complex structures J_1 and J_2 on M are deformation-inequivalent.

The proof begins with the observation that $\mathbb{CP}_2\#8\overline{\mathbb{CP}}_2$ is h-cobordant to the complex surface S of general type discovered by Barlow [4]. While Barlow's explicit complex structures on S do not have ample canonical bundle, we show in §§2–3 that suitable small deformations of these complex structures do indeed have $c_1 < 0$. The theory the complex Monge-Ampère equation [2, 23, 24, 27] and standard facts about h-cobordisms [22, 25] then imply that Einstein's equation can be solved on $S \times S$ both when $\lambda > 0$ and when $\lambda < 0$.

Incidentally, the low-dimensional failure of the h-cobordism theorem is illustrated by the fact that S and $\mathbb{CP}_2\#8\overline{\mathbb{CP}}_2$ are themselves non-diffeomorphic [17]; cf. [21, 19]. On the other hand, Freedman's work [12] tells us that just enough of the h-cobordism theorem survives to conclude that the Einstein manifolds S and $\mathbb{CP}_2\#8\overline{\mathbb{CP}}_2$ are *homeomorphic.* Thus, while the Besse conjecture still stands in dimension 4, it survives only by virtue of the existence of exotic differentiable structures on 4-manifolds.

So far as we know, these are also the first examples of smooth compact manifolds which admit both Kähler metrics of positive Ricci curvature and Kähler metrics of negative Ricci curvature. However, Lohkamp [20], generalizing an earlier result of Gao and Yau [13], has shown that absolutely every smooth manifold of dimension $n > 2$ admits Riemannian metrics of negative Ricci curvature, so it is only the Kähler condition that gives this last observation any new interest. Rethinking the 3-dimensional case in light of these last results should serve to remind the reader that the Einstein condition is immeasurably stronger than the mere requirement that the Ricci curvature have a fixed sign.

2 Deformations of the Barlow Surface

Let F be a quintic surface in $\mathbb{CP}_3$ with exactly 20 nodes and no other singularities. Each of the nodes is locally modeled by $\mathbb{C}^2/\mathbb{Z}_2$, since the quadratic map $\mathbb{C}^2 \to \odot^2\mathbb{C}^2 : v \mapsto v \otimes v$ identifies $\mathbb{C}^2/\mathbb{Z}_2$ with the nodal surface $\det = 0$ in $\odot^2\mathbb{C}^2 \cong \mathbb{C}^3$. Thus F naturally carries the structure of a complex orbifold. We will be interested in the case in which F is a global orbifold:

$$F = Y/\mathbb{Z}_2$$

for some compact complex 2-manifold Y with a holomorphic involution. Using the theory of Hilbert modular surfaces, one can [8], for example, show that this is true of the specific 20-nodal quintic surface

$$\begin{aligned} \sum_{j=1}^{5} z_j^5 &= \frac{5}{4}(\sum_{j=1}^{5} z_j^2)(\sum_{j=1}^{5} z_j^3) \qquad (1) \\ \sum_{j=1}^{5} z_j &= 0, \end{aligned}$$

in a hyperplane of $\mathbb{CP}_4$. For more details, see §3.

We will need two basic facts about the surface Y. The first of these is rather obvious:

Lemma 2 *In the above situation, Y has ample canonical line bundle.*

Proof. The canonical line bundle bundle of Y is the pull-back of $\mathcal{O}(1)$ from $\mathbb{CP}_3$ via the tautological map $Y \to F$. Since this map is finite-to-one, it does not collapse any curves. The canonical line bundle is therefore positive by Nakai's criterion. ∎

The second basic fact is, by contrast, quite delicate:

Theorem 3 *In the above situation, $H^2(Y, TY) = 0$.*

In the interest of clarity, we defer the rather long proof to §3.

Now consider the particular quintic F given by equation 1. The symmetric group $\mathcal{S}_5$ acts on F, and in particular we have an action of the dihedral group $D_{10} \subset \mathcal{S}_5$ of pentagonal isometries, generated by (25)(34) and (12345). This action lifts to Y in such a way that [8] the cyclic subgroup $\mathbb{Z}_5 \subset D_{10}$ generated by (12345) acts freely on Y and such that [4] the involution (25)(34) acts with exactly 20 fixed points. The so-called Catanese surface $X = Y/\mathbb{Z}_5$ is therefore non-singular, and comes equipped with an involution $\alpha : X \to X$ with exactly 4 fixed points. From the above basic facts we immediately read off

Lemma 4 *The Catanese surface X has ample canonical bundle and satisfies $H^2(X, TX) = 0$.*

Proof. These follow from general facts about unramified covers. For example, since $-c_1(Y)$ is represented by a positive 2-form, it is also, by averaging, represented by a positive $\mathbb{Z}_5$-invariant 2-form; such a form descends to X, and represents $-c_1(X)$. In the same vein, the pull-back maps such as $H^2(X, TX) \to H^2(Y, TY)$ are necessarily injective, since the pull-back of a harmonic representative is harmonic with respect to an invariant metric. ∎

Dividing X by the action of the the involution α, we obtain a surface $X/\mathbb{Z}_2$ whose only singularities are four nodes. The *Barlow surface* is by definition the minimal resolution S of $X/\mathbb{Z}_2$. One can show [4] that S is a minimal, simply connected complex surface of general type, with $c_1^2 = 1$ and $p_g = 0$.

By construction, S contains four (-2)-curves $B_1, \ldots, B_4$, arising from the nodes of $X/\mathbb{Z}_2$; in particular, S does *not* have ample canonical bundle. On the other hand,

Lemma 5 *The only (-2)-curves in S are $B_1, \ldots, B_4$.*

Proof. Because the orbifold $X/\mathbb{Z}_2$ has K ample, the underlying variety has a projective embedding in projective space [3] which lifts to a pluri-canonical map of S collapsing only the B_j. ∎

Let us use B to denote the union of the four (-2)-curves $B_1, \ldots, B_4$ in S, and let $TS(-\log B)$ denote the sheaf of vector fields on S with trivial normal component along the curve B.

Lemma 6 $H^2(S, TS(-\log B)) = 0$.

Proof. The Serre dual of $H^2(S, TS)(-\log B)$ is $H^0(S, \Omega^1(\log B) \otimes \Omega^2)$. The latter is [9, Prop. 3.1] exactly the $\mathbb{Z}_2$-invariant subspace of $H^0(\tilde{X}, \Omega^1 \otimes \Omega^2) = H^0(X, \Omega^1 \otimes \Omega^2)$. However, the latter is Serre dual to $H^2(X, TX)$, which vanishes by Lemma 4. ∎

Theorem 7 *The Barlow surface S has a smooth versal deformation space of complex dimension 8. A general point in this space corresponds to a surface with ample canonical bundle.*

Proof. Let ν denote the normal bundle of B. We then have an exact sequence

$$0 \to TS(-\log B) \to TS \to \mathcal{O}_B(\nu) \to 0$$

of sheaves on S. Thus Lemma 6 tells us, in particular, that $H^2(S, TS) = 0$. Kodaira-Spencer theory thus gives us a smooth versal deformation of S with tangent space $H^1(S, TS)$, the dimension of which is $h^1(S, TS) = -\chi(S, TS) = (5c_2 - 7c_1^2)/6 = (5 \cdot 11 - 7 \cdot 1)/6 = 8$. (Note that $H^0(S, TS) = 0$ because S is of general type.)

Now for each $j = 1, \ldots, 4$, define $TS(-\log B_j)$ to be the sheaf of holomorphic vector fields with trivial normal component along the curve B_j. We then have exact sequences

$$0 \to TS(-\log B) \to TS(-\log B_j) \to \mathcal{O}_{\bar{B}_j}(\nu) \to 0,$$

where $\bar{B}_j = \cup_{j \neq k} B_k$ is the complement of $B_j \subset B$. Thus the natural maps $H^1(S, TS(-\log B_j)) \to H^1(S, TS)$ have images of codimension 1. Choose a curve through the base-point of the Kodaira-Spencer family, parameterized by an embedding of the unit disk $\Delta \subset \mathbb{C}$, which is not tangent to any of these hyperplanes. Let $\mathcal{S} \to \Delta$ be the induced family with central fiber S. By construction, the Kodaira-Spencer map of this family at zero does not take values in $H^1(S, TS(-\log B_j))$ for any $j \in \{1, 2, 3, 4\}$. Now there are only finitely many classes $a \in H^2(S, \mathbb{Z})$ such that $c_1 \cdot a = 0$ and $a \cdot a = -2$, since $c_1^2 > 0$ and the intersection form is of Lorentz type. Since $q(S) = 0$, each such a corresponds to a unique holomorphic line bundle $L_{a,t} \to S_t$ for each $t \in \Delta$, where S_t is the fiber over t. We have $h^0(S_t, L_{a,t}) \leq 1$ for all (a, t), since (-2) curves are necessarily isolated, so semi-continuity [14, §10.5] implies that, for each a, the set of $t \in \Delta$ where $h^0(S_t, L_{a,t}) = 1$ is closed in the analytic Zariski topology. Now suppose there were such an a such that $0 \in \Delta$ wasn't isolated in $\{t \in \Delta \mid h^0(S_t, L_{a,t}) = 1\}$. Then we would have $h^0(S_t, L_{a,t}) = 1\ \forall t \in \Delta$, and the direct image theorem [14, §10.5] would assert the existence of a section of $L_{t,a}$ which is non-trivial for each t and depends holomorphically on t.

Thus there would be a family of (-2)-curves in the fibers of $\mathcal{S} \to \Delta$ depending smoothly on the base, and in particular it would follow that $a = [B_j]$ for some j. Moreover, we would have a family of curves $B_{j,t} \subset S_t$ depending holomorphically on t, so that $\mathcal{S}$ actually comes from a deformation of the pair (S, B_j). But its Kodaira-Spencer map at the base-point would therefore take values in $H^1(S, TS(-\log B_j))$, in contradiction to our assumptions. This shows that there is a non-empty subset of points in the Kodaira-Spencer space corresponding to surfaces without (-2)-curves; and by semi-continuity, this non-empty subset is the complement of an analytic subset. It follows [5] that the general point of the Kodaira-Spencer space corresponds to a surface with ample canonical bundle. ∎

For the experts, the above elementary argument may be considerably shortened by invoking the results of [7, 10]. Indeed, from this point of view, Lemma 6 asserts that there is a smooth submersion Def $(X/\mathbb{Z}_2) \stackrel{\psi}{\to} \mathcal{L}_{X/\mathbb{Z}_2}$, where $\mathcal{L}_{X/\mathbb{Z}_2}$ is the local deformation space of the four nodes of $X/\mathbb{Z}_2$. Thus by [10, (1.8)], we have a fiber product of smooth manifolds

$$\begin{array}{ccc} \text{Def }(S) & \to & \mathcal{L}_S \\ \downarrow & & \downarrow_\beta \\ \text{Def }(X/\mathbb{Z}_2) & \stackrel{\psi}{\to} & \mathcal{L}_{X/\mathbb{Z}_2} \end{array}$$

where $\beta : \mathbb{C}^4 \to \mathbb{C}^4$ is given by $(z_j) \mapsto (z_j^2)$. Thus a general deformation of S has a smooth canonical model — i.e. has ample canonical bundle.

3 Vanishing Theorems for Nodal Surfaces

In this section, we will prove the key vanishing result Theorem 3. We will do this by first proving some general results concerning the following class of nodal surfaces:

Definition 1 *Let $x \mapsto [a_{ij}(x)]$ be a linear map from $\mathbb{C}^4$ to the symmetric $n \times n$ matrices, and let $F \subset \mathbb{CP}_3$ denote the surface of degree n defined by* $\det([a_{ij}(x)]) = 0$. *Suppose that the only singular points at which* $\operatorname{rank}[a_{ij}(x)] \leq n-2$ *are nodes. Then we shall say that F is a* nodal linearly symmetric surface *of degree n.*

It turns out that there are precisely $\binom{n+1}{3}$ nodes [8], which we shall denote by $p_j \in F$, $j = 1, \dots, \binom{n+1}{3}$. Let $\tilde{F}$ be the smooth surface obtained by blowing up these nodes, let $\pi : \tilde{F} \to F$ be the blowing-down map, and set $A_j = \pi^{-1}(p_j)$. The A_j are thus disjoint (-2)-curves, and we denote their union by $A = \cup_j A_j$. Setting $\mathcal{O}_{\tilde{F}}(H) = \pi^*\mathcal{O}_F(1)$ and $r = [\frac{n}{2}]$, there is then a divisor L on $\tilde{F}$ such that $2L = A + (2r + 1 - n)H$. It was shown in [8, Theorem 2.19] that $h^0(\tilde{F}, \mathcal{O}(rH -$

$L)) = n$, and the base locus of the system $|rH - L|$ is empty. We thus have a holomorphic map $\varepsilon : \tilde{F} \to \mathbb{CP}_3 \times \mathbb{CP}_{n-1}$ given by $|H| \times |rH - L|$, and ε is an embedding, since $|H|$ alone suffices away from the nodes and $|rH - L|$ is a degree-1 embedding on each A_j. It was also shown that, relative to a suitable basis v_j, $j = 1, \ldots, n$ for $H^0(\tilde{F}, \mathcal{O}(rH - L))$ that the image of ε is exactly given by the equations

$$\sum_{j=1}^{n} a_{ij}(x) v_j = 0, \ i = 1, \ldots, n.$$

As was pointed out by L. Ein, this proves the following useful fact:

Lemma 8 *The blow-up $\tilde{F}$ of such a surface F of degree n at its nodes is a complete intersection of n hypersurfaces in $\mathbb{CP}_3 \times \mathbb{CP}_{n-1}$ of bidegree $(1,1)$.*

In particular, if $\tilde{F}$ is smooth, its normal bundle in $\mathbb{CP}_3 \times \mathbb{CP}_{n-1}$ is just the restriction of $[\mathcal{O}(1,1)]^{\oplus n}$ to $\tilde{F}$. We also get the following useful vanishing result:

Lemma 9 *If $n > 4$, then*

$$H^1(\tilde{F}, \mathcal{O}(a,b)) = 0$$

provided that

- $a, b \geq -1$; *or*
- $b < 2$, $a < n - 2$; *or*
- $-1 \leq a \leq n - 3$; *or*
- $-1 \leq b \leq 1$.

Moreover,

$$H^0(\mathbb{CP}_3 \times \mathbb{CP}_{n-1}, \mathcal{O}(a,b)) \longrightarrow H^0(\tilde{F}, \mathcal{O}(a,b))$$

is surjective provided that

- $a, b \geq 0$; *or*
- $b < 3$, $a < n - 1$; *or*
- $0 \leq a \leq n - 2$; *or*
- $0 \leq b \leq 2$.

Proof. Let $E = [\mathcal{O}_{\mathbb{CP}_3 \times \mathbb{CP}_{n-1}}(-1,-1)]^{\oplus n}$. Because $\tilde{F}$ is a complete intersection, we have the Koszul complex

$$0 \to \Lambda^n E \to \cdots \to \Lambda^2 E \to E \to \mathcal{O}_{\mathbb{CP}_3 \times \mathbb{CP}_{n-1}} \to \mathcal{O}_{\tilde{F}} \to 0,$$

and we may tensor this with $\mathcal{O}(a,b)$ to obtain an exact complex

$$0 \to \mathcal{E}^{-n} \to \cdots \to \mathcal{E}^{-2} \to \mathcal{E}^{-1} \to \mathcal{O}(a,b) \to \mathcal{O}_{\tilde{F}}(a,b) \to 0.$$

Since a sum of line bundles on a projective space can only have non-trivial cohomology in the top or bottom dimension, the Künneth formula tells us that this complex always satisfies $H^i(\mathcal{E}^{-j}) = 0$ when $i \notin \{0, 3, n-1, n+2\}$. By breaking the complex into short exact sequences, it thus follows that $H^1(\tilde{F}, \mathcal{O}(a,b)) = 0$ whenever $H^3(\mathcal{E}^{-2}) = H^{n-1}(\mathcal{E}^{-(n-2)}) = 0$. But $\mathcal{E}^{-2}$ is a direct sum of copies of $\mathcal{O}(a-2, b-2)$, so that $H^3(\mathcal{E}^{-2}) = 0$ provided that $a > -2$ or $b < 2$. Similarly, since $\mathcal{E}^{-(n-2)}$ is a direct sum of copies of $\mathcal{O}(a-(n-2), b-(n-2))$, we have $H^{n-1}(\mathcal{E}^{-(n-2)}) = 0$ provided that $a < n-2$ or $b > -2$.

The same splicing argument shows that the surjectivity of

$$H^0(\mathbb{CP}_3 \times \mathbb{CP}_{n-1}, \mathcal{O}(a,b)) \longrightarrow H^0(\tilde{F}, \mathcal{O}(a,b))$$

follows whenever $H^3(\mathcal{E}^{-3}) = H^{n-1}(\mathcal{E}^{-(n-1)}) = 0$. Since $\mathcal{E}^{-3}$ and $\mathcal{E}^{-(n-1)}$ are respectively sums of copies of $\mathcal{O}(a-3, b-3)$ and $\mathcal{O}(a-(n-1), b-(n-1))$, one has $a > -1$ or $b < 3 \Rightarrow H^3(\mathcal{E}^{-3}) = 0$, and $a < n-2$ or $b > -1 \Rightarrow H^{n-1}(\mathcal{E}^{-(n-1)}) = 0$. ∎

This, incidentally, yields a sharpening of [8, Proposition 2.25]:

Corollary 10 *Let $p_j \in \mathbb{CP}_3$ be the node points of F, as before, and for each $j = 1, \ldots, \binom{n+1}{3}$, consider the condition of passing through p_j as a linear constraint on the linear system of surfaces of degree $(n-1)$ in $\mathbb{CP}_3$. Then these $\binom{n+1}{3}$ conditions are linearly independent.*

Proof. The restriction map

$$H^0(\mathbb{CP}_3 \times \mathbb{CP}_{n-1}, \mathcal{O}(0,2)) \to H^0(\tilde{F}, \mathcal{O}(0,2))$$

is surjective by Lemma 9, and it is injective by the same reasoning. However, $\mathcal{O}_{\tilde{F}}((n-1)H - A) = \mathcal{O}_{\tilde{F}}(0,2)$ by construction. Thus

$$h^0(\tilde{F}, \mathcal{O}((n-1)H - A)) = h^0(\mathbb{CP}_3 \times \mathbb{CP}_{n-1}, \mathcal{O}(0,2)) = \binom{n+1}{2}.$$

But the restrict-and-pull-back map $H^0(\mathbb{CP}_3, \mathcal{O}(n-1)) \to H^0(\tilde{F}, \mathcal{O}((n-1)H))$ is also an isomorphism, so $H^0(\tilde{F}, \mathcal{O}((n-1)H - A))$ can be identified with the subspace of $H^0(\mathbb{CP}_3, \mathcal{O}(n-1))$ consisting of sections which vanish at the p_j. Since

$$\binom{n+1}{2} = \binom{n+2}{3} - \binom{n+1}{3} = h^0(\mathbb{CP}_3, \mathcal{O}(n-1)) - \#\{p_j\},$$

the claim therefore follows. ∎

We now specialize to the $n = 5$ case. Let F be a linearly symmetric quintic whose only singularities are its 20 nodes; an example is [8] given by 1. Thus $\tilde{F}$ is smooth. Moreover, $A = 2L$, so there is a double branched cover $\tilde{Y}$ ramified exactly over the exceptional curves A_j. The inverse image of each A_j is an (-1)-curve E_j, and we may blow these 20 disjoint (-1)-curves down to obtain a smooth surface Y, which comes equipped with a natural involution $\Phi : Y \to Y$ corresponding to the sheet-interchanging map $\tilde{\Phi} : \tilde{Y} \to \tilde{Y}$ of $\tilde{Y} \to \tilde{F}$. We thus have $F = Y/\mathbb{Z}_2$, realizing F as a global orbifold, exactly as claimed in §2.

We are now prepared to prove the key vanishing result used in the previous section:

Theorem 3 *Let Y be as above. Then $H^2(Y, TY) = 0$.*

Proof. The Serre dual of $H^2(Y, TY)$ is $H^0(Y, \Omega^1 \otimes \Omega^2)$, and the latter can be identified with $H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2)$ as a consequence of Hartogs' theorem. Let us express the latter in terms of the (± 1)-eigenspaces of the action of $\tilde{\Phi}$:

$$H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2) = H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2)^+ \oplus H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2)^-.$$

By standard formulae [9, Prop. 3.1], these eigenspaces have direct interpretations on $\tilde{F}$:

$$\begin{aligned} H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2)^+ &= H^0(\tilde{F}, \Omega^1(\log A) \otimes \Omega^2) \\ H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2)^- &= H^0(\tilde{F}, \Omega^1 \otimes \Omega^2 \otimes L). \end{aligned}$$

In fact, however, by [10, Proposition 1.6], the first statement has an ostensible strengthening:

$$H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2)^+ = H^0(\tilde{F}, \Omega^1 \otimes \pi^*\omega_F) = H^0(\tilde{F}, \Omega^1 \otimes \Omega^2).$$

This strengthening is Serre dual to the equality

$$H^2(\tilde{F}, T\tilde{F}(-\log A)) = H^2(\tilde{F}, T\tilde{F}),$$

which follows directly [7, Corollary 1.3] from the observation that F has deformations which smooth all its nodes.

Let us now compute $H^0(\tilde{Y}, \Omega^1 \otimes \Omega^2)^+ = H^0(\tilde{F}, \Omega^1 \otimes \Omega^2 \otimes L)$. By Lemma 8, $\tilde{F}$ is the complete intersection of five hypersurfaces in $\mathbb{CP}_3 \times \mathbb{CP}_4$ of bidegree $(1, 1)$. The canonical line bundle of $\tilde{F}$ is therefore

$$\Omega^2_{\tilde{F}} = \mathcal{O}_{\tilde{F}}(-4, -5) \otimes \mathcal{O}_{\tilde{F}}(5, 5) = \mathcal{O}_{\tilde{F}}(1, 0),$$

whereas

$$\mathcal{O}(L) = \mathcal{O}(2H) \otimes [\mathcal{O}(2H - L)]^* = \mathcal{O}_{\tilde{F}}(2, -1).$$

Thus our objective is to show that $H^0(\Omega^1_{\tilde{F}}(3,-1))$ vanishes. Because $H^1(\tilde{F},\mathcal{O}(2,-2)) = 0$ by Lemma 9, tensoring the conormal bundle sequence

$$0 \to [\mathcal{O}_{\tilde{F}}(-1,-1)]^{\oplus 5} \to \hat{\Omega}^1 \to \Omega^1_{\tilde{F}} \to 0$$

with $\mathcal{O}(3,-1)$ shows that it is enough to ascertain the vanishing of $H^0(\tilde{F},\hat{\Omega}^1(3,-1))$, where $\hat{\Omega}^1 = \Omega^1_{\mathbb{P}\times\mathbb{P}} \otimes \mathcal{O}_{\tilde{F}}$ is the restriction of the cotangent bundle of $\mathbb{CP}_3 \times \mathbb{CP}_4$ to $\tilde{F}$. However, tensoring the Euler exact sequence

$$0 \to \hat{\Omega}^1 \to [\mathcal{O}(-1,0)]^{\oplus 4} \oplus [\mathcal{O}(0,-1)]^{\oplus 5} \to \mathcal{O} \oplus \mathcal{O} \to 0$$

with $\mathcal{O}_{\tilde{F}}(3,-1)$ tells us that

$$H^0(\tilde{F},\hat{\Omega}^1(3,-1)) \subset [H^0(\tilde{F},\mathcal{O}(3,-2))]^{\oplus 4} \oplus [H^0(\tilde{F},\mathcal{O}(2,-1))]^{\oplus 5}.$$

On the other hand, $H^0(\mathbb{CP}_3\times\mathbb{CP}_4,\mathcal{O}(3,-2)) \to H^0(\tilde{F},\mathcal{O}(3,-2))$ and $H^0(\mathbb{CP}_3\times\mathbb{CP}_4,\mathcal{O}(2,-1)) \to H^0(\tilde{F},\mathcal{O}(2,-1))$ are both surjective by Lemma 9; and since the relevant line bundles on $\mathbb{CP}_3 \times \mathbb{CP}_4$ obviously do not admit non-trivial sections, we conclude that $H^0(\tilde{Y},\Omega^1\otimes\Omega^2)^- = 0$.

To finish the proof, we now compute $H^0(\tilde{Y},\Omega^1\otimes\Omega^2)^+ = H^0(\tilde{F},\Omega^1\otimes\Omega^2)$ by the same method. Indeed, $\Omega^1_{\tilde{F}}\otimes\Omega^2_{\tilde{F}} = \Omega^1_{\tilde{F}}(1,0)$, and the conormal sequence tells us that

$$H^0(\tilde{F},\hat{\Omega}^1(1,0)) \to H^0(\tilde{F},\Omega^1(1,0)) \to [H^1(\tilde{F},\mathcal{O}(0,-1))]^{\oplus 5}$$

is exact. Since $H^1(\tilde{F},\mathcal{O}(0,-1)) = 0$ by Lemma 9, we need only verify that $H^0(\tilde{F},\hat{\Omega}^1(1,0))$ vanishes, too. Now the Euler sequence tells us that

$$H^0(\tilde{F},\hat{\Omega}^1(1,0)) = \ker \left(\begin{array}{c} [H^0(\tilde{F},\mathcal{O})]^{\oplus 4} \\ \oplus \\ [H^0(\tilde{F},\mathcal{O}(1,-1))]^{\oplus 5} \end{array} \right) \to \left(\begin{array}{c} H^0(\tilde{F},\mathcal{O}(1,0)) \\ \oplus \\ H^0(\tilde{F},\mathcal{O}(1,0)) \end{array} \right).$$

However, $H^0(\tilde{F},\mathcal{O}(1,-1)) = 0$ by Lemma 9, and $[H^0(\tilde{F},\mathcal{O})]^{\oplus 4} \to H^0(\tilde{F},\mathcal{O}(1,0))$ is manifestly injective. Thus $H^0(\tilde{Y},\Omega^1\otimes\Omega^2)^+ = 0$, and the result follows. ∎

It is worth noting that the above result depends heavily on the assumption that F is quintic. Indeed, one can use much the same argument to show that the the vanishing assertion definitely fails iff $n \gg 5$.

4 Einstein Manifolds and Smooth Topology

We begin by recalling that two smooth compact oriented n-manifolds are said to be h-cobordant [22] if there is a compact oriented $(n+1)$-manifold W whose boundary is $M \cup \bar{N}$, and such that the inclusion of either boundary component is a homotopy equivalence; here $\bar{N}$ denotes the orientation-reversed version of N. One then says that W is an h-cobordism from M to N. This defines an equivalence relation on the set of all n-manifolds, since gluing two h-cobordisms end-to-end yields a new h-cobordism. This equivalence relation is also compatible with Cartesian products:

Lemma 11 *Let M and N be h-cobordant manifolds. Then their Cartesian self-products $M \times M$ and $N \times N$ are also h-cobordant. Moreover, the iterated self-products $M^{(k)} = M \times \cdots \times M$ and $N^{(k)} = N \times \cdots \times N$ are h-cobordant for any $k \geq 1$.*

Proof. If W is a an h-cobordism between M and N, then $M \times W$ is an h-cobordism between $M \times M$ and $M \times N$, whereas $W \times N$ is an h-cobordism between $M \times N$ and $N \times N$; gluing these end-to-end then gives the desired h-cobordism.

The iterated case follows similarly, using induction. Namely, if W_{k-1} is an h-cobordism from $M^{(k-1)}$ to $N^{(k-1)}$, then $M \times W_{k-1}$ is an h-cobordism between $M^{(k-1)}$ and $M \times N^{(k-1)}$, whereas $W \times N^{(k-1)}$ is an h-cobordism between $M \times N^{(k-1)}$ and $N^{(k)}$. ∎

This observation allows one to prove the following useful fact:

Proposition 12 *Let M and N be two smooth simply connected compact 4-manifolds with equal Euler characteristics $\chi(M) = \chi(N)$ and signatures $\tau(M) = \tau(N) \not\equiv 0 \bmod 16$. Then $M \times M$ is diffeomorphic to $N \times N$. Moreover, the iterated Cartesian self-products $M^{(k)}$ and $N^{(k)}$ are diffeomorphic for any $k > 1$.*

Proof. By Rochlin's theorem [1], M and N are not spin, and so have odd intersection form. Since $b_2(M) = b_2(N)$ and $\tau(M) = \tau(N)$, the intersection forms of M and N are therefore isomorphic by the Minkowksi-Hasse classification [16] and Donaldson's thesis [11]. A theorem of Wall [25] therefore tells us that M and N are h-cobordant. By the previous lemma, it follows that of $M^{(k)}$ and $N^{(k)}$ are h-cobordant for any k. But for any $k > 1$, these are simply connected compact smooth manifolds of dimension > 5, so Smale's h-cobordism theorem asserts that they are diffeomorphic, as claimed. ∎

Corollary 13 *Let S denote the Barlow surface, and let R denote the rational complex surface $\mathbb{CP}_2\#8\overline{\mathbb{CP}}_2$. Then $S^{(k)}$ is diffeomorphic to $R^{(k)}$ for all $k > 1$. In particular, $S \times S$ is diffeomorphic to $R \times R$.*

Proof. Both R are S are simply connected complex surfaces with $c_1^2 = 1$ and $p_g = 0$. Thus both have $\chi = 11$ and $\tau = -7 \not\equiv 0 \bmod 16$. The claim thus follows from the preceding result. ∎

Theorem 14 *Let $R = \mathbb{CP}_2 \# 8\overline{\mathbb{CP}}_2$. Then the 8-manifold $R \times R$ admits Einstein metrics of both positive and negative scalar curvatures. Similarly, the $4k$-manifold $R^{(k)}$ admits Einstein metrics of both positive and negative scalar curvatures for all $k > 1$. Moreover, the relevant Einstein metrics can be constructed so as to be Kähler with respect to suitable complex structures.*

Proof. By Theorem 7, the Barlow surface (S, J_0) has deformations with ample canonical line bundle. If J_t is any complex structure on S with this property, the work of Aubin and Yau [2, 27] tells us that there is a unique Einstein metric g_S on S which is Kähler with respect to J_t and has scalar curvature -1. On the other hand, if $\tilde{J}$ is a complex structure on R corresponding to a blow-up of $\mathbb{CP}_2$ at 8 points in general position, then Tian [23], building on his joint work with Yau [24], has shown that R admits an Einstein metric g_R of scalar curvature $+1$ which is Kähler with respect to $\tilde{J}$.

Now because the Ricci tensor of any Riemannian product is the direct sum of the Ricci tensors of the factors, any Riemannian product of two Einstein manifolds with the same value of λ is again Einstein. In particular, Cartesian self-products of Einstein manifolds are always Einstein, and, moreover, Cartesian self-products of Kähler-Einstein manifolds are always Kähler-Einstein. Thus $R^{(k)}$ admits Kähler-Einstein metrics with positive scalar curvature, and $S^{(k)}$ admits Kähler-Einstein metrics with negative scalar curvature. But by the above Corollary, these manifolds are diffeomorphic, and the relevant Einstein metrics may therefore both be considered as living on $R^{(k)}$. ∎

Acknowledgements. The first author would like to thank L. Ein for pointing out that $\tilde{F}$ is a complete intersection. He would also like to to thank P. Supino for his queries regarding Theorem 3.

References

[1] M. Atiyah and F. Hirzebruch, *Riemann-Roch Theorems for Differentiable Manifolds*, **Bull. Am. Math. Soc. 65** (1959) 276–281.

[2] T. Aubin, *Equations du Type Monge-Ampère sur les Variétés Kählériennes Compactes*, **C. R. Acad. Sci. Paris 283A** (1976) 119-121.

[3] W. Baily, *On the Imbedding of V-manifolds in Projective Space*, **Amer. J. Math. 79** (1957) 403–430.

[4] R. Barlow, *A Simply Connected Surface of General Type with $p_g = 0$*, *Invent. Math. 79* (1985) 293–301.

[5] W. Barth, C. Peters and A. Van de Ven, **Compact Complex Surfaces**, Springer-Verlag, 1984.

[6] A. Besse, **Einstein Manifolds**, Springer-Verlag, 1987.

[7] D. Burns and J. Wahl, *Local Contributions to Global Deformations of Surfaces*, **Inv. Math. 26** (1974) 67–88.

[8] F. Catanese, *Babbage's Conjecture, Contact of Surfaces, Symmetrical Determinental Varieties and Applications*, **Inv. Math. 63** (1981) 433–466.

[9] F. Catanese, *On the Moduli Spaces of Surfaces of General Type*, **J. Diff. Geom 19** (1984) 483–515.

[10] F. Catanese, *Everywhere Nonreduced Moduli Spaces*, **Inv. Math. 98** (1989) 293–310.

[11] S. Donaldson, *An Application of Gauge Theory to Four-Dimensional Topology*, **J. Diff. Geom. 18** (1983) 279–315.

[12] M. Freedman, *On the Topology of 4-Manifolds*, **J. Diff. Geom. 17** (1982) 357–454.

[13] L. Z. Gao and S.-T. Yau, *The Existence of Negatively Ricci Curved Metrics on Three-Manifolds*, **Invent. Math. 85** (1986) 637–652.

[14] H. Grauert and R. Remmert, **Coherent Analytic Sheaves**, Springer, 1984.

[15] N.J. Hitchin, *On Compact Four-Dimensional Einstein Manifolds*, **J. Diff. Geom. 9** (1974) 435–442.

[16] D. Husemoller and J. Milnor, *Symmetric Quadratic Forms*, Springer, 1973.

[17] D. Kotschick, *On Manifolds Homeomorphic to* $\mathbf{CP}_2\#8\overline{\mathbf{CP}}_2$, **Invent. Math. 95** (1989) 591–600.

[18] C. LeBrun, *Einstein Metrics and Mostow Rigidity,* **Math. Res. Lett. 2** (1995) 1–8.

[19] C. LeBrun, *On the Scalar Curvature of Complex Surfaces*, **Geom. Func. An. 5** (1995) 619–628.

[20] J. Lohkamp, *Metrics Of Negative Ricci Curvature*, **Ann. Math. 140** (1994) 655–683.

[21] C. Okonek and A. Van de Ven, Γ-*Type-Invariants Associated to* PU(2)-*bundles and the Differentiable Structure of Barlow's Surface*, **Invent. Math. 95** (1989) 601–614.

[22] S. Smale, *On the Structure of Manifolds*, **Am. J. Math. 87** (1964) 387–399.

[23] G. Tian, *On Calabi's Conjecture for Complex Surfaces with Positive First Chern Class,* **Inv. Math. 101** (1990) 101-172.

[24] G. Tian and S.T. Yau, *Kähler-Einstein Metrics on Complex Surfaces with* $\mathbf{c}_1 > 0$, **Comm. Math. Phys. 112** (1987) 175-203.

[25] C.T.C. Wall, *On Simply Connected 4-Manifolds*, **J. Lond. Math. Soc. 39** (1964) 141–149.

[26] E. Witten, *Monopoles and Four-Manifolds*, **Math. Res. Lett. 1** (1994) 809–822.

[27] S.-T. Yau, *On the Ricci-Curvature of a Complex Kähler Manifold and the Complex Monge-Ampère Equations*, **Comment. Pure Appl. Math. 31** (1978) 339–411.

Authors' addresses:

Prof. Fabrizio Catanese
Mathematisches Institut
Bunsenstraße 3/5
37073 Göttingen, Germany

Prof. Claude LeBrun
Department of Mathematics
SUNY at Stony Brook
Stony Brook, NY 11794-3651 USA