Gravitational Instantons, Weyl Curvature, and Conformally Kähler Geometry

Olivier Biquard, Sorbonne Université
Paul Gauduchon, École Polytechnique
and
Claude LeBrun*, Stony Brook University

1 Introduction

In the late 1970s, Stephen Hawking [16] and Gary Gibbons [15], along with a small group of other gravitational physicists at Cambridge, first began their systematic exploration of the multiverse of complete, non-compact, Ricci-flat Riemannian 4-manifolds. They termed such spaces gravitational instantons, in the expectation that these would eventually come to represent tunneling modes in a theory of quantum gravity. Their striking discoveries included the construction of two infinite families of half-conformally-flat gravitational instantons, respectively generalizing the Eguchi-Hanson metric [14] and the Euclidean Taub-NUT metric, and specific properties of these unanticipated families turned out to have long-term implications for the direction of differential-geometric research. Indeed, the fact that the Ricci-flat metrics in these new families were all anti-self-dual then allowed Hitchin [19] to pioneer an essentially independent approach to them, based on Penrose's nonlinear graviton construction [28]. Soon afterwards, it was then realized these metrics were also hyper-Kähler, in the new sense that had recently been introduced by Calabi [8], and considering them in this broader context not only gave rise to new methods [20, 21] of constructing such spaces, but also led to a satisfyingly complete classification of asymptotically locally Euclidean (ALE) hyper-Kähler gravitational instantons [22]. This inspired repeated flurries of intensive mathematical activity [10, 12, 17, 18, 27, 30] during the following decades, eventually resulting in an essentially complete classification of hyper-Kähler gravitational instantons with curvature in L^2.

*Supported in part by NSF grant DMS-2203572.

(However, since there are known examples [3] of complete hyper-Kähler 4-manifolds whose curvature is *not* in L^2, the current advanced state of the subject may still not represent the end of the story.) Such results also allow one to deduce classification theorems for complete Ricci-flat 4-manifolds that are Kähler [31] or anti-self-dual [33], because either of these hypotheses suffices to imply that the universal cover of the manifold is hyper-Kähler.

However, while the mathematical papers cited above generally streamline their terminology by building the assumption of being hyper-Kähler into the very definition of a "gravitational instanton," Gibbons, Hawking, and their physicist colleagues certainly never intended for the term to be narrowed in this way. Indeed, one of Hawking's first examples [16] of a gravitational instanton was the Riemannian Schwarzschild metric, a complete Ricci-flat manifold diffeomorphic to $S^2 \times \mathbb{R}^2$ that is gotten from the simplest Lorentzian black-hole solution by formally replacing time t with it. This example is not even locally hyper-Kähler, but, in view with its close relationship with gravitational physics, it most certainly deserves to be called a "gravitational instanton." The same must also be said of the more general Riemannian Kerr metrics, which are again complete Ricci-flat metrics on $S^2 \times \mathbb{R}^2$, and which are formally obtained from their Lorentzian spinning-black–hole analogues by multiplying both time and the angular-momentum parameter by i.

While these last-mentioned metrics are not even locally Kähler, they turn out to be *conformally* Kähler, and so are, in particular, *Hermitian.* They are also asymptotically locally flat (ALF), in the sense of Definition 1 below; in particular, they have cubic volume growth. Finally, they are *toric*, in the sense that their isometry groups contain a 2-torus $\mathbb{T}^2$.

Complete oriented Ricci-flat Riemannian manifolds with all of these properties were recently classified [7] by the first two authors of this paper. They fall into just four smooth connected families, thereby realizing exactly four different diffeotypes. In addition to the Kerr family alluded to above, the other possibilities are the Taub-bolt metric, the reverse-oriented Taub-NUT metric, and a family discovered by Chen and Teo [11] in 2011. This Chen-Teo family had been completed unanticipated by the physics community, and the Hermitian nature of these metrics was only discovered later, by Aksteiner and Andersson [1]. In fact, Biquard and Gauduchon [7] did much more than merely classify such metrics; indeed, by building on earlier work by Paul Tod [32], they showed that these metrics can always be constructed from axisymmetric harmonic functions on Euclidean 3-space.

However, one disquieting feature of this otherwise compelling story is that it only concerns metrics that are invariant under an isometric $\mathbb{T}^2$-action.

The aim of this paper is to replace this symmetry assumption with an open, purely Riemannian condition. Indeed, the Ricci-flat metrics occurring in the above classification turn out to all have the property that their self-dual Weyl tensors $W^+ : \Lambda^+ \to \Lambda^+$ satisfy $\det(W^+) > 0$ everywhere. Einstein metrics with this property are said to satisfy *Wu's criterion*, in honor of Peng Wu [34], who first discovered that compact Einstein manifolds with this property are necessarily conformally Kähler. Our main observation here is that the first author's proof [24] of Wu's criterion can be adapted to the context of ALF gravitational instantons, provided one imposes fall-off conditions on the metric that are stringent enough to provide good control the boundary terms. Our first main result is the following:

Theorem A. *Let (M, h_0) be any toric, Hermitian, but non-Kähler ALF gravitational instanton. If h is another Ricci-flat Riemannian metric on M that is sufficiently C^3_1-close to h_0, then (M, h) is also a Hermitian ALF gravitational instanton, and carries a non-trivial Killing field ξ. Moreover, h is conformally related to a complete, strictly extremal Kähler metric g.*

For the proof, see §3 below, where the C^3_1 norm is also defined.

For two of the four families of toric Hermitian gravitational instantons, the metric satisfies both $\det(W^+) > 0$ and $\det(W^-) > 0$, and our methods can therefore be applied for both orientations. Doing so then leads to a stronger result in these cases:

Theorem B. *Let (M, h_0) be a Kerr or Taub-bolt gravitational instanton, and let h be another Ricci-flat metric on M that is sufficiently C^3_1-close to h_0. Then (M, h) is once again a Kerr or Taub-bolt gravitational instanton.*

For the proof, see §4 below. That section also highlights the result of Aksteiner, Anderson, Dahl, Nilsson and Simon [2] which originally led us to expect for Theorem B to hold. In addition, we have added a brief discussion of a recent preprint[1] by Minyang Li [25] that indicates that Theorem B could be generalized to also cover the two other families of gravitational instantons that appear in the Biquard-Gauduchon classification.

2 Wu's Criterion, Revisited

In this section, we will see that a remarkable open criterion introduced by Peng Wu [34] in the context compact Einstein 4-manifolds also leads to

[1] The first version of our own paper was submitted to the arXiv during the weekend between the submission of Li's e-print and its public posting.

compelling results in other contexts. Wu originally observed that a compact oriented simply-connected Einstein 4-manifold with Einstein constant $\lambda > 0$ has $\det(W^+) > 0$ if the metric is conformally Kähler, and then gave a rather opaque argument to show that the converse is also true. This prompted the third author of the present paper to give an entirely different proof [24] of Wu's criterion that actually proves more; for example, it turns out that a compact oriented Riemannian 4-manifold (M, h) with $b_+(M) \neq 0$ satisfies $\delta W^+ = 0$ and $\det(W^+) > 0$ if and only if $b_+(M) = 1$ and $h = s^{-2}g$ for some Kähler metric g on M with scalar curvature $s > 0$. In what follows, we will localize many key steps in [24] in order to obtain results that are better adapted to the study of gravitational instantons.

Suppose that (M, h) is an oriented Riemannian 4-manifold whose self-dual Weyl curvature tensor W^+ is *harmonic*, in the sense that

$$\delta W^+ := -\nabla \cdot W^+ = 0. \tag{1}$$

For example, if h is Einstein, equation (1) follows as a consequence of the second Bianchi identity. However, independent of such considerations, the Dirac-type equation (1) always implies [29, equation (6.8.40)] that

$$0 = \nabla^*\nabla W^+ + \frac{s}{2}W^+ - 6W^+ \circ W^+ + 2|W^+|^2 I, \tag{2}$$

a Weitzenböck formula that has often been eclipsed by its very useful contraction [6, equation (16.73)] with W^+. Next, let $f : M \to \mathbb{R}^+$ be a smooth positive function on M, and consider the corresponding conformal rescaling $g = f^{-2}h$ of the original metric h. Owing to the weighted conformal invariance [29, equation (6.8.8)] of the Dirac-type equation (1), one then finds that

$$\delta(fW^+) = 0 \tag{3}$$

with respect to the conformally rescaled metric g. The same calculation [29, equation (6.8.35)] that proves (2) therefore now gives us a Weitzenböck formula

$$0 = \nabla^*\nabla(fW^+) + \frac{s}{2}fW^+ - 6fW^+ \circ W^+ + 2f|W^+|^2 I \tag{4}$$

for fW^+ with respect to the rescaled metric g.

We will next need to clearly understand when an oriented Riemannian 4-manifold has $\det(W^+) > 0$. Since $W^+ : \Lambda^+ \to \Lambda^+$ is self-adjoint, we can diagonalize W^+ at any point $p \in M$ as

$$W^+ = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

by choosing a suitable orthonormal basis for Λ^+; and, after re-ordering our basis if necessary, we may arrange that $\alpha \geq \beta \geq \gamma$ at p. However, by its very definition, the self-dual Weyl curvature $W^+ : \Lambda^+ \to \Lambda^+$ automatically satisfies $\mathrm{Trace}(W^+) = 0$, and it therefore follows that

$$\alpha + \beta + \gamma = 0.$$

Hence $\alpha > 0$ and $\gamma < 0$ at any point where $W^+ \neq 0$, and we thus see that $\det W^+ = \alpha\beta\gamma$ always has the same sign as $-\beta$, where β is once again the middle eigenvalue. In other words, $\det W^+ > 0$ everywhere iff exactly one of the eigenvalues, namely α, is positive at each point, while both the other two are negative:

$$W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}.$$

This in particular implies that the positive eigenvalue α has multiplicity one everywhere. If we let $S(\Lambda^+) \subset \Lambda^+$ denote the sphere bundle defined by $|\omega|^2 = 2$, then the the smooth function $\mathsf{Q} : S(\Lambda^+) \to \mathbb{R}$ defined by $\mathsf{Q}(\omega) = W^+(\omega, \omega)$ has non-degenerate fiberwise Hessian along the set $\mathscr{P} \subset S(\Lambda^+)$ of fiberwise maxima. Equivalently, the restriction of $d\mathsf{Q}$ to the fibers, considered as a section of the vertical cotangent bundle of $S(\Lambda^+)$, is transverse along $\mathscr{P}$ to the zero section of this vector bundle. The implicit function theorem therefore guarantees that $\mathscr{P}$ is a submanifold of $S(\Lambda^+)$, and that it is moreover transverse to the fibers of $S(\Lambda^+) \to M$. It therefore follows that the set $\mathscr{P}$ of α-eigenforms $\omega \in S(\Lambda^+)$ can locally be parameterized by a system of smooth local sections of $\Lambda^+ \to M$. Since there are exactly two choices of such an ω at each point of M, differing only by sign, we conclude that $\mathscr{P} := \{\omega \in \Lambda^+ \mid W^+_h(\omega) = \alpha_h \omega, \quad |\omega|^2_h = 2\}$ is a smooth principal $\mathbb{Z}_2$-bundle over M. More importantly, $\alpha : M \to \mathbb{R}$ actually defines a smooth positive function $\alpha = W^+(\omega, \omega)/2$ on M, obtained by taking $\pm\omega$ to be the two local sections of $\mathscr{P} \subset \Lambda^+$ near an arbitrary point of M.

Remark. Because $x = \alpha$ is the unique positive solution of the depressed cubic equation

$$0 = \det(xI - W^+) = x^3 - \left[\frac{1}{2}|W^+|^2\right] x - \det(W^+), \tag{5}$$

Cardano's formula provides an explicit expression

$$\alpha = 2^{2/3} \Re e \sqrt[3]{\det(W^+) + i\sqrt{\frac{|W^+|^6}{54} - [\det(W^+)]^2}} \tag{6}$$

for the eigenvalue α in terms of $|W^+|^2$ and $\det(W^+)$, where the cube root means the principal branch, with positive real part. However, this explicit formula does not make it obvious that α remains smooth at points where $|W^+|^3 = 3\sqrt{6}\det(W^+) > 0$, or in other words, where $\beta = \gamma < 0$. Because the smoothness of $\alpha : M \to \mathbb{R}^+$ plays such a central role in what follows, we have therefore chosen to emphasize the above abstract explanation of its regularity, rather than focusing on the explicit formula (6). $\diamondsuit$

For simplicity, we will henceforth assume that M is simply connected. This then guarantees that the principal $\mathbb{Z}_2$-bundle $\mathscr{P} \to M$ is in fact trivial. Consequently, there is then a smooth globally-defined self-dual 2-form ω on M with $W^+(\omega) = \alpha\omega$ and $|\omega|^2 = 2$ at every point; moreover, the only other such global 2-form is $-\omega$, so this ω is actually unique up to overall sign. Equivalently, our assumption that M is simply connected implies that the α-eigenspace $L \subset \Lambda^+$ of W^+ is a trivial real line-bundle $L \to M$.

Now the condition $\det(W^+) > 0$ is conformally invariant, so the above discussion also applies to the conformal rescaling $g = f^{-2}h$ of h defined by any smooth positive function $f : M \to \mathbb{R}^+$. However, the endomorphism $W^+ : \Lambda^+ \to \Lambda^+$ is defined by raising an index

$$\varphi_{ab} \longmapsto [W^+(\varphi)]_{cd} := \frac{1}{2} W^{+ab}{}_{cd}\, \varphi_{ab},$$

and so carries a conformal weight, even though the naïve Weyl tensor $W^{+a}{}_{bcd}$ is literally conformally invariant. Consequently, replacing h with $g = f^{-2}h$ rescales the top eigenvalue by a factor of f^2:

$$\alpha_g = f^2 \alpha_h.$$

We will henceforth impose the interesting choice

$$\boxed{f = \alpha_h^{-1/3}} \tag{7}$$

for our conformal factor f, which then has the consequence that

$$\alpha_g = f^2 \alpha_h = \alpha_h^{1/3} = f^{-1}.$$

In particular, $\alpha := \alpha_g$ therefore satisfies

$$\alpha f \equiv 1 \tag{8}$$

for this carefully chosen conformally-rescaled metric $g = f^{-2}h$.

Since our assumption that M is simply-connected again implies that the α-eigenbundle $L \subset \Lambda^+$ of W^+ is a trivial real line-bundle, there once again exists a global self-dual 2-form ω on M that satisfies

$$W_g^+(\omega) = \alpha_g \ \omega, \qquad |\omega|_g^2 = 2 \tag{9}$$

at every point of M; moreover, this ω is unique up to overall sign. Here, normalizing $\omega \in \Lambda^+$ so that $|\omega|_g^2 = 2$ is equivalent to requiring that

$$\omega = g(J\cdot, \cdot)$$

for a unique almost-complex structure J that is compatible with both g and the given orientation. If we can show, under suitable circumstances, that $\nabla\omega = 0$ with respect to the Levi-Civita connection ∇ of g, it will then follow that J is integrable, and that (M, g, J) is Kähler, with Kähler form ω. When $\delta W^+ = 0$, our strategy for proving this will be based on a careful study of the inner product

$$0 = \left\langle \nabla^*\nabla(fW^+) + \frac{s}{2}fW^+ - 6fW^+ \circ W^+ + 2f|W^+|^2 I, \ \omega \otimes \omega \right\rangle \tag{10}$$

of (4) with $\omega \otimes \omega$. To make headway on this, we will need a few key facts about self-dual 2-forms and the Weyl curvature, starting with the following:

Lemma 1. *Let (M, h) be a simply-connected oriented Riemannian 4-manifold for which $\det(W^+) > 0$ everywhere. Let $g = f^{-2}h$ be a conformal rescaling of h, and let ω be a self-dual 2-form that satisfies (9) everywhere. Then*

$$W^+(\nabla^a\omega, \nabla_a\omega) \leq 0, \tag{11}$$

everywhere, where every term is understood to be defined with respect to g.

Proof. The covariant derivative $\nabla\omega$ of ω belongs to $\Lambda^1 \otimes \omega^\perp \subset \Lambda^1 \otimes \Lambda^+$ because ω has constant norm with respect to g. The result therefore follows from the fact that $W^+(\phi, \phi) \leq 0$ for any $\phi \in \omega^\perp \subset \Lambda^+$. □

We will also need the following standard algebraic observation:

Lemma 2. *At any point p of an oriented Riemannian 4-manifold (M, g),*

$$|W^+|^2 \geq \frac{3}{2}\alpha^2 \tag{12}$$

where $\alpha = \alpha_g$ is the the top eigenvalue of W_g^+ at p.

Proof. Because Trace $(W^+) = 0$,

$$|W^+|^2 = \alpha^2 + \beta^2 + (-\alpha - \beta)^2 = \frac{3}{2}\alpha^2 + 2(\beta + \frac{1}{2}\alpha)^2 \geq \frac{3}{2}\alpha^2$$

where β is the middle eigenvalue of W_g^+ at p. $\square$

Finally, we will also need the Weitzenböck formula [29, equation (6.8.38)]

$$(d + d^*)^2\omega = \nabla^*\nabla\omega - 2W^+(\omega) + \frac{s}{3}\omega \tag{13}$$

for the Hodge Laplacian on self-dual 2-forms; cf. [5, p. 324].

Lemma 3. *If (M, h) is a simply-connected oriented Riemannian 4-manifold that satisfies $\det(W^+) > 0$, then the conformally rescaled metric $g = f^{-2}h$ defined by (7) and the self-dual 2-form ω defined by (9) together satisfy*

$$\langle \nabla^*\nabla(fW^+), \omega \otimes \omega\rangle \geq 2|\nabla\omega|^2, \tag{14}$$

at every point of M, where both sides are computed relative to g.

Proof. Since $f\alpha := f\alpha_g \equiv 1$, $W^+(\omega) = \alpha\omega$, and $|\omega|^2 \equiv 2$, we have

$$\begin{aligned}
\langle \nabla^*\nabla(fW^+), \omega \otimes \omega\rangle &= \langle -\nabla^a\nabla_a(fW^+), \omega \otimes \omega\rangle \\
&= -\nabla^a\nabla_a\langle fW^+, \omega \otimes \omega\rangle + 2\nabla_a\langle fW^+, \nabla^a(\omega \otimes \omega)\rangle \\
&\qquad - f\langle W^+, \nabla^a\nabla_a(\omega \otimes \omega)\rangle \\
&= \Delta(f\alpha|\omega|^2) + 4\nabla_a\langle fW^+, \omega \otimes \nabla^a\omega\rangle \\
&\qquad - f\langle W^+, \nabla^a\nabla_a(\omega \otimes \omega)\rangle \\
&= 4\nabla_a\langle (f\alpha)\omega, \nabla^a\omega\rangle - f\langle W^+, \nabla^a\nabla_a(\omega \otimes \omega)\rangle \\
&= 2\nabla_a\nabla^a|\omega|^2 - f\langle W^+, \nabla^a\nabla_a(\omega \otimes \omega)\rangle \\
&= -f\langle W^+, \nabla^a\nabla_a(\omega \otimes \omega)\rangle \\
&= -2f\langle W^+, \omega \otimes \nabla^a\nabla_a\omega\rangle - 2f\langle W^+, (\nabla^a\omega) \otimes (\nabla_a\omega)\rangle \\
&= -2(f\alpha)\langle \omega, \nabla^a\nabla_a\omega\rangle - 2fW^+(\nabla^a\omega, \nabla_a\omega) \\
&= \Delta|\omega|^2 + 2|\nabla\omega|^2 - 2fW^+(\nabla^a\omega, \nabla_a\omega) \\
&= 2|\nabla\omega|^2 - 2fW^+(\nabla^a\omega, \nabla_a\omega)
\end{aligned}$$

Since $\det(W^+) > 0$, Lemma 1 then implies the promised inequality (14). $\square$

Proposition 1. *Let (M, h) be a simply-connected oriented Riemannian 4-manifold that satisfies $\delta W^+ = 0$ and $\det(W^+) > 0$ everywhere. Then the* *conformally-rescaled metric $g = f^{-2}h$ defined by* (7) *and the self-dual* 2*-form* ω *defined by* (9) *together satisfy*

$$0 \geq \frac{1}{2}|\nabla\omega|^2 \;+ \frac{3}{2}\Big\langle \omega, (d+d^*)^2\omega \Big\rangle \tag{15}$$

at every point of M*, where all terms are to be computed with respect to* g*.*

Proof. By applying (8), (9), (11), (12), (13), and (14) to (10), we have

$$\begin{aligned}
0 &= \Big\langle \nabla^*\nabla f W^+ + \frac{s}{2} f W^+ - 6 f W^+ \circ W^+ + 2f|W^+|^2 I, \omega\otimes\omega \Big\rangle \\
&= \langle \nabla^*\nabla(fW^+), \omega\otimes\omega\rangle + \Big[\frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\Big] f \\
&\geq 2|\nabla\omega|^2 + \Big[\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2\Big] f \\
&\geq 2|\nabla\omega|^2 + \Big[\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 3\alpha^2|\omega|^2\Big] f \\
&\geq 2|\nabla\omega|^2 + \Big[\frac{s}{2}|\omega|^2 - 3\alpha|\omega|^2\Big](\alpha f) \\
&= 2|\nabla\omega|^2 + \Big[\frac{s}{2}|\omega|^2 - 3\alpha|\omega|^2\Big] \\
&= \frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \nabla^*\nabla\omega\rangle - \frac{3}{4}\Delta|\omega|^2 + \Big[\frac{s}{2}|\omega|^2 - 3\alpha|\omega|^2\Big] \\
&= \frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\Big[\langle\omega, \nabla^*\nabla\omega\rangle - 2W^+(\omega,\omega) + \frac{s}{3}|\omega|^2\Big] \\
&= \frac{1}{2}|\nabla\omega|^2 \;+ \frac{3}{2}\Big\langle \omega, (d+d^*)^2\omega \Big\rangle,
\end{aligned}$$

thus proving the promised pointwise inequality. □

Lemma 4. *Under the hypotheses of Proposition 1, the* 2*-form* ω *satisfies*

$$(2\sqrt{6}|W^+| - s) \geq 2|\nabla\omega|^2$$

with respect to the rescaled metic g*, at every point of* M*.*

Proof. Since $|\omega|^2 \equiv 2$, the proof of Proposition 1 shows, in particular, that

$$0 \geq 2|\nabla\omega|^2 + \left(\frac{s}{2} - 3\alpha\right)|\omega|^2 = 2|\nabla\omega|^2 + (s - 6\alpha)$$

everywhere. Because $\sqrt{\frac{2}{3}}|W^+| \geq \alpha$ by (12), the claim therefore follows. □

On the other hand, notice that

$$\begin{aligned}\star\Big\langle \omega,(d+d^*)^2\omega\Big\rangle &= \omega\wedge (d+d^*)^2\omega\\ &= \omega\wedge[d(-\star d\star)+(-\star d\star)d]\omega\\ &= -2\omega\wedge[d\star d\omega]\\ &= \star 2|d\omega|^2-2\, d[\omega\wedge\star d\omega].\end{aligned}$$

This allows us to rewrite the pointwise inequality (15) as

$$3\, d[\omega\wedge\star d\omega]\geq \star\Big(\frac{1}{2}|\nabla\omega|^2+3\,|d\omega|^2\Big). \tag{16}$$

Integrating (16) on a precompact domain with smooth boundary, and then applying Stokes' Theorem, we therefore obtain the following result:

Proposition 2. *Let (M,h) be a simply-connected oriented Riemannnian 4-manifold that satisfies $\delta W^+=0$ and $\det(W^+)>0$. Consider the conformally rescaled metric $g=f^{-2}h$ defined by (7), and let ω be one of the two self-dual 2-forms on M that satisfies (9). Then for any precompact domain $U\subset M$ with smooth boundary $\partial U=\overline{U}-U$, we have*

$$3\int_{\partial U}\omega\wedge\star d\omega\geq\int_U\Big[\frac{1}{2}|\nabla\omega|^2\ +3\,|d\omega|^2\Big]d\mu_g. \tag{17}$$

This now allows us to deduce the following:

Proposition 3. *Let (M,h) be an oriented, simply-connected Riemannian 4-manifold that satisfies $\delta W^+=0$ and $\det(W^+)>0$ everywhere. Suppose, moreover, that M is a nested union $M=\cup_j U_j$ of precompact domains $U_1\Subset U_2\Subset\cdots\Subset U_j\Subset\cdots$ with smooth boundary such that*

$$\lim_{j\to\infty}\int_{\partial U_j}\omega\wedge\star d\omega=0,$$

where ω is the self-dual 2-form defined by (9) relative to the conformally rescaled metric $g=f^{-2}h$ defined by (7). Then (M,g) is a Kähler manifold.

Proof. Applying (17) to each U_j yields

$$3\int_{\partial U_j}\omega\wedge\star d\omega\geq\int_{U_j}\Big[\frac{1}{2}|\nabla\omega|^2\ +3|d\omega|^2\Big]d\mu_g\geq\frac{1}{2}\int_{U_j}|\nabla\omega|^2 d\mu_g.$$

Since the right-hand side is non-negative, our hypothesis therefore implies that $\lim_{j\to\infty}\int_{U_j}|\nabla\omega|^2 d\mu_g=0$. But $U_j\subset U_{j+1}$, so the terms $\int_{U_j}|\nabla\omega|^2 d\mu_g$ in this sequence are also non-decreasing in j. Thus $\int_{U_j}|\nabla\omega|^2 d\mu_g=0$ for all j, and hence $\nabla\omega\equiv$ on each U_j. But since $M=\cup_j U_j$, this implies that $\nabla\omega\equiv 0$ on all of M. It follows that (M,g) is a Kähler manifold. $\square$

3 Gravitational Instantons

In order to be able to invoke Proposition 3 in concrete circumstances, we next show that the relevant boundary hypothesis will automatically hold if certain geometric conditions are fulfilled.

Theorem 1. *Let (M, h) be an oriented, simply-connected, Ricci-flat 4-manifold that satisfies $\det(W^+) > 0$ everywhere, and suppose that M is expressed as a nested union $M = \cup_j U_j$ of precompact domains*

$$U_1 \Subset U_2 \Subset \cdots \Subset U_j \Subset \cdots$$

with smooth boundary. Let $g = f^{-2}h$ be the conformally rescaled metric defined by (7), and let $d\breve{\mu}_g$ denote the 3-dimensional volume measure on each of these boundaries ∂U_j induced by the restriction of g. Also suppose that the g-induced 3-dimensional volumes of these boundaries are uniformly bounded, while the integrals of $|W_g^+|$ and s_g on these boundaries tend to zero with respect to this same 3-dimensional volume-measure:

$$\int_{\partial U_j} 1 \, d\breve{\mu}_g \quad < \quad \mathsf{C}, \tag{18}$$

$$\lim_{j\to\infty} \int_{\partial U_j} |W_g^+| \, d\breve{\mu}_g \quad = \quad 0, \tag{19}$$

$$\lim_{j\to\infty} \int_{\partial U_j} |s_g| \, d\breve{\mu}_g \quad = \quad 0. \tag{20}$$

Then (M, g) is a strictly extremal Kähler manifold, while the given Ricci-flat 4-manifold (M, h) is Hermitian, and carries a non-trivial Killing field.

Proof. Because ω is a self-dual 2-form of norm $\equiv \sqrt{2}$, one has

$$|\omega \wedge \star d\omega| = |\delta\omega| = |\nabla \cdot \omega|$$

with respect to g, and it is therefore relatively easy to see that

$$2\sqrt{2}|\nabla\omega| \geq |\omega \wedge \star d\omega| \tag{21}$$

at every point of M. (Indeed, after detailed calculation, the constant $2\sqrt{2}$ in (21) can actually be replaced by 1; but the gist of what follows merely depends on the fact that there is some universal constant for which such an inequality holds.) Now, Lemma 4 tells us that the inequality

$$(2\sqrt{6}|W^+| - s) \geq 2|\nabla\omega|^2$$

also holds at every point of M. Consequently, inequality (21) implies that

$$2\left[2\sqrt{6}|W^+| + |s|\right]^{1/2} \geq |\omega \wedge \star d\omega|$$

at every point of M, and we therefore deduce that

$$2\int_{\partial U_j}\left[2\sqrt{6}|W^+| + |s|\right]^{1/2} d\check{\mu}_g \geq \left|\int_{\partial U_j} \omega \wedge \star d\omega\right|$$

holds for each j. The Cauchy-Schwarz inequality thus implies that

$$2\left[\int_{\partial U_j} 1\ d\check{\mu}_g\right]^{1/2}\left[2\sqrt{6}\int_{\partial U_j}|W^+|\ d\check{\mu}_g + \int_{\partial U_j}|s|\ d\check{\mu}_g\right]^{1/2} \geq \left|\int_{\partial U_j} \omega \wedge \star d\omega\right|$$

for every j. Thus, our hypotheses (18), (19), and (20) now imply that

$$\lim_{j\to\infty}\int_{\partial U_j} \omega \wedge \star d\omega = 0.$$

Since (M, h) also satisfies $\delta W^+ = 0$ and $\det(W^+) > 0$, this means that all the hypotheses of Proposition 3 are fulfilled, and it therefore follows that g is a Kähler metric. In particular, the conformally related metric $h = f^2 g$ is Hermitian. However, since h is also Ricci-flat, and hence Bach-flat, the conformal invariance of the latter condition guarantees that g is Bach-flat, too. This forces [13, 23] the Kähler metric g to be extremal in the sense of Calabi. However, since $\det(W^+) > 0$, the Kähler metric g also satisfies $s/6 = \alpha = f^{-1} > 0$. But $h = f^2 g$ is scalar-flat, whereas g has positive scalar curvature. The conformal factor $f = 6s^{-1}$, and hence the scalar curvature s of g, must therefore be non-constant. This shows that g must be a *strictly* extremal Kähler metric. In particular, $\xi := J\nabla s$ must be a non-trivial Killing field — not only for g, but also for its ξ-invariant conformal rescaling $h = 36 s^{-2} g$. □

Our goal is now to apply Theorem 1 to asymptotically locally flat (ALF) gravitational instantons, thereby putting the first two authors' main classification result [7, Theorem 8.2] in a new and broader context.

Definition 1. *Let (M, h) be a complete, Ricci-flat Riemannian 4-manifold (M, h). Then (M, h) will be called an* ALF gravitational instanton *if*

- *there is a compact subset $\mathfrak{C} \subset M$ such that $M - \mathfrak{C}$ is diffeomorphic to $\mathbb{R}^+ \times \Sigma$, where Σ^3 is oriented, and finitely covered by $S^2 \times S^1$ or S^3;*

- *Σ is equipped with a sign-ambiguous pair $\pm(T, \eta)$ of a vector field T and a 1-form η which satisfy $T \lrcorner\, \eta = 1$ and $T \lrcorner\, d\eta = 0$ on, at worst, a double cover of Σ;*

- *Σ is also equipped with a positive-semi-definite symmetric 2-tensor field $\gamma \in \Gamma(\odot^2 T^*\Sigma)$ such that $\mathcal{L}_T \gamma = 0$ and $\ker \gamma = \operatorname{span} T$, and which moreover locally defines a Gauss-curvature $+1$ metric on the space of leaves of the foliation tangent to T; and*

- *after pulling back via a suitable diffeomorphism $\mathbb{R}^+ \times \Sigma \to M - \mathfrak{C}$, the metric h takes the form*

  $$h = d\varrho^2 + \varrho^2 \gamma + \eta^2 + \mho,$$

  where the standard coordinate ϱ on $\mathbb{R}^+$ and the tensor fields γ and η^2 on Σ have been pulled back to $\mathbb{R}^+ \times \Sigma$ via the first- and second-factor projections, and where the error term $\mho$ satisfies the fall-off condition

  $$\mathbb{D}^j \mho = O(\varrho^{-1-j}) \tag{22}$$

  for $0 \leq j \leq 3$, where $\mathbb{D}$ denotes the Levi-Civita connection of the background metric $d\varrho^2 + \varrho^2\gamma + \eta^2$ on $\mathbb{R}^+ \times \Sigma$.

For the gravitational instantons that will primarily concern us here, T and η are both single-valued on Σ, without any need to pass to a double cover. However, Definition 1 has been carefully worded to avoid excluding e.g. the ALF hyper-Kähler gravitational instantons [10, 26] of type D_k. On the other hand, because Definition 1 assumes from the outset that (M^4, h) is complete and Ricci-flat, we have simplified [7, Definition 1.1] by assuming that M only has one end, since the Cheeger-Gromoll splitting theorem [9] would otherwise lead to a contradiction, e.g. by forcing (M^4, h) to be a flat Riemannian product $\mathbb{R} \times \Sigma$, even though Σ cannot admit flat metrics.

Now, in the spirit of our fall-off hypothesis (22), and after choosing some base-point $p \in M$, we will say that a C^k function or tensor field $\mho$ on an ALF gravitational instanton (M, h_0) is of weighted class C^k_1 if

$$\|\mho\|_{C^k_1} := \sup_M \sum_{j=0}^{k} (1 + \mathsf{dist})^{j+1} |\nabla^j \mho|_{h_0}$$

is finite, where ∇ denotes the Levi-Civita connection of h_0. We will also say that a second C^k metric h on M is within C^k_1-distance ε of h_0 if, in terms of this weighted norm, $\|h - h_0\|_{C^k_1} < \varepsilon$. Having fixed these conventions, we now formulate and prove a concrete implementation of Theorem 1:

Theorem A. *Let (M, h_0) be any toric, Hermitian, but non-Kähler ALF gravitational instanton. If h is another Ricci-flat Riemannian metric on M that is sufficiently C^3_1-close to h_0, then (M, h) is also a Hermitian ALF gravitational instanton, and carries a non-trivial Killing field ξ. Moreover, h is conformally related to a complete, strictly extremal Kähler metric g.*

Proof. By the main classification result [7, Theorem 8.2] of Biquard and Gauduchon, the given gravitational instanton h_0 must be a Kerr, Chen-Teo, Taub-bolt, or (reverse-oriented) Taub-NUT metric. Because these are all ALF gravitational instantons in the sense of Definition 1, their curvature tensors $\mathcal{R}$ satisfy $\mathcal{R} = O(\varrho^{-3})$ and $\nabla\mathcal{R} = O(\varrho^{-4})$. Moreover, each of these metrics is conformal to a Kähler metric $g_0 = u^{-2}h_0$, where the positive function u is related to the scalar curvature of g_0 by $u^{-1} = ks_{g_0}$ for some constant k. It follows that u is proper, because $s_{g_0} \to 0$ at infinity [7, §2]. This then implies that $s_{g_0} > 0$ everywhere[2]. Indeed, since $h_0 = u^2 g_0$ is Ricci-flat, and hence scalar-flat, the Yamabe equation tells us that

$$0 = s_{h_0}u^3 = (6\Delta_{g_0} + s_{g_0})u,$$

where $\Delta = d^*d = -\nabla \cdot \nabla$ is the geometric Laplacian of g_0. But since $u > 0$ is proper, it must have some minimum $p \in M$, and at p we therefore have

$$s_{g_0}(p) = 6u^{-1}(\nabla \cdot \nabla u)|_p = 6u^{-1}(\mathrm{Trace\,Hess}\, u)|_p \geq 0.$$

Since s_{g_0} is continuous and nowhere zero, this shows that $s_{g_0} > 0$ everywhere, as claimed. If we now choose the normalization $k = 1/6$, note that u then becomes the function $f_{g_0} = \alpha_{g_0}^{-1} = \alpha_{h_0}^{-1/3}$ assigned to h_0 by (7).

It follows that g_0 and h_0 both satisfy $\det(W^+) > 0$, and indeed that $\det(W^+_{h_0}) = \frac{1}{4}\alpha^3_{h_0} = \frac{1}{3\sqrt{6}}|W^+_{h_0}|^3 > 0$ everywhere. Meanwhile, the fall-off condition (22) implies that the Riemann tensor of h_0 satisfies

$$\mathcal{R} = O(\varrho^{-3}) \quad \text{and} \quad \nabla\mathcal{R} = O(\varrho^{-4}),$$

so we also, for instance, have $|W^+_{h_0}| = O(\varrho^{-3})$ in the end region. But in fact, $|W^+_{h_0}| = \sqrt{\frac{3}{2}}\alpha_{h_0} = \sqrt{\frac{3}{2}}\alpha^3_{g_0}$ actually has *precisely* ϱ^{-3} fall-off, because α_{g_0} is an affine defining function of the moment polygon's "edge at infinity," and so is asymptotically greater [7, p. 394] than a constant times ϱ^{-1}.

Now suppose that h is a second Ricci-flat metric on M which is close to h_0 in the C^3_1 sense. It then follows that h also satisfies the fall-off conditions

[2] The fact that $g_0 = g_K$ has positive scalar curvature was never made explicit in [7], but was, for example, implicit in the conclusion $A > 0$ of [7, Corollary 5.2].

(22) relative to the model metric $d\varrho^2 + \varrho^2\gamma + \eta^2$. In particular, (M,h) is itself an ALF gravitational instanton, and its curvature tensor also satisfies $\mathcal{R} = O(\varrho^{-3})$ and $\nabla\mathcal{R} = O(\varrho^{-4})$. Consequently, the self-dual Weyl curvature of h satisfies the fall-off conditions

$$|W_h^+| = O(\varrho^{-3}), \quad \nabla|W_h^+| = O(\varrho^{-4}),$$

and

$$\det(W_h^+) = O(\varrho^{-9}), \quad \nabla\det(W_h^+) = O(\varrho^{-10}).$$

Since these same fall-off rates apply to h_0, and since $3\sqrt{6}\det(W^+) = |W_{h_0}^+|^3$ is bounded above and below by positive constant multiples of $(1+\mathsf{dist})^{-9}$, we will automatically have $\det(W_h^+) > 0$ and $2|W_{h_0}^+| > |W_h^+| > \frac{1}{2}|W_{h_0}^+|$ if $\|h - h_0\|_{C_1^3} < \varepsilon$ for ε sufficiently small. Consequently, $\alpha_h > |W_h^+| > \frac{1}{2}|W_{h_0}^+|$ is then also larger than a positive constant times ϱ^{-3} when $\varrho \gg 0$.

Now, after writing (5) as

$$\left(\alpha^2 - \frac{1}{2}|W^+|^2\right)\alpha = \det(W^+)$$

and then putting the first derivative of this equation in the form

$$\left(3\alpha^2 - \frac{1}{2}|W^+|^2\right)\nabla\alpha = \frac{1}{2}\alpha\nabla|W^+|^2 + \nabla\det(W^+)$$

the fact that

$$\det(W^+) > 0 \quad\Longrightarrow\quad \frac{2}{3}|W^+|^2 \geq \alpha^2 > \frac{1}{2}|W^+|^2 \tag{23}$$

now yields *a priori* fall-off rates for α_h:

$$\alpha_h = O(\varrho^{-3}) \quad\text{and}\quad \nabla\alpha_h = O(\varrho^{-4}). \tag{24}$$

The function $\alpha_g := \alpha_h^{1/3}$ consequently has fall-off

$$\alpha_g = O(\varrho^{-1}), \quad |\nabla\alpha_g|_h = O(\varrho^{-2}) \tag{25}$$

and therefore belongs to C_1^1. Moreover, $\alpha_g = \alpha_h^{1/3}$ is also larger than a positive constant times ϱ^{-1} for all $\varrho \gg 0$.

Using this, we will now show that the scalar curvature s_g is an L^1 function on (M,g), or in other words that

$$\int_M |s_g|\, d\mu_g < \infty.$$

To see this, let us first observe that

$$|s_g| = 2s_+ - s,$$

where $s_+ = \max(s_g, 0)$ is the positive part of s_g. For я $\in \mathbb{R}^+$, now let $M_я \subset M$ be the compact 4-manifold-with-boundary that is the union of the subset $\varrho^{-1}((-\infty, я])$ of the end region and the compact complement $\mathfrak{C}$ of end. We then have

$$\int_{M_я} |s_g|\ d\mu_g \leq 2\int_{M_я} s_+\, d\mu_g + \left|\int_{M_я} s_g\ d\mu_g\right|.$$

It will thus suffice to show that both $\int_{M_я} s_+ d\mu_g$ and $|\int_{M_я} s_g d\mu_g|$ remain uniformly bounded as я $\to \infty$.

We begin by considering $\int s_g d\mu_g$, which we will analyze by means of the Yamabe equation

$$s_g\ d\mu_g = 6\alpha_g(\Delta_h\alpha_g)\ d\mu_h,$$

incorporating the facts that $s_h = 0$, $g = \alpha_g^2 h$, and $d\mu_g = \alpha_g^4 d\mu_h$. Thus

$$\int_{M_я} s_g d\mu_g = 6\int_{M_я} |\nabla\alpha_g|_h^2\ d\mu_h - 6\int_{\partial M_я} \alpha_g(\nabla_\nu\alpha_g)\ d\check{\mu}_h,$$

and hence

$$\left|\int_{M_я} s_g\ d\mu_g\right| \leq 6\int_{M_я} |\nabla\alpha_g|_h^2\ d\mu_h + 6\int_{\partial M_я} \alpha_g\,|\nabla\alpha_g|_h\ d\check{\mu}_h,$$

where $d\check{\mu}_h$ is the volume 3-form on $\partial M_я$ induced by the restriction of h, and where ν denotes the outward-pointing unit normal of $\partial M_я$ with respect to h. On the other hand, since (M, h) is an ALF gravitational instanton, in the sense of Definition 1, the metric has the asymptotic form

$$h = d\varrho^2 + \varrho^2\gamma + \eta^2 + O(\varrho^{-1}).$$

Thus, in the end region, our fall-off (25) on α_g guarantees that

$$\begin{aligned} |\nabla\alpha_g|_h^2\ d\mu_h &\leq \mathsf{B}\varrho^{-2}|d\rho\wedge\Omega\wedge\eta| \\ \alpha_g|\nabla\alpha_g|_h\ d\check{\mu}_h &\leq \mathsf{B}\varrho^{-1}\ |\Omega\wedge\eta| \end{aligned}$$

for all $\varrho \gg 0$, where Ω is the area form of γ, and B is a sufficiently large constant. Thus $|\int_{M_я} s_g d\mu_g|$ behaves like $\mathsf{const} + O(я^{-1})$ for я $\gg 0$, and is therefore uniformly bounded in я.

On the other hand, since $g = \alpha_g^2 h$, and since, after possibly increasing the size of B, $\alpha_g < \frac{1}{2}\mathsf{B}\varrho^{-1}$ in the asymptotic region by (25), we also have

$$g < \mathsf{B}^2 \left[\gamma + \frac{d\varrho^2 + \eta^2}{\varrho^2}\right] \tag{26}$$

for all $\varrho > \varrho_0$, where the pointwise inequality is to be understood in the sense of quadratic forms. In particular, the 4-dimensional volume measure of g satisfies

$$d\mu_g < \mathsf{B}^4 \varrho^{-2} |d\varrho \wedge \Omega \wedge \eta|$$

for all $\varrho > \varrho_0$. On the other hand, Lemma 4 guarantees that

$$2\sqrt{6}\ |W_g^+| \geq \max(s_g, 0) = s_+$$

at every point. When combined with (23), this then implies that

$$4\sqrt{3}\ \alpha_g \geq s_+$$

everywhere. Since $\alpha_g < \frac{1}{2}\mathsf{B}\varrho^{-1}$ in the asymptotic region, we therefore have

$$s_+ d\mu_g \leq 2\sqrt{3}\ \mathsf{B}^5\ \varrho^{-3} |d\varrho \wedge \Omega \wedge \eta|$$

for all $\varrho > \varrho_0$. It follows that $\int_{M_я} s_+ d\mu_g$ behaves like $\mathsf{const} + O(я^{-2})$ for $я \gg 0$, and is therefore uniformly bounded in $я$. Since $|\int_{M_я} s_g d\mu_g|$ has also been shown to be uniformly bounded, we thus conclude that $\int_{M_я} |s_g|\ d\mu_g$ is uniformly bounded, too. Hence

$$\int_M |s_g|\ d\mu_g = \sup_я \int_{M_я} |s_g|\ d\mu_g < \infty,$$

as claimed.

However, since we also know that $\alpha_g > 2\mathsf{b}\ \varrho^{-1}$ when $\varrho > \varrho_0$, for some positive constant b, we also have

$$g > \mathsf{b}^2 \left[\gamma + \frac{d\varrho^2 + \eta^2}{\varrho^2}\right]$$

for all $\varrho > \varrho_0$, so the 4-dimensional volume measure $d\mu_g$ and the 3-dimensional volume measure $d\check{\mu}_g$ of the $\varrho = \mathsf{const}$ hypersurfaces jointly satisfy

$$d\mu_g > \mathsf{A}\varrho^{-1} |d\varrho \wedge d\check{\mu}_g| = |dt \wedge d\check{\mu}_g|,$$

for all $\varrho > \varrho_0$, where $\mathsf{A} = \mathsf{b}^4/\mathsf{B}^3$, and where we have set $t := \mathsf{A}\log\varrho$. For each t, we now let $\Sigma_t \approx \Sigma$ be the level-set $\varrho = e^{t/\mathsf{A}}$, and then define

$$\mathcal{F}(t) = \int_{\Sigma_t} |s_g| \; d\check{\mu}_g.$$

Setting $t_0 = \mathsf{A}\log\varrho_0$, we then have

$$\int_{t_0}^{\infty} \mathcal{F}(t)\; dt < \int_{\varrho\geq\varrho_0} |s_g| \; d\mu_g < \int_M |s_g| \; d\mu_g < \infty.$$

Since $\mathcal{F}(t)$ is a continuous positive function, this means that there must be an increasing sequence $t_j \to \infty$ with $\mathcal{F}(t_j) \to 0$. Setting $я_j = e^{t_j/\mathsf{A}}$ and defining U_j to be the interior of $M_{я_j}$ thus defines an exhaustion of $M = \cup_j U_j$ by nested pre-compact domains

$$U_1 \Subset U_2 \Subset \cdots \Subset U_j \Subset \cdots$$

with smooth boundary such that condition (20) is satisfied:

$$\lim_{j\to\infty} \int_{\partial U_j} |s_g| \; d\check{\mu}_g = 0.$$

On the other hand, since $\alpha_g < \frac{1}{2}\mathsf{B}\varrho^{-1}$, inequalities (23) and (26) tell us that

$$\begin{aligned} |d\check{\mu}_g| &< \mathsf{B}^3\varrho^{-1}|\Omega\wedge\eta| \\ |W_g^+|\,|d\check{\mu}_g| &< \mathsf{B}^4\varrho^{-2}|\Omega\wedge\eta| \end{aligned}$$

for all $\varrho > \varrho_0$, so there is a positive constant C such that

$$\begin{aligned} \int_{\partial U_j} 1 \; d\check{\mu}_g &< \frac{\mathsf{C}}{я_j} \\ \int_{\partial U_j} |W_g^+| \; d\check{\mu}_g &< \frac{\mathsf{C}}{я_j^2} \end{aligned}$$

for all $j \gg 0$. Since $\lim_{j\to\infty} я_j = \infty$, this shows that conditions (18) and (19) are also satisfied by our exhaustion U_j of (M, g). Theorem 1 therefore tells us that (M, g) is a strictly extremal Kähler manifold, and that the ALF gravitational instanton (M, h) is consequently Hermitian, and carries a non-trivial Killing field ξ, as claimed. $\square$

4 Rigidity Results

We will now use Theorem A in conjunction with the Biquard-Gauduchon classification [7, Theorem 8.2] to prove various rigidity results. To do this, however, we will first need to check that these two machines actually mesh correctly.

Lemma 5. *For (M,h) as in Theorem A, the limit at infinity of the Killing field ξ is a non-zero constant multiple of the vector field T on Σ. In particular, the action on Σ induced by ξ at infinity preserves the triple (T,η,γ).*

Proof. By construction, the Killing field ξ is a given by

$$\xi^a = J_b{}^a g^{bc}\nabla_c s_g = 6J_b{}^a g^{bc}\nabla_c\alpha_g = 6J_b{}^a f^2 h^{bc}\nabla_c f^{-1} = -6J_b{}^a h^{bc}\nabla_c f,$$

where $f = \alpha_g^{-1}$. Our asymptotics for α guarantee that, for $\varrho \gg 0$, α_g is bigger than a positive constant times ϱ^{-1}, and that $|\nabla\alpha_g|_h$ is less than a constant times ϱ^{-2}, so it follows that $|\nabla f|_h = \alpha_g^{-2}|\nabla\alpha_g|_h$ is uniformly bounded. It therefore follows that $|\xi| = 6|\nabla f|_h$ is uniformly bounded, too.

On the other hand, since $\alpha_g \to 0$ at infinity, $f = \alpha_g^{-1}$ is a smooth proper function on M, and therefore achieves its minimum at some $p \in M$. Since $|df|_p = 0$, it therefore follows that $\xi = -6J\nabla f$ has a zero at p. The flow of the Killing field ξ therefore preserves the distance to p, so it follows that ξ is orthogonal to every geodesic passing through this base-point p.

However, because ξ is a Killing field, it automatically satisfies

$$\nabla_a\nabla_b\xi^c = \mathcal{R}^c{}_{bad}\xi^d \tag{27}$$

on (M,h), primarily as a reflection of the fact that its restriction to any geodesic is a Jacobi field. Since $|\xi|_h$ is uniformly bounded and $|\mathcal{R}|_h = O(\varrho^{-3})$, it therefore follows that $\nabla\nabla\xi = O(\varrho^{-3})$. On the other hand, the asymptotic local model metric $d\varrho^2+\varrho^2\gamma+\eta^2$ has a Riemannian submersion to Euclidean $\mathbb{R}^3$ whose fibers are tangent to T. By restricting this equation to geodesics and integrating, it therefore follows that, along any slice transverse to T, the projection $\xi \bmod T$ differs from an affine-linear function $\mathbb{R}^3 \to \mathbb{R}^3$ by terms of order ϱ^{-1}; and since $\xi \bmod T$ is uniformly bounded, it therefore follows that $\xi \bmod T$ actually tends to a constant (i.e. parallel) vector field on Euclidean $\mathbb{R}^3$. But since ξ is orthogonal to every geodesic through p, and since the tangent directions of such geodesics project to an open cone in $\mathbb{R}^3$, this leads to an immediate contradiction unless this constant field is zero. This shows that $\xi = uT + \mho$ for some smooth bounded function u, where the error term $\mho$ satisfies $\mho = O(\varrho^{-1})$ and $\nabla\mho = O(\varrho^{-2})$. However, because

T is a bounded Killing field for the model metric $d\varrho^2 + \varrho^2\gamma + \eta^2$, our fall-off hypothesis (22) guarantees that $\nabla_{(a}T_{b)} = O(\varrho^{-2})$ with respect to h. Hence

$$0 = \nabla_{(a}\xi_{b)} = \nabla_{(a}uT_{b)} + O(\varrho^{-2}) = T_{(a}\nabla_{b)}u + O(\varrho^{-2})$$

and we therefore must have $u = \text{const} + O(\varrho^{-1})$. Moreover, the relevant constant must be non-zero, because $|\xi| = 6|\nabla f|$, and (25) forces $f = \alpha^{-1}$ to be asymptotically bigger than some positive constant times ϱ. Thus, some constant multiple $\widehat{\xi}$ of ξ satisfies $\widehat{\xi} = T + O(\varrho^{-1})$. In particular, the action on Σ induced at infinity by ξ coincides, up to reparameterization, with the action of T, and so preserves (T, η, γ), as claimed. $\square$

Corollary 1. *Let (M, h) be as described by Theorem A, and suppose that that there is an isometric action of the 2-torus $\mathbb{T}^2$ on (M, h), such that the constructed Killing field ξ arises from some element of the Lie algebra $\mathfrak{t}^2$ of $\mathbb{T}^2$. Then (M, h) is a non-Kähler toric Ricci-flat Hermitian ALF manifold, in the precise technical sense required by Biquard and Gauduchon.*

Proof. Lemma 5 guarantees that there is an element $\hat{\xi} \in \mathfrak{t}^2$ whose action at infinity coincides with the flow of the vector field T on Σ, exactly as required by [7, Definition 1.2]. $\square$

With these basic facts in hand, we now prove our first rigidity result.

Theorem 2. *Let (M, h) be as described by Theorem A, and suppose that the constructed Killing field ξ is not periodic. Then (M, h) is one of the toric ALF gravitational instantons classified by Biquard and Gauduchon [7].*

Proof. The action of the identity component $\mathrm{Iso}_0(M, h)$ of the isometry group preserves the self-dual Weyl curvature W_h^+, and hence its top eigenvalue $\alpha_h : M \to \mathbb{R}^+$, and hence the smooth proper function $f = \alpha_h^{-1/3}$. The proof of Lemma 5 shows that $|\nabla f|$ tends to a non-zero constant at infinity, so the set X of critical points of f is therefore compact. However, this X is the zero set of both the Killing field $\xi = -6J\nabla f$ and the holomorphic vector field $\xi^{1,0}$. Each connected component of the $\mathrm{Iso}_0(M, h)$-invariant subset $X \subset M$ is therefore either a point or a totally geodesic compact complex curve. Since we also know that $\chi(X) = \chi(M) > 0$, it follows that some component of X_0 of X, and hence some orbit $Y \subset X_0$, is either a point or a $\mathbb{CP}_1$. Equivariance of the exponential map from the normal bundle of Y to M then guarantees that this gives us a faithful representation $\mathrm{Iso}_0(M, h) \hookrightarrow \mathbf{U}(2)/\mathbb{Z}_\ell$ of the isometry group into some finite quotient of $\mathbf{U}(2)$.

Let us now consider the non-trivial Killing field ξ as an element of the Lie algebra $\mathfrak{iso}_0(M,h)$, and examine the 1-parameter subgroup $\exp(\mathbb{R}\xi) = \{\exp(t\xi) \mid t \in \mathbb{R}\} \subset \mathrm{Iso}_0(M,h)$ that it determines. The closure of this subgroup is then compact, Abelian, and connected, and so is a torus $\mathbb{T} = \overline{\exp(\mathbb{R}\xi)} \subset \mathrm{Iso}_0(M,h) \subset \mathbf{U}(2)/\mathbb{Z}_\ell$. If ξ is periodic, this torus will just be a circle. Otherwise, $\mathbb{T}$ must be a 2-torus, since $\mathbf{U}(2)$ has rank 2. In the latter case, the ALF gravitational instanton (M,h) then becomes Hermitian, non-Kähler, and toric, and so, by Corollary 1, falls within the purview of the Biquard-Gauduchon classification [7, Theorem 8.2]. □

Corollary 2. *Let (M,h_0) be a toric Hermitian ALF gravitational instanton for which the corresponding vector field T on Σ is not periodic. Then any Ricci-flat metric h on M which is sufficiently C^3_1 close to h_0 must be one of the toric ALF gravitational instantons classified by Biquard-Gauduchon.*

Proof. The proof of Lemma 5 shows that the vector field T arising from h_0 is also the limit at infinity of a constant multiple the constructed Killing field ξ of any C^3_1-close Ricci-flat metric h. If T is not periodic, it thus follows that the constructed Killing field ξ of h cannot be periodic, either. The claim is therefore an immediate consequence of Theorem 2. □

A theorem of Aksteiner, Andersson, Dahl, Nilsson, and Simon [2] classifies those ALF gravitational instantons that carry an isometric S^1-action and are diffeomorphic to either a Kerr or a Taub-bolt space. Quoting their result in conjunction with Theorem A would now allow us to prove a rigidity result in these two cases. However, we will instead buttress the claims of [2] by proving this rigidity theorem in a self-contained way, by building directly on the results already obtained in this article.

Theorem B. *Let (M,h_0) be a Kerr or Taub-bolt gravitational instanton, and let h be another Ricci-flat metric on M that is sufficiently C^3_1-close to h_0. Then (M,h) is once again a Kerr or Taub-bolt gravitational instanton.*

Proof. If (M,h_0) is Taub-bolt or belongs to the Kerr family, then both (M,h_0) and its reverse-oriented version $(\overline{M},h_0)$ are non-Kähler and Hermitian toric ALF, and so, by Corollary 1, both fall under the purview of the Biquard-Gauduchon classification. In particular, (M,h_0) then satisfies both $\det(W^+) > 0$ and $\det(W^-) > 0$, and one can therefore apply Theorem A with respect to either orientation of M. If h is a Ricci-flat metric on M that is sufficiently C^3_1 close to h_0, one therefore deduces that M admits a pair of complex structures $\{J_+, J_-\}$ that are respectively compatible with the two different orientations of M, and a pair of extremal Kähler metrics $\{g_+, g_-\}$

that are compatible with $J_\pm$, respectively, where $h = \alpha_\pm^{-2} g_\pm$ with $\alpha_\pm > 0$ the top eigenvalues of $W^\pm_{g_\pm}$, respectively, which is to say that $\alpha^3_\pm$ are the top eigenvalues of $W^\pm_h$. In the terminology of [4], the Riemannian manifold (M, h) is therefore ambi-Kähler. By [4, Proposition 12], it therefore follows that the Bach-flat manifold (M, h) is at least *locally ambitoric*, and what follows is simply a verification that this conclusion actually follows *globally* in our case.

Indeed, since the conformal class $[h]$ is Bach-flat, both of the Kähler metrics $g_\pm$ must be extremal, and this then gives rise to two non-trivial Killing fields $J_\pm \nabla^{g_\pm} \alpha_\pm$ on (M, h). If these Killing fields are linearly independent, (M, h) is toric, and we are done. On the other hand, if they are linearly dependent, one can produce a second Killing field using case (iii) of the proof of [4, Proposition 11]. Indeed, let $\xi = J_+ \nabla^{g_+} \alpha_+$ be the Killing field associated with g_+. After multiplying g_- by a suitable constant, and replacing J_- with $-J_-$ if necessary, we may then arrange to also have $\xi = J_- \nabla^{g_-} \alpha_-$. Now define $\mathscr{S} \in \mathrm{End}(TM)$ by

$$\mathscr{S} = \frac{1}{2}\left(\frac{1}{\alpha_+^2} + \frac{1}{\alpha_-^2}\right) I + \frac{1}{\alpha_+ \alpha_-} J_+ \circ J_- .$$

It then follows from [4, Appendix B.5] that $g(\mathscr{S}\cdot, \cdot)$ is a Killing tensor, and that $\mathscr{S}(\xi)$ is therefore a Killing field that commutes with ξ. If $\mathscr{S}(\xi)$ is not the zero field, then (M, h) toric. Otherwise, by [4, Proposition 12], (M, g, J_+) is the product of two extremal Kähler curves, one of which has constant curvature, and our asymptotics then guarantee that the latter curve is moreover a round 2-sphere. In every possible case, (M, h) is therefore toric, and, in light of Corollary 1, the Biquard-Gauduchon classification [7] therefore applies. The diffeotype of M therefore forces (M, h) either to be Taub-bolt, or to belong to the Kerr family. □

Added note. Contemporaneously with the appearance of the first version of this article on the arXiv, an e-print by Mingyang Li [25] announced a proof that Hermitian ALF gravitational instantons are always toric. If we take this result for granted, Theorem B can then be improved to just assume that (M, h_0) is a gravitational instanton appearing in the Biquard-Gauduchon classification, and then conclude that (M, h) must also belong to the same family. Of course, in light of Corollary 2, the gist of this improvement only concerns the case when T is *periodic*. In this periodic case, Li's proof begins by compactifying (M, J) as an orbifold complex surface, and then proceeds to deduce properties of the isometry group of (M, h) from properties of the complex automorphism group of the compactification.

References

[1] S. Aksteiner and L. Andersson, *Gravitational instantons and special geometry.* e-print arXiv:2112.11863 [gr-qc], 2021.

[2] S. Aksteiner, L. Andersson, M. Dahl, G. Nilsson, and W. Simon, *Gravitational instantons with S^1 symmetry.* e-print arXiv:2306.14567 [math.DG], 2023.

[3] M. T. Anderson, P. B. Kronheimer, and C. LeBrun, *Complete Ricci-flat Kähler manifolds of infinite topological type*, Comm. Math. Phys., 125 (1989), pp. 637–642.

[4] V. Apostolov, D. M. J. Calderbank, and P. Gauduchon, *Ambitoric geometry I: Einstein metrics and extremal ambikähler structures*, J. Reine Angew. Math., 721 (2016), pp. 109–147.

[5] A. L. Besse, *Géométrie riemannienne en dimension 4*, vol. 3 of Textes Mathématiques [Mathematical Texts], CEDIC, Paris, 1981.

[6] ——, *Einstein Manifolds*, vol. 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

[7] O. Biquard and P. Gauduchon, *On toric Hermitian ALF gravitational instantons*, Comm. Math. Phys., 399 (2023), pp. 389–422.

[8] E. Calabi, *Isometric families of Kähler structures*, in The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), Springer, New York, 1980, pp. 23–39.

[9] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom., 6 (1971), pp. 119–128.

[10] G. Chen and X.X. Chen, *Gravitational instantons with faster than quadratic curvature decay. I*, Acta Math., 227 (2021), pp. 263–307.

[11] Y. Chen and E. Teo, *A new AF gravitational instanton*, Phys. Lett. B, 703 (2011), pp. 359–362.

[12] S. A. Cherkis and N. J. Hitchin, *Gravitational instantons of type D_k*, Comm. Math. Phys., 260 (2005), pp. 299–317.

[13] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math., 49 (1983), pp. 405–433.

[14] T. Eguchi and A. J. Hanson, *Self-dual solutions to Euclidean gravity*, Ann. Physics, 120 (1979), pp. 82–106.

[15] G. W. Gibbons and S. W. Hawking, *Classification of gravitational instanton symmetries*, Comm. Math. Phys., 66 (1979), pp. 291–310.

[16] S. W. Hawking, *Gravitational instantons*, Phys. Lett. A, 60 (1977), pp. 81–83.

[17] H.-J. Hein, *Gravitational instantons from rational elliptic surfaces*, J. Amer. Math. Soc., 25 (2012), pp. 355–393.

[18] H.-J. Hein, S. Sun, J. Viaclovsky, and R. Zhang, *Nilpotent structures and collapsing Ricci-flat metrics on the K3 surface*, J. Amer. Math. Soc., 35 (2022), pp. 123–209.

[19] N. J. Hitchin, *Polygons and gravitons*, Math. Proc. Cambridge Philos. Soc., 85 (1979), pp. 465–476.

[20] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys., 108 (1987), pp. 535–589.

[21] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom., 29 (1989), pp. 665–683.

[22] ——, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom., 29 (1989), pp. 685–697.

[23] C. LeBrun, *Bach-flat Kähler surfaces*, J. Geom. Anal., 30 (2020), pp. 2491–2514.

[24] ——, *Einstein manifolds, self-dual Weyl curvature, and conformally Kähler geometry*, Math. Res. Lett., 28 (2021), pp. 127–144.

[25] M. Li, *Classification results for Hermitian non-Kähler gravitational instantons.* e-print arXiv:2310.13197 [math.DG], 2023.

[26] V. Minerbe, *On the asymptotic geometry of gravitational instantons*, Ann. Sci. Éc. Norm. Supér. (4), 43 (2010), pp. 883–924.

[27] ——, *Rigidity for multi-Taub-NUT metrics*, J. Reine Angew. Math., 656 (2011), pp. 47–58.

[28] R. Penrose, *Nonlinear gravitons and curved twistor theory*, General Relativity and Gravitation, 7 (1976), pp. 31–52.

[29] R. Penrose and W. Rindler, *Spinors and space-time. Vol. 2*, Cambridge University Press, Cambridge, 1986. Spinor and twistor methods in space-time geometry.

[30] S. Sun and R. Zhang, *Collapsing geometry of hyperkähler 4-manifolds and applications.* e-print arXiv:2108.12991v2 [math.DG], 2022.

[31] I. Şuvaina, *ALE Ricci-flat Kähler metrics and deformations of quotient surface singularities*, Ann. Global Anal. Geom., 41 (2012), pp. 109–123.

[32] P. Tod, *One-sided type-D Ricci-flat metrics.* e-print arXiv:2003.03234 [qr-gq], 2021.

[33] E. P. Wright, *Quotients of gravitational instantons*, Ann. Global Anal. Geom., 41 (2012), pp. 91–108.

[34] P. Wu, *Einstein four-manifolds with self-dual Weyl curvature of nonnegative determinant*, Int. Math. Res. Not. IMRN, (2021), pp. 1043–1054.

Acknowledgements. All three authors are grateful for the hospitality of the Institut Mittag-Leffler during its workshop, Einstein Spaces and Special Geometry, where our collaboration on this project began. The third author would also like to thank Lars Andersson, Vestislav Apostolov, Mingyang Li, Song Sun, and Edward Witten for useful discussions of the problem.

Olivier Biquard
Sorbonne Université & Université Paris Cité,
CNRS, IMJ-PRG, F-75005 Paris
Email address: olivier.biquard@sorbonne-universite.fr

Paul Gauduchon
École Polytechniqe, CNRS, CMLS, F-91120 Palaiseau
Email address: paul.gauduchon@polytechnique.fr

Claude LeBrun
Stony Brook University (SUNY), Stony Brook, NY 11794-3651
Email address: claude@math.stonybrook.edu