Gravitational Instantons,

Weyl Curvature, &

Conformally Kähler Geometry

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Differential Geometry Seminar
University of California, Berkeley
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Joint work with

Joint work with

Olivier Biquard

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Olivier Biquard
Sorbonne Université

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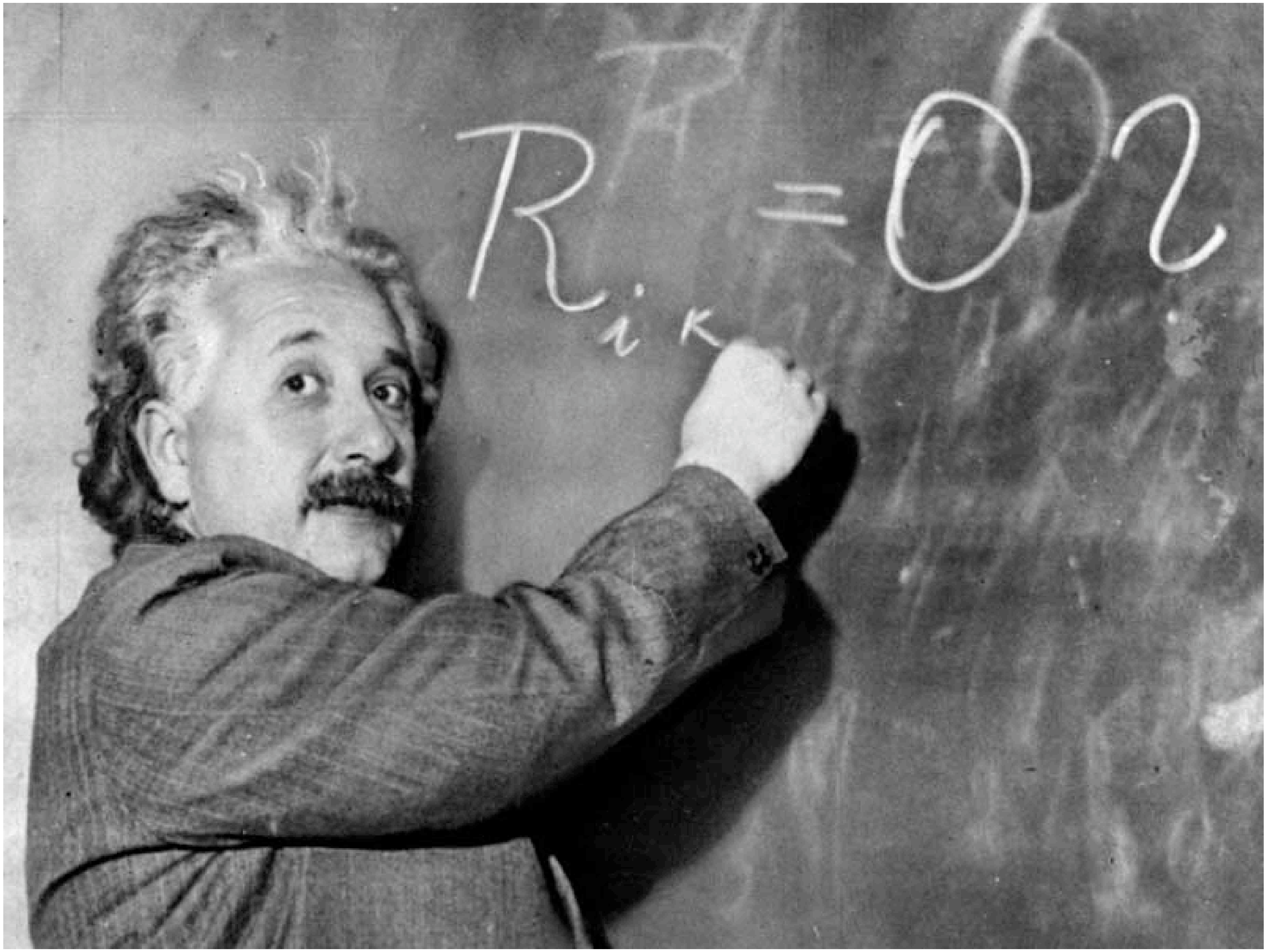

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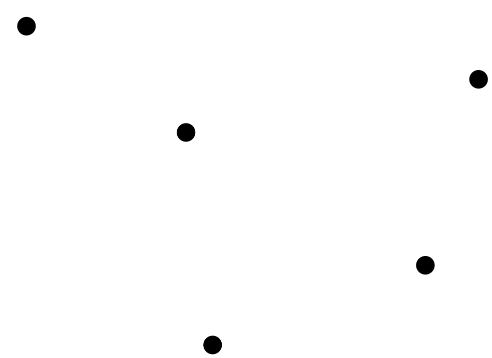

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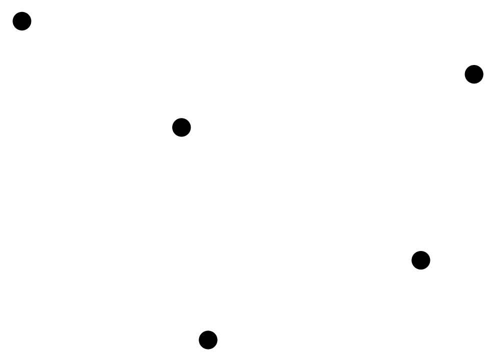

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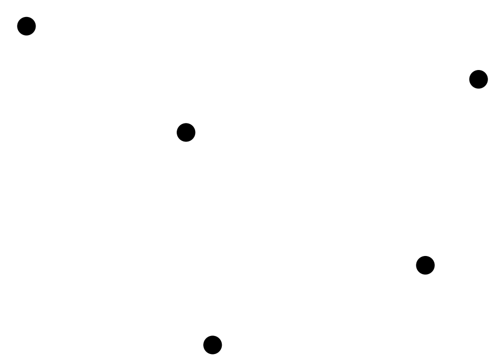

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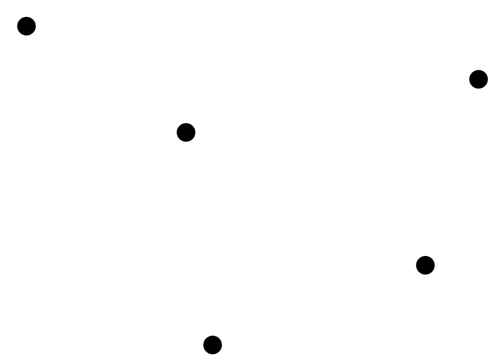

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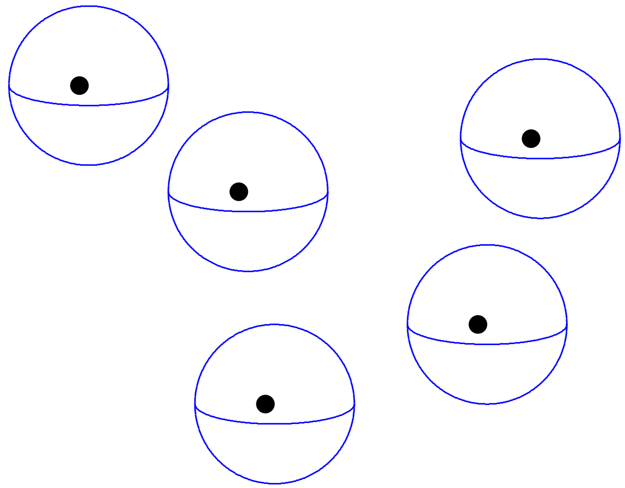

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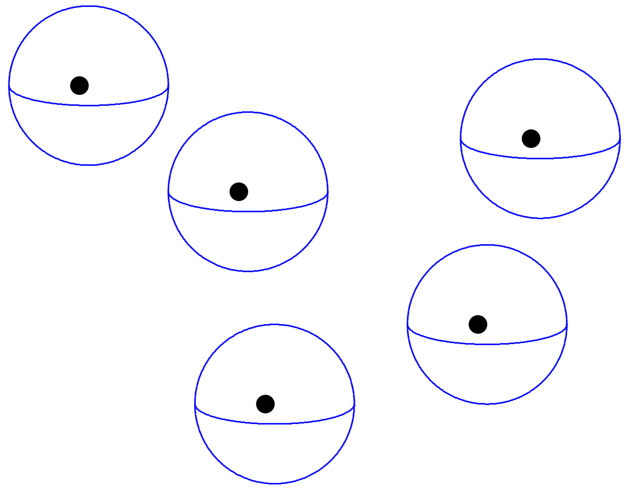

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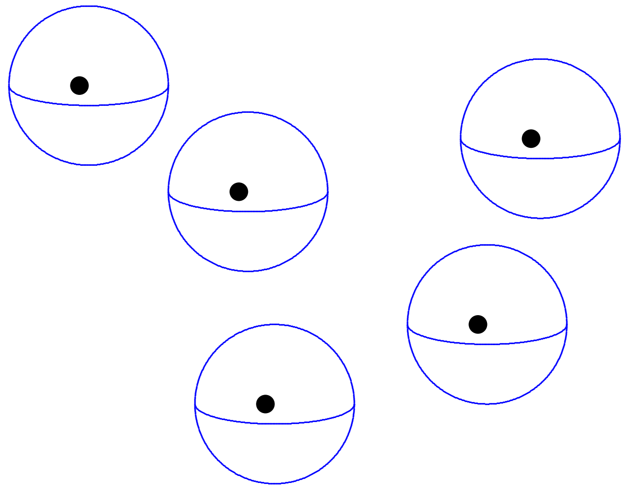

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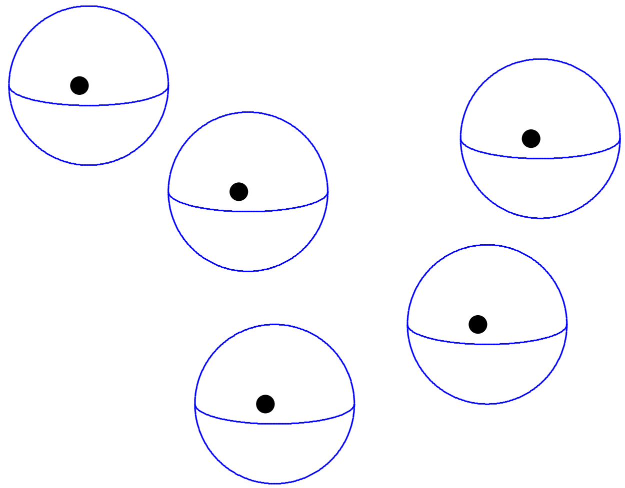

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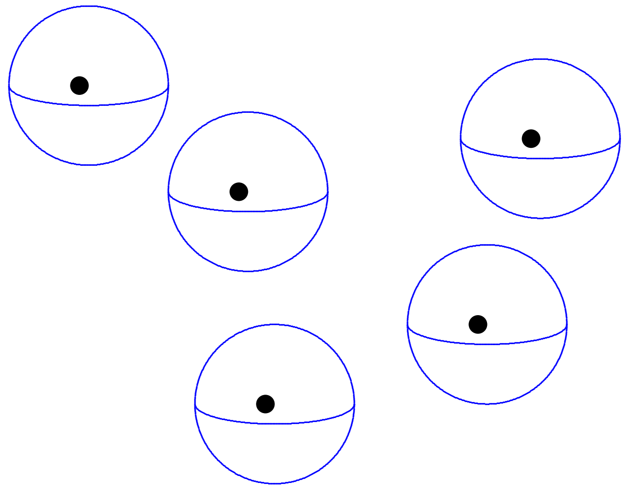

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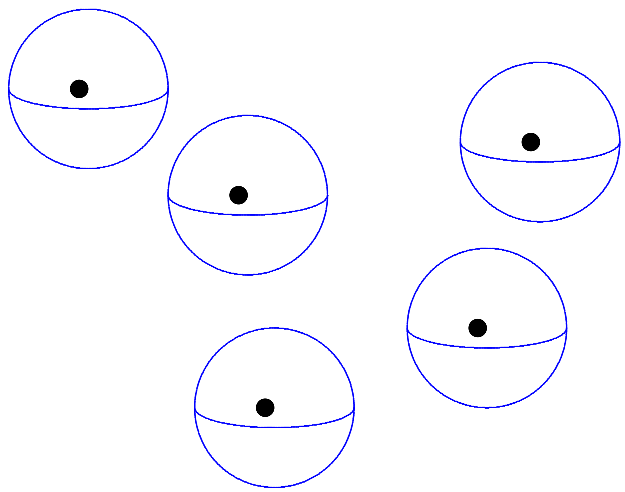

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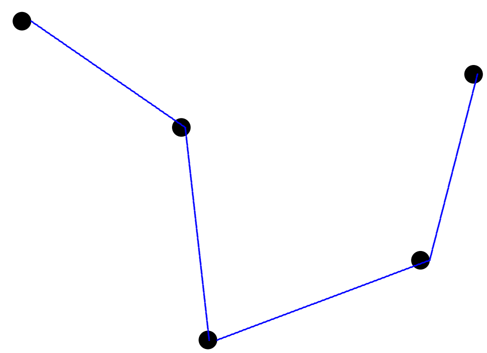

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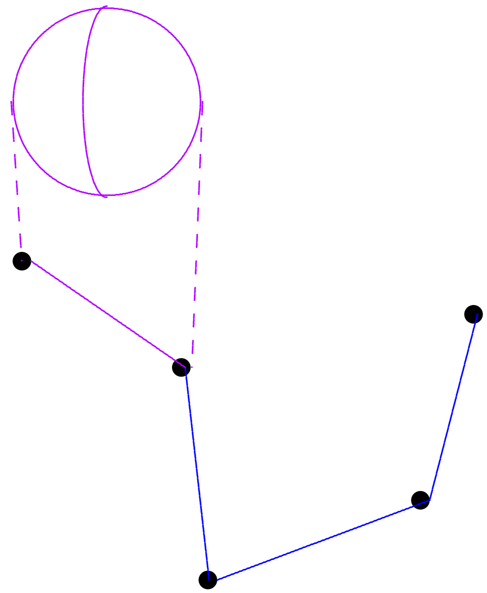

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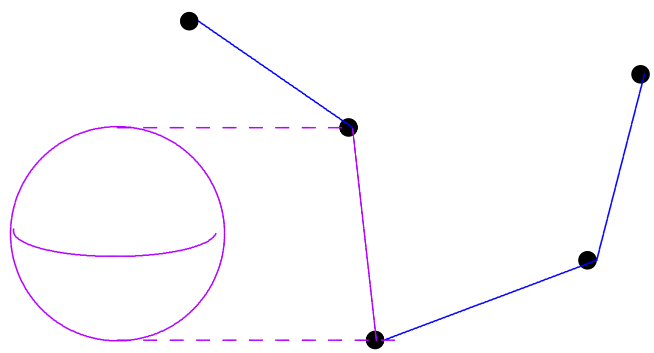

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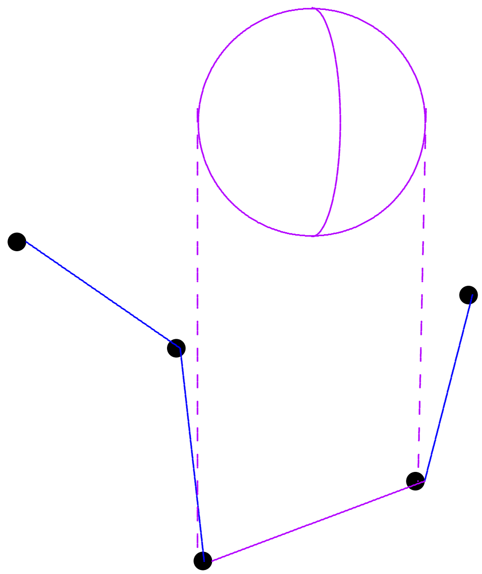

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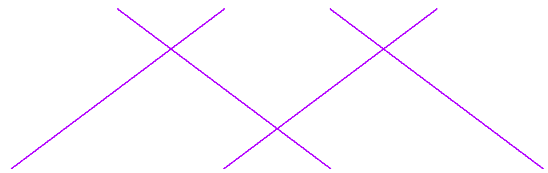

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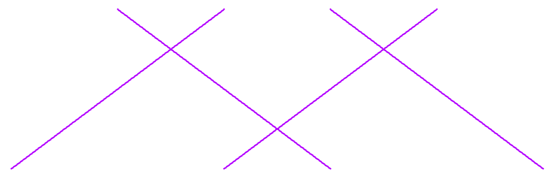

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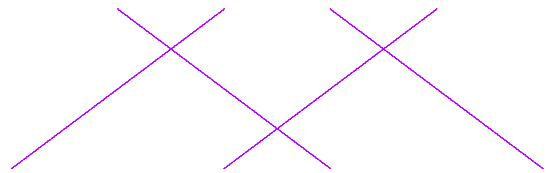

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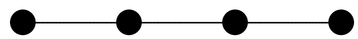

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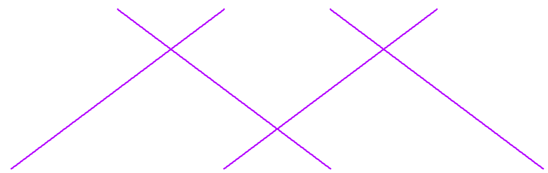

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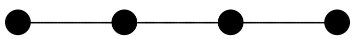

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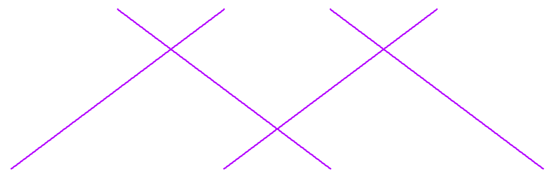

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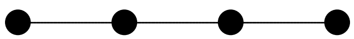

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Plumb together k copies of T^*S^2
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cf. Bishop-Gromov inequality!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, . . .

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Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

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Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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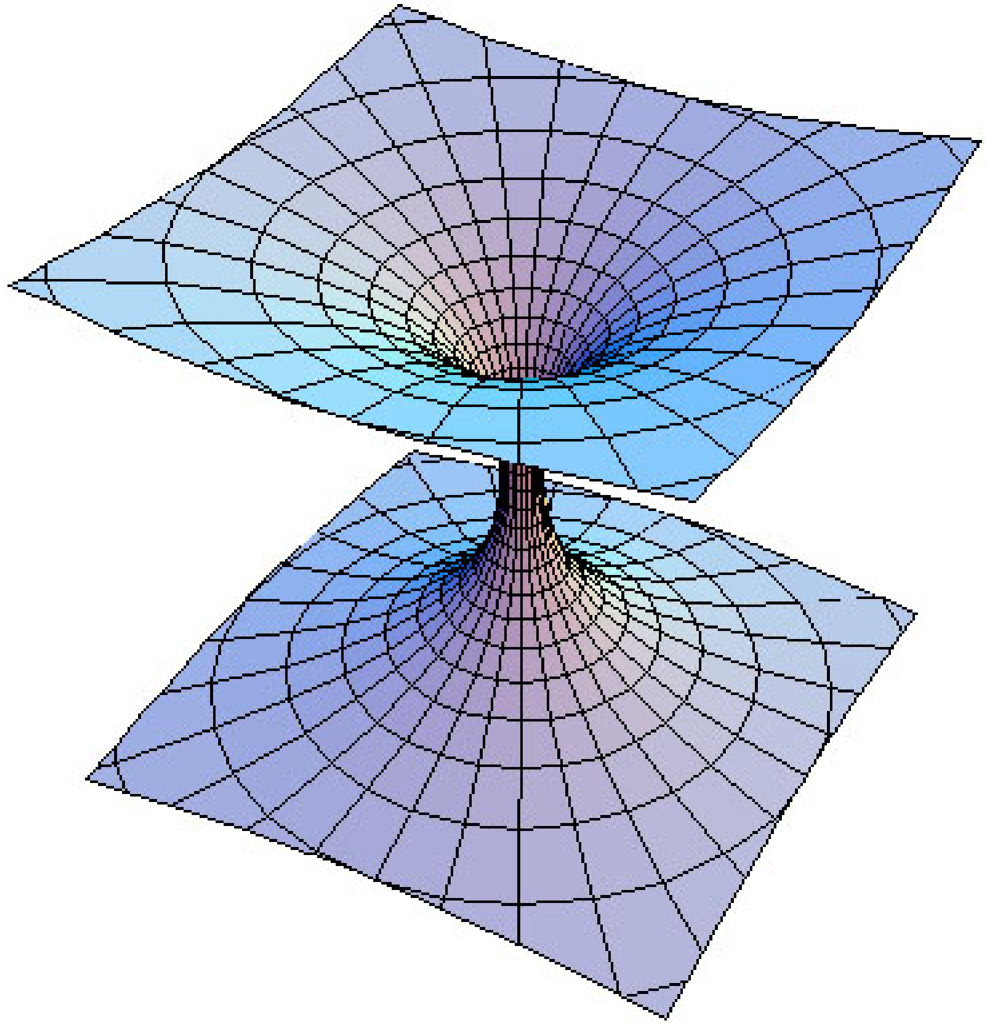
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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Hitchin, Kronheimer, Cherkis-Hitchin, Minerbe, Hein, Chen-Chen, Hein-Sun-Viaclovsky-Zhang. . .

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This might lend some credence to the aphorism. . .

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Y. Chen & E. Teo, 2011

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It is false in all higher dimensions!

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$$[g] := \{u^2 g \mid u : M \to \mathbb{R}^+\}.$$

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$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

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Then (M, h) *is an extremal Kähler manifold with non-constant, positive scalar curvature.*

Theorem A. *Let* (M, g_0) *be one of the ALF toric Hermitian gravitational instantons featured in Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M *which is sufficiently* C^3_1*-close to* g

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However, if g_0 is Kerr or Taub-bolt, we can prove a more definitive rigidity result, because g_0 then has both $\det(W^+) > 0$ and $\det(W^-) > 0$.

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Theorem B. *Let (M, g_0) be a Taub-bolt or Kerr gravitational instanton, and let g be another Ricci-flat metric on M that is sufficiently C^3_1-close to g_0. Then (M, g) is once again a Taub-bolt or Kerr gravitational instanton.*

M admits complex structures $J_\pm$ for 2 orientations,

and Kähler metrics $h_\pm$ in conformal class $[g]$.

Apostolov-Calderbank-Gauduchon '16:

"ambi-Kähler" structure.

Ambi-Kähler Einstein metrics are ambi-toric!

Theorem B. *Let* (M, g_0) *be a Taub-bolt or Kerr gravitational instanton, and let* g *be another Ricci-flat metric on* M *that is sufficiently* C^3_1*-close to* g_0*. Then* (M, g) *is once again a Taub-bolt or Kerr gravitational instanton.*

Thank you for inviting me!

It's a pleasure to be here!