Desingularizations of

Conformally Kähler,

Einstein Orbifolds

Claude LeBrun
Stony Brook University

Kähler Geometry: Past, Present, and Future.
Banff International Research Station.
Banff, Alberta, Canada. September 22, 2025

Joint work with

Joint work with

Tristan Ozuch

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Massachusetts Institute of Technology

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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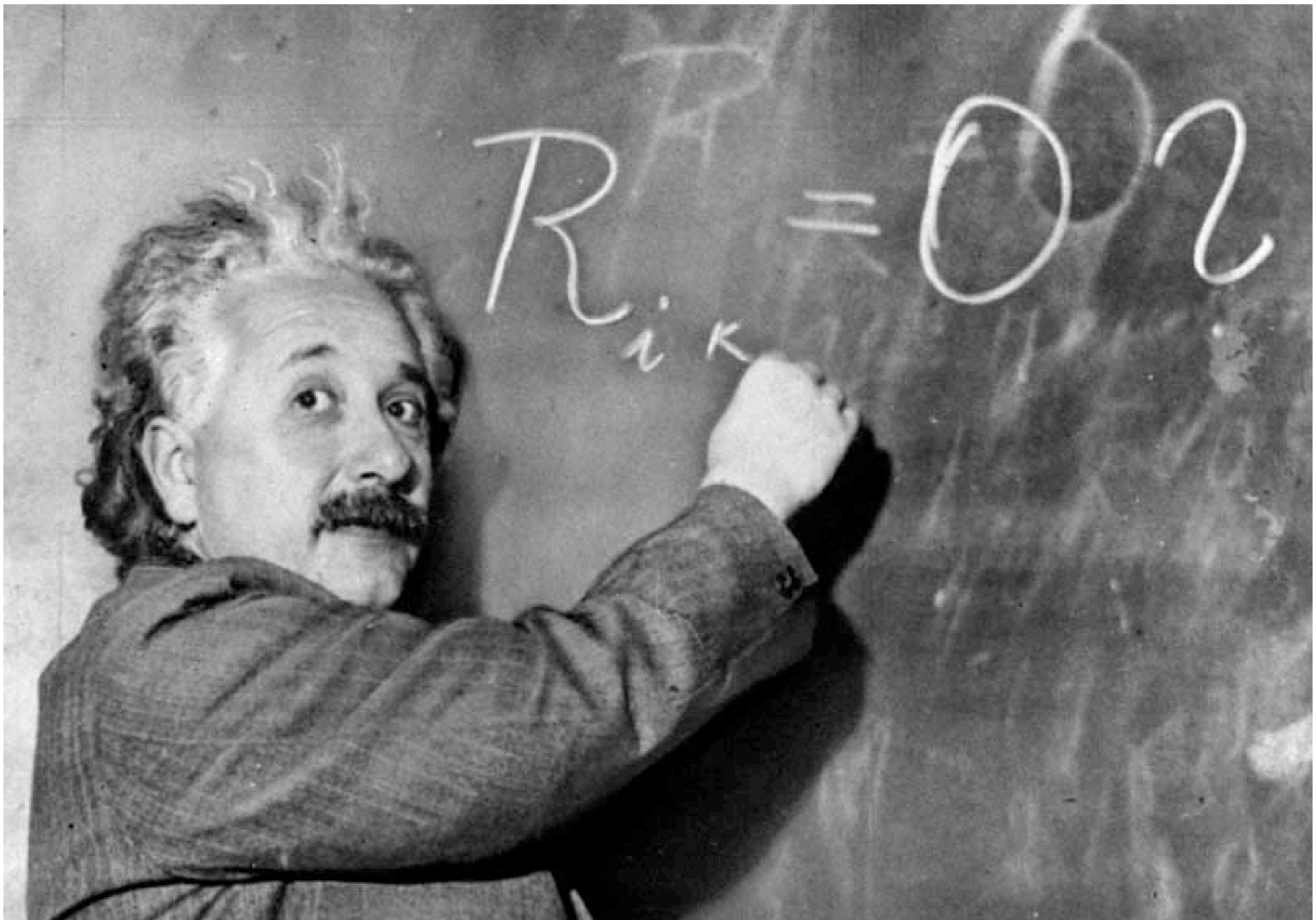

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Has same sign as the *scalar curvature*

$$s = r_j{}^j = \mathcal{R}^{ij}{}_{ij}.$$

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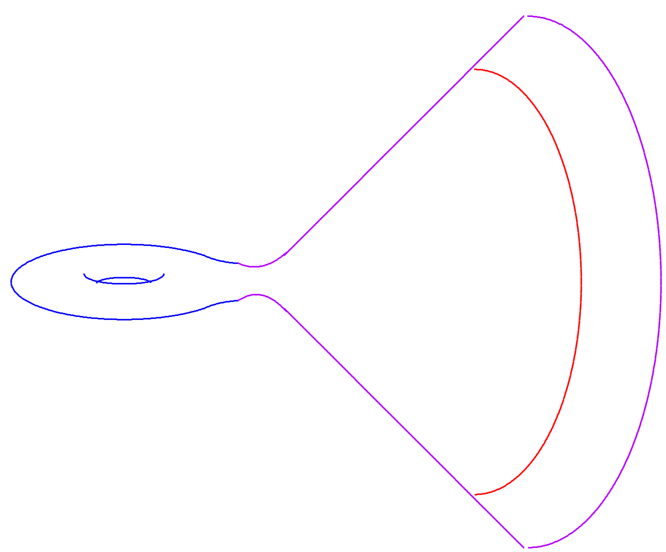

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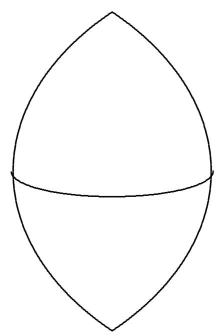

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Goal: Show that this doesn't change anything!

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All known examples all have $W_+ = 0 \ldots$

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Reversing orientation interchanges $\Lambda^+ \leftrightsquigarrow \Lambda^-$.

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splits into 4 irreducible pieces:

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$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

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We avoid this question by means of a definition!

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We can now recast this definition in a much more concrete and effective form by citing classification results of **Ioana Şuvaina** and **Evan Wright**.

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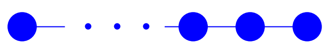
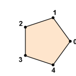
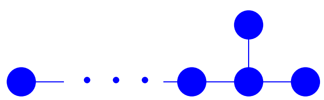
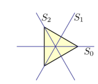
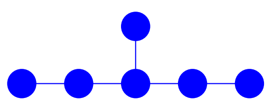
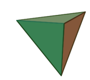
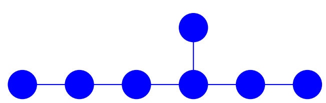
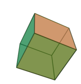
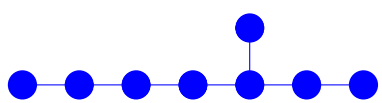
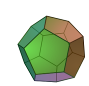
Dynkin Diagram		$\Gamma < SU(2)$	*Classical Figure*
A_k		*Cyclic*	
D_k		*Binary Dihedral*	
E_6		*Binary Tetrahedral*	
E_7		*Binary Octahedral*	
E_8		*Binary Icosahedral*	

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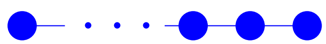
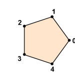
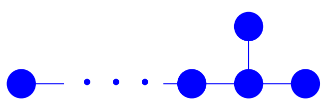
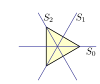
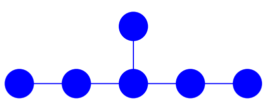
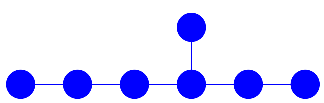
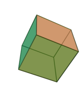
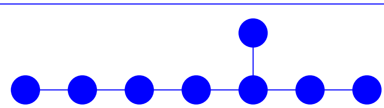
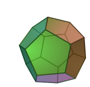
A_k		$\mathbb{Z}_{k+1}$	
D_k		Dih^*_{k-2}	
E_6		*Binary Tetrahedral*	
E_7		*Binary Octahedral*	
E_8		*Binary Icosahedral*	

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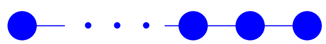
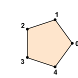
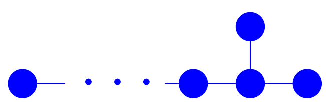
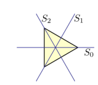
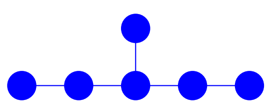
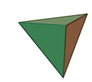
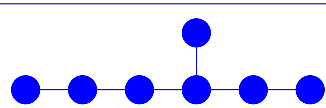
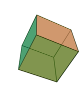
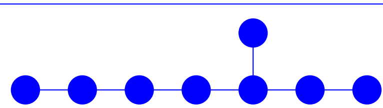
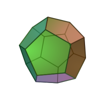
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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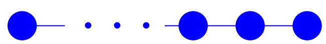
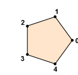
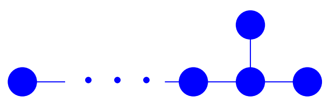
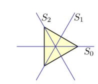
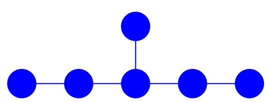
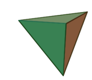
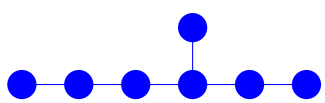
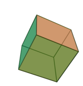
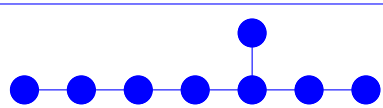
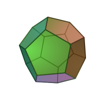
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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Standard shorthand: $\frac{1}{\ell m^2}(1, \ell mn - 1)$.

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These examples form part of a systematic picture. . .

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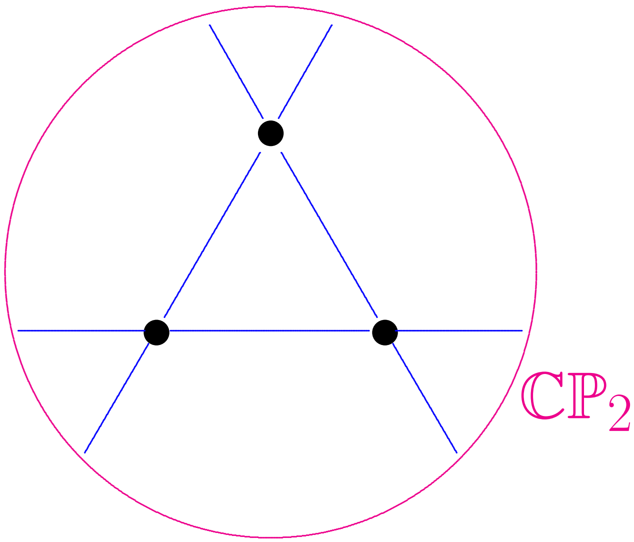

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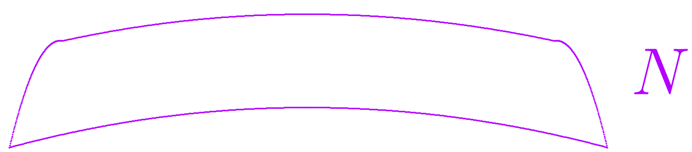

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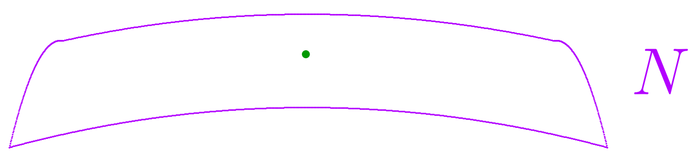

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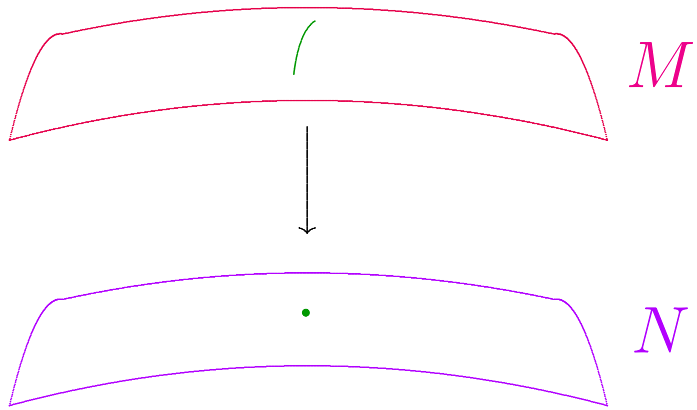

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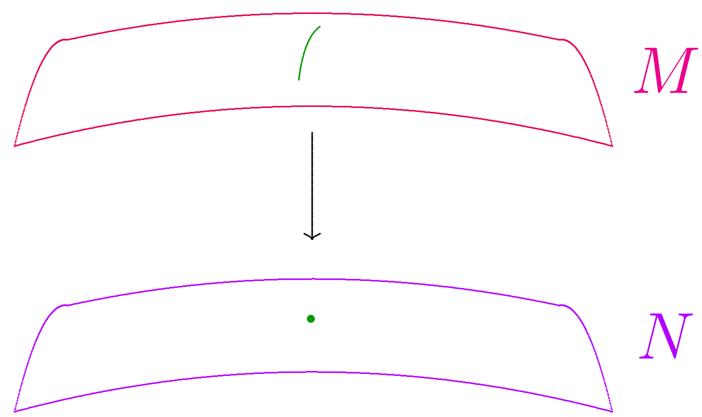

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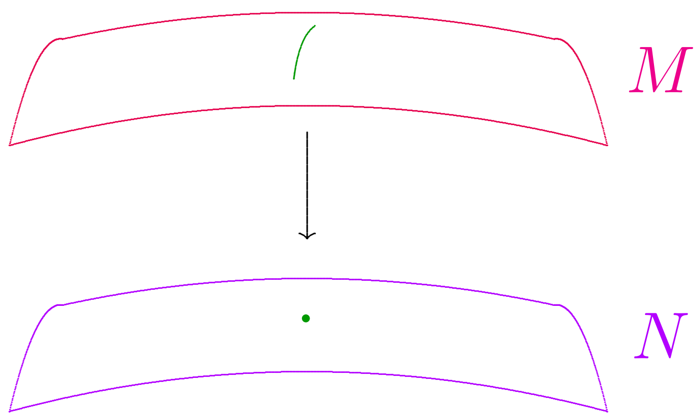

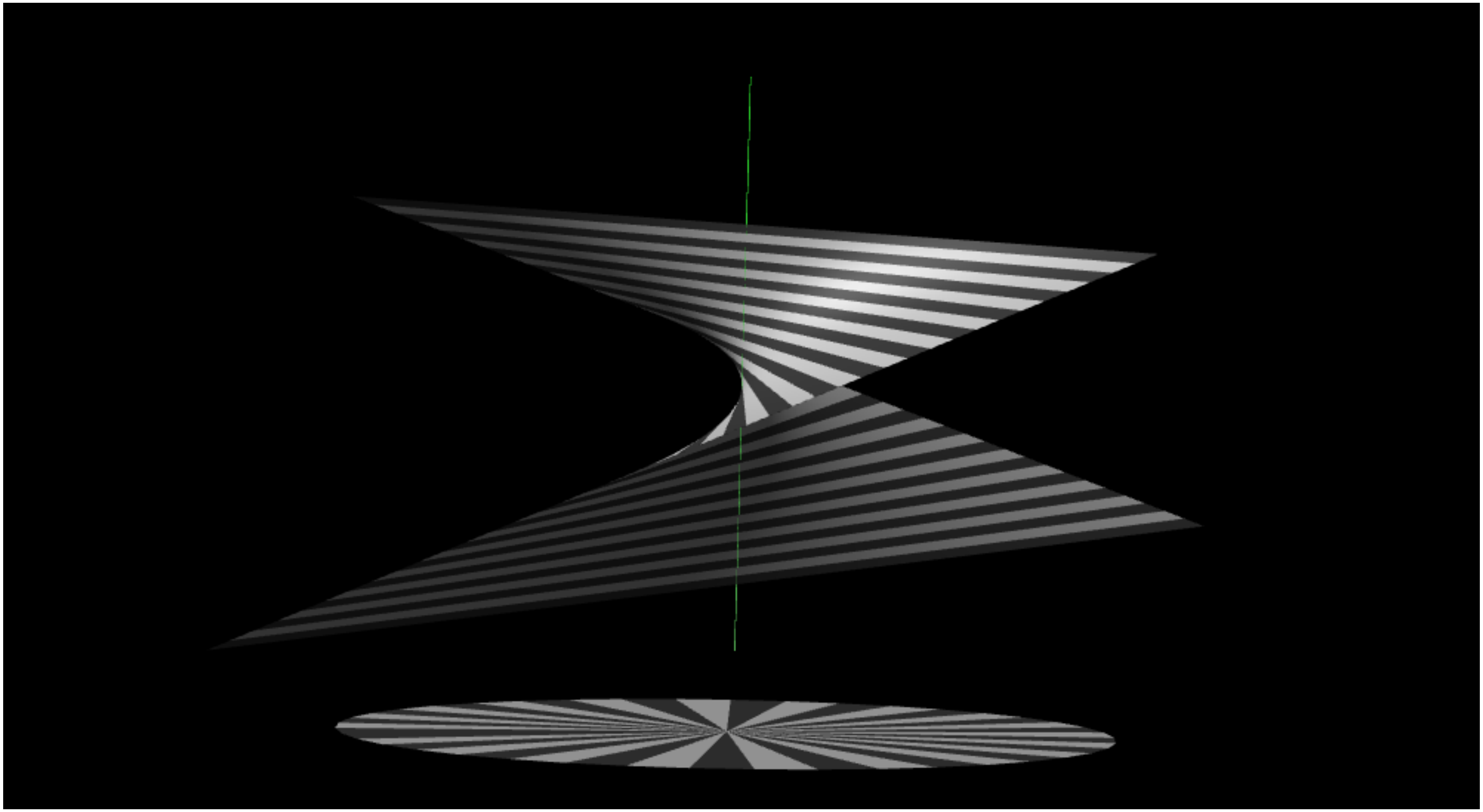

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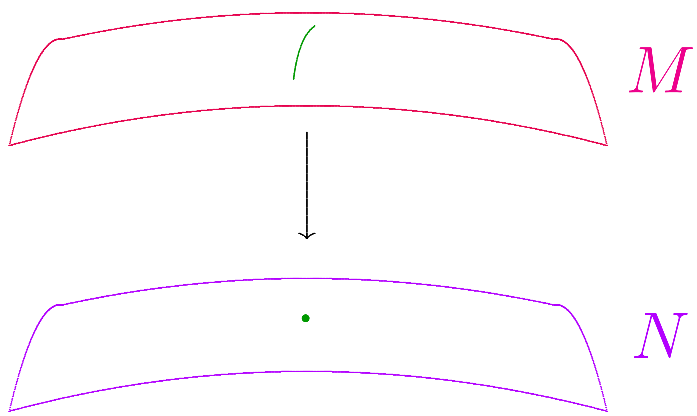

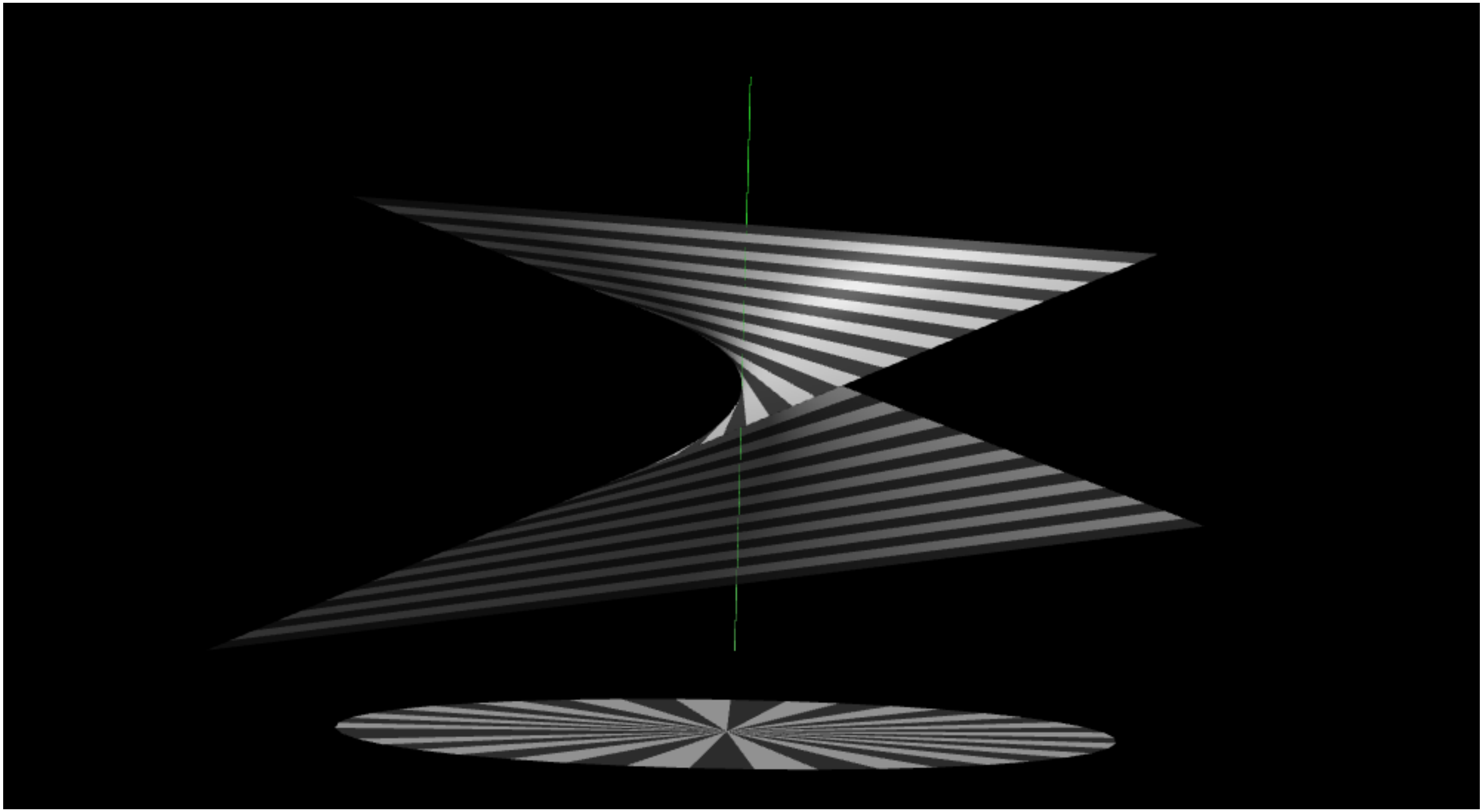

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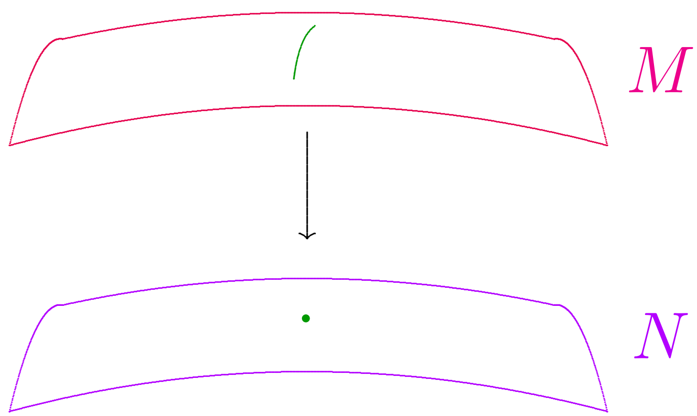

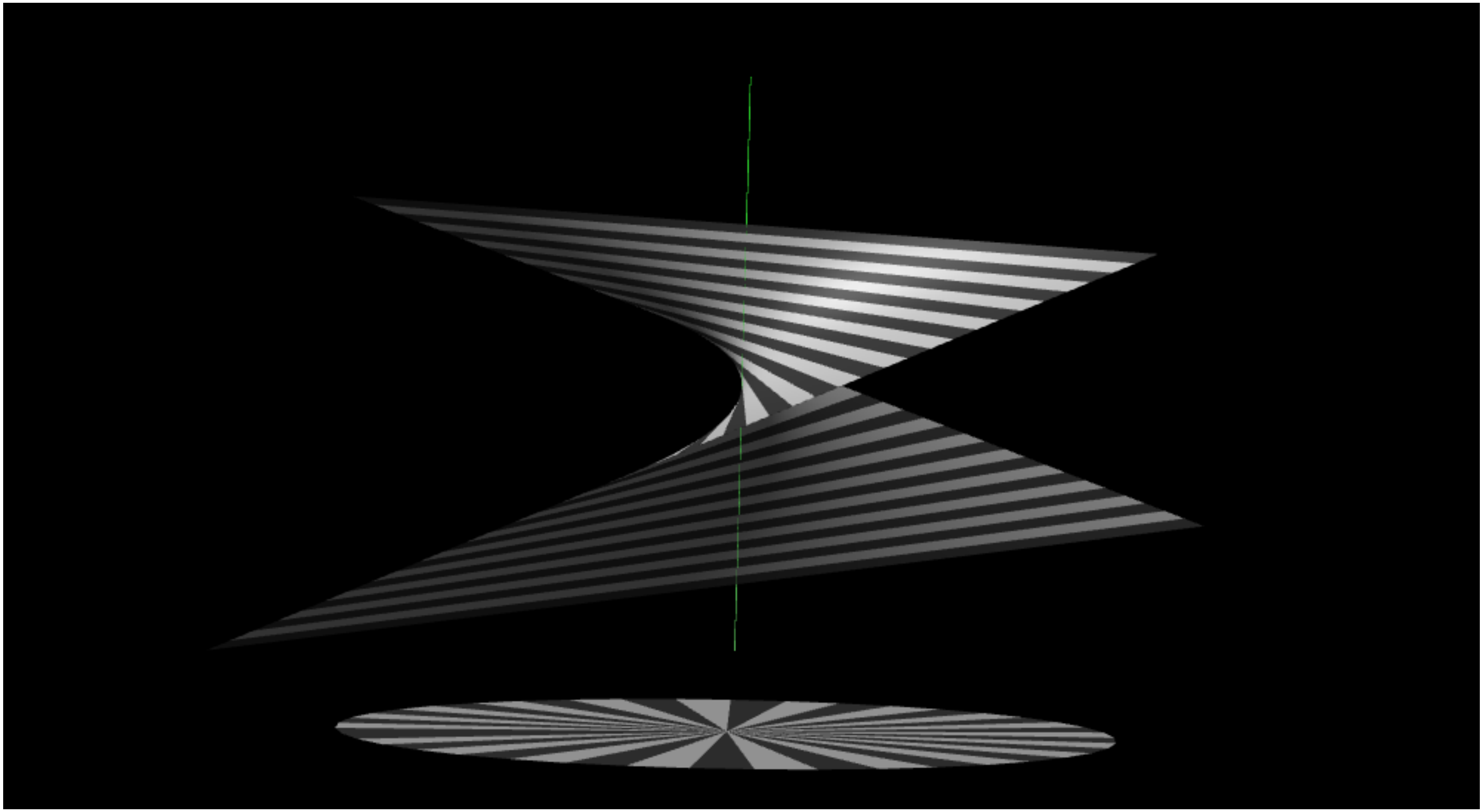

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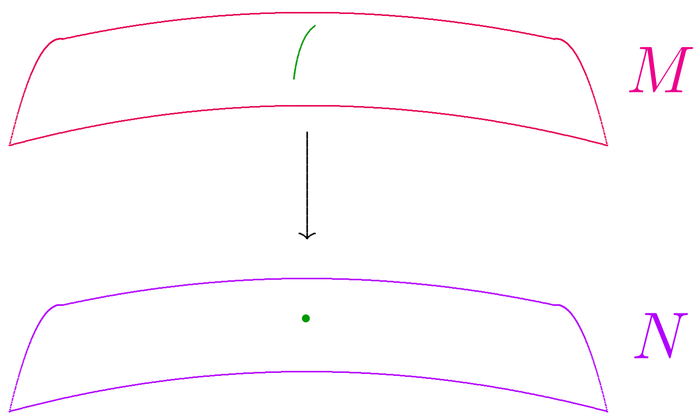

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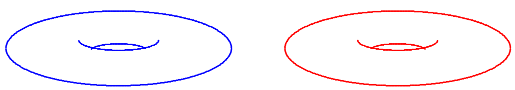

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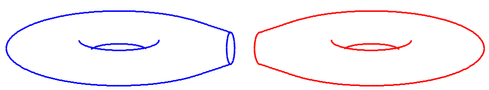

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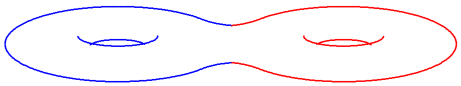

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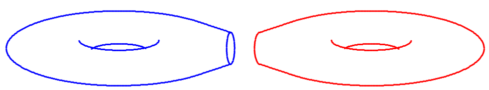

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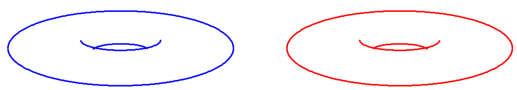

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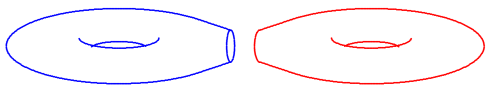

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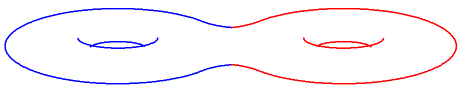

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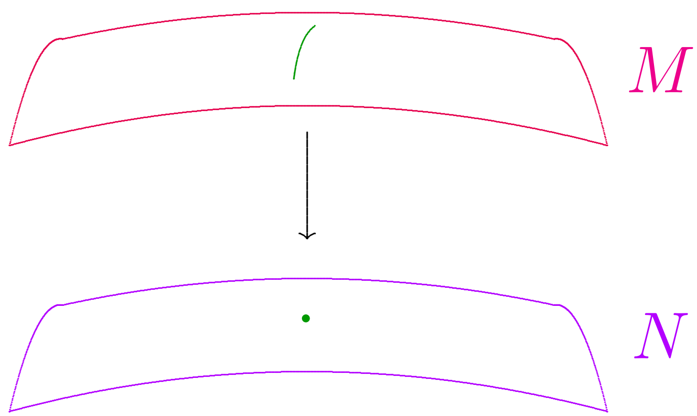

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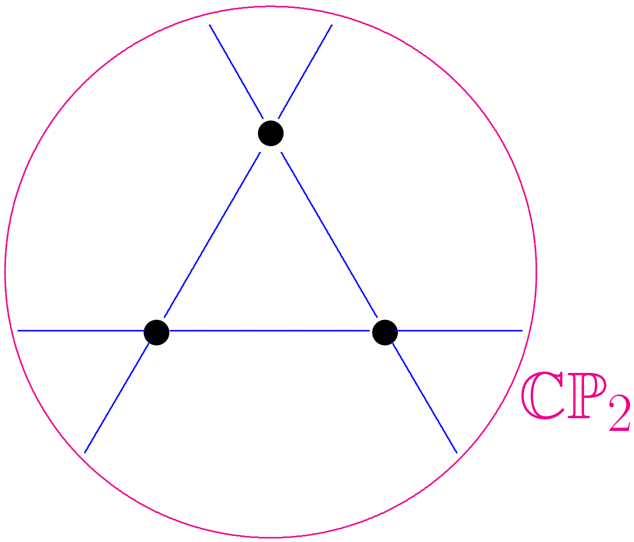

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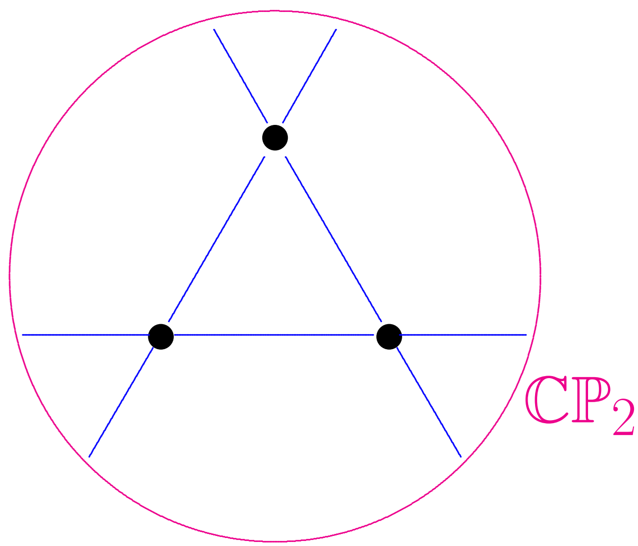
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



No 3 on a line,

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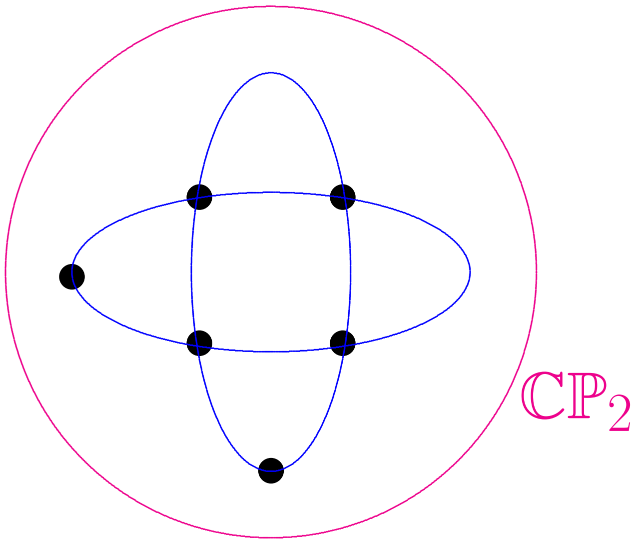

No 3 on a line, no 6 on conic,

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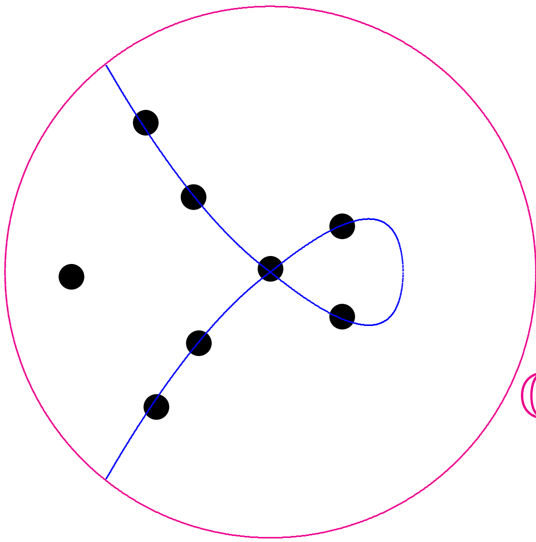

$\mathbb{CP}_2$

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$$g = g(J\cdot, J\cdot)$$

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Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is unique*

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L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi.

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Existence: Page

1978

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

1990

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

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2008

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Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \qquad 5 \leq k \leq 8,$$

because the others are rigid.

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- $\mathbb{CP}_2 \# 8 \overline{\mathbb{CP}}_2$: only (!) $A_1, \ldots, A_{10}$, D_4, $\frac{1}{4}(1,1)$, $\frac{1}{8}(1,3)$, or $\frac{1}{9}(1,2)$ singularities arise.

Example.

Example. $(S^2 \times S^2)/\mathbb{Z}_2$

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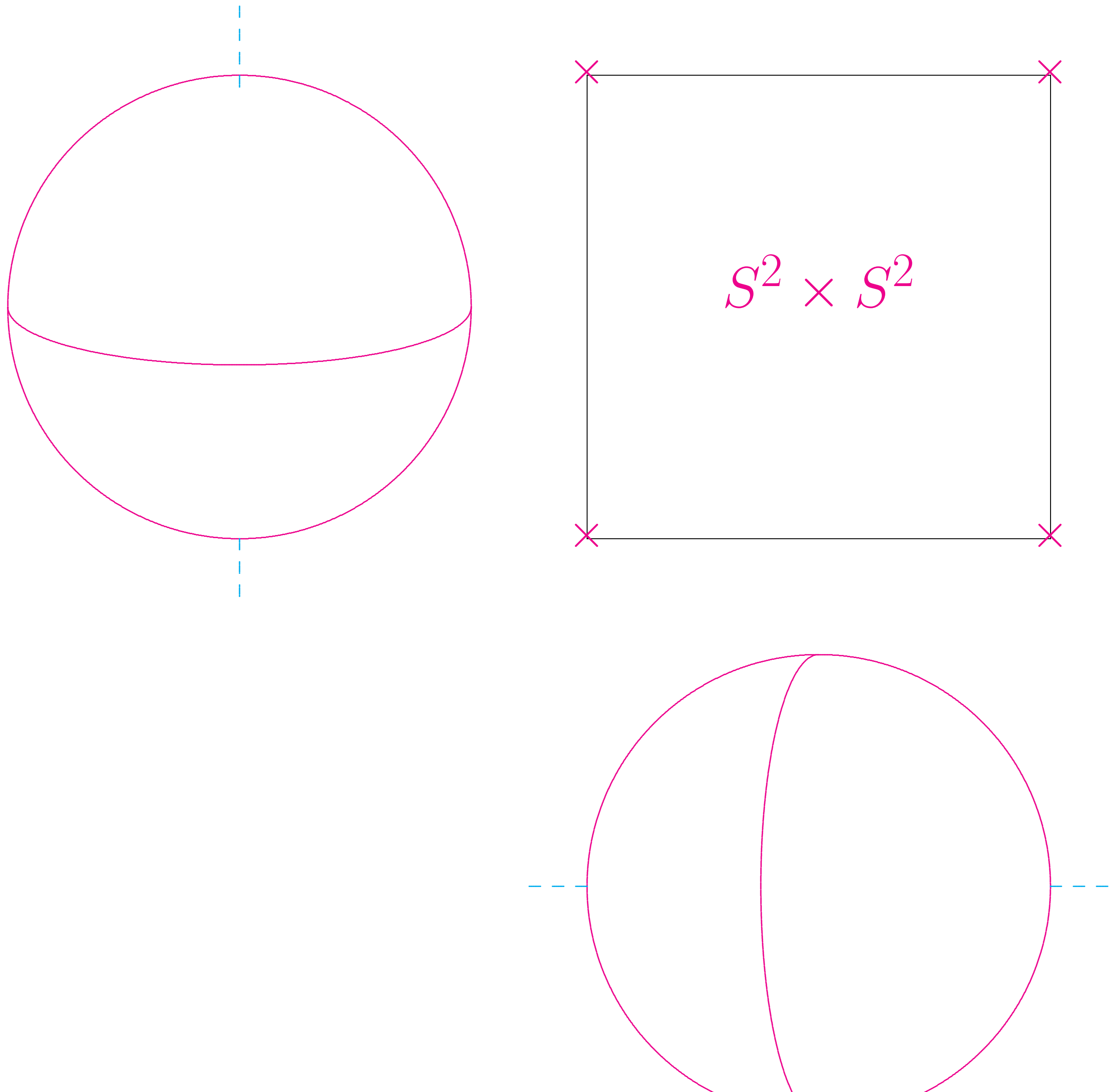

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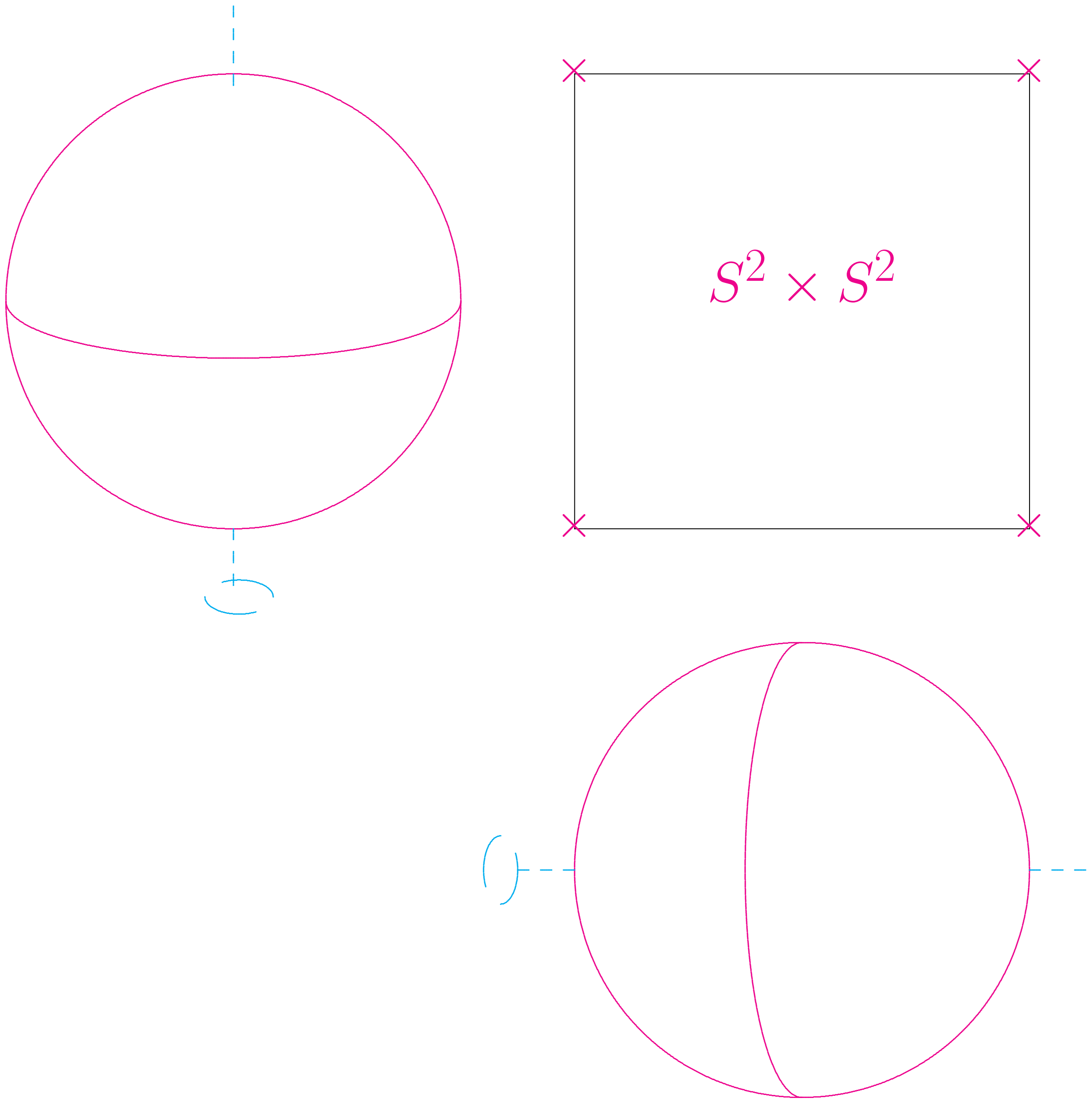

Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$

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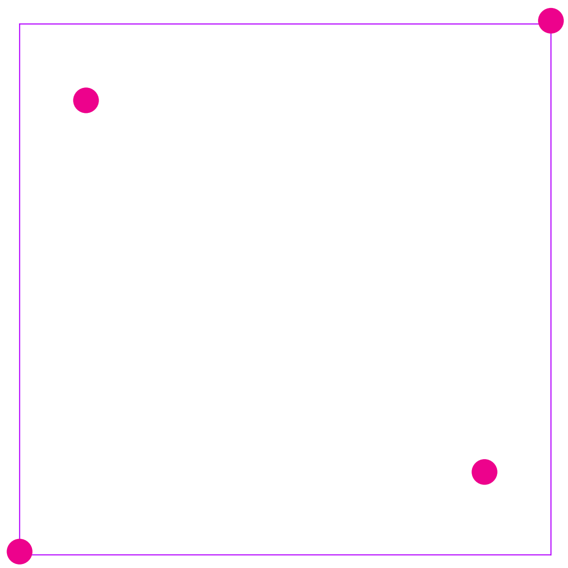

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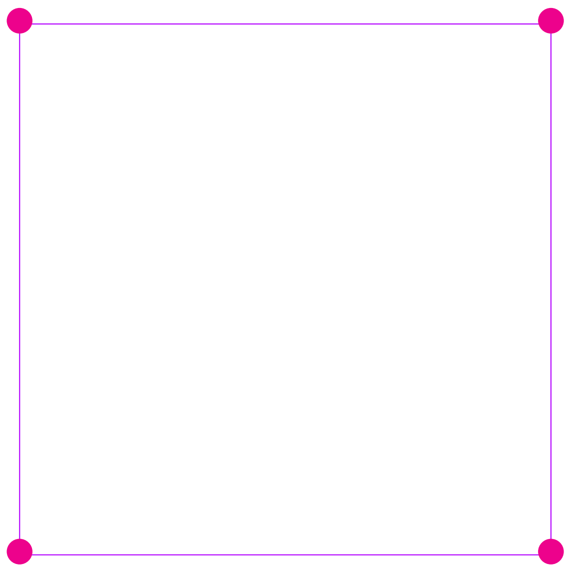

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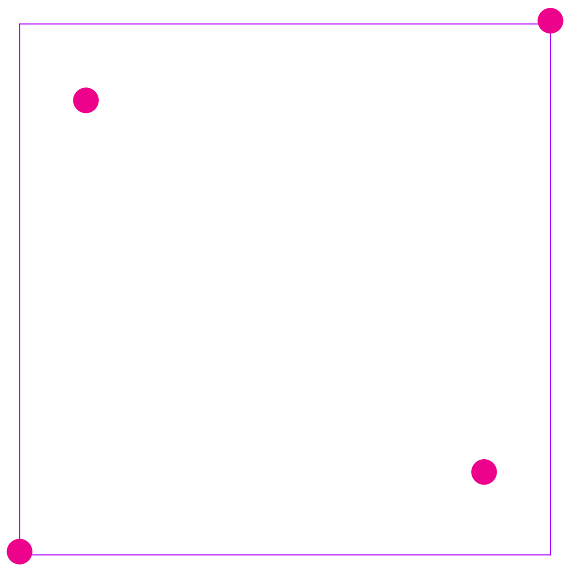

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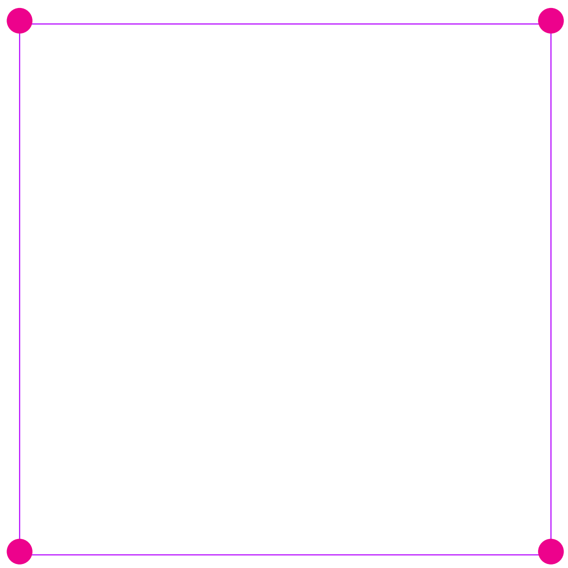

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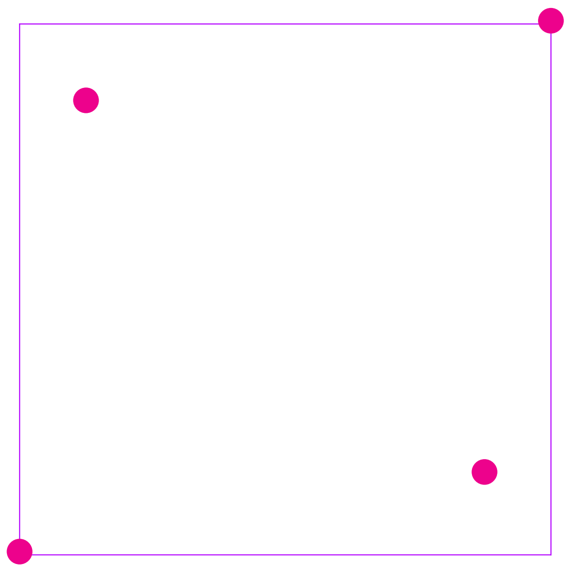

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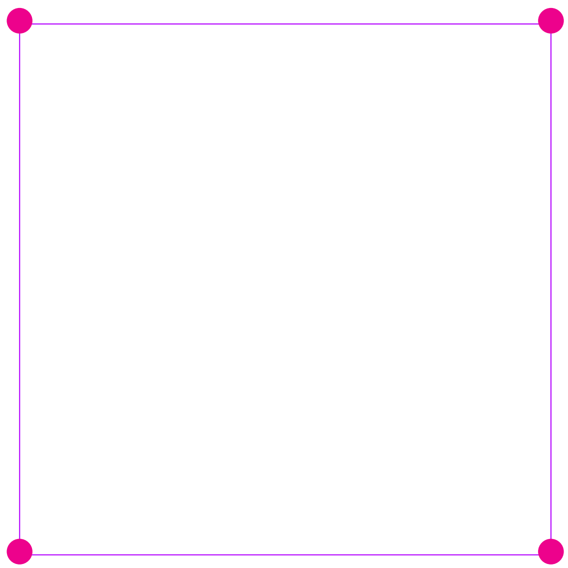

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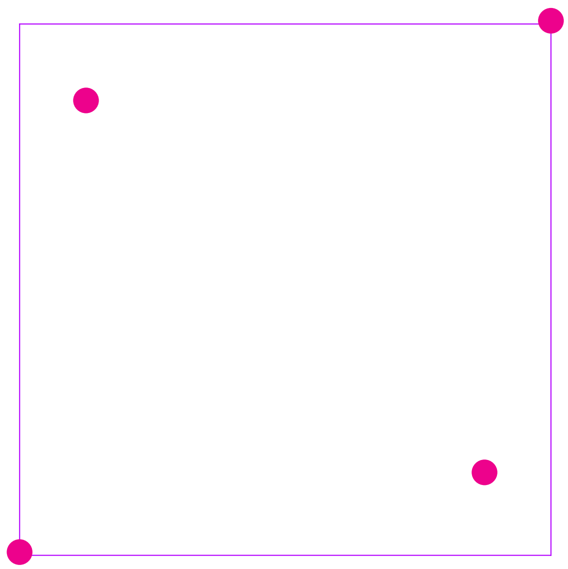

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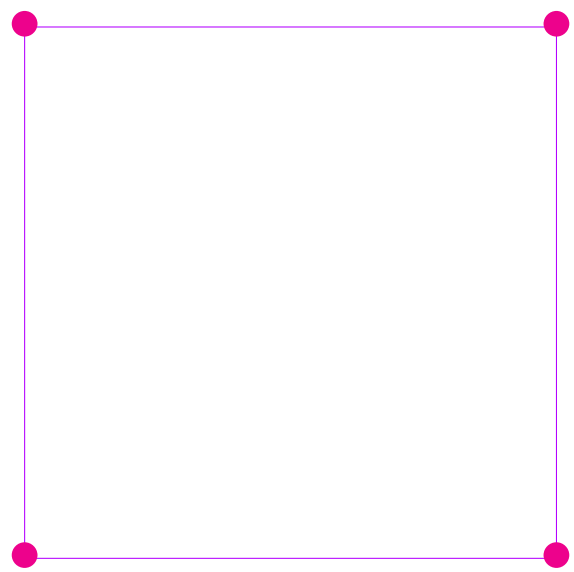

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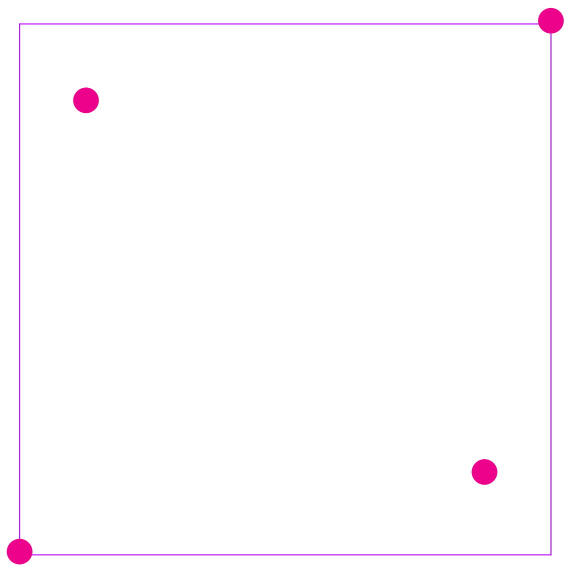

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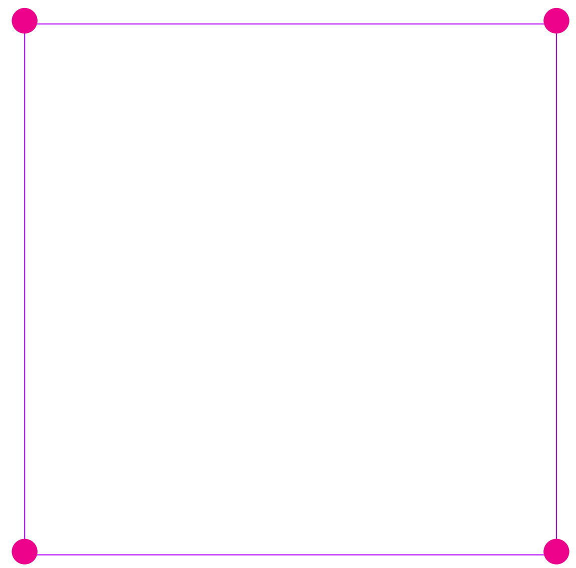

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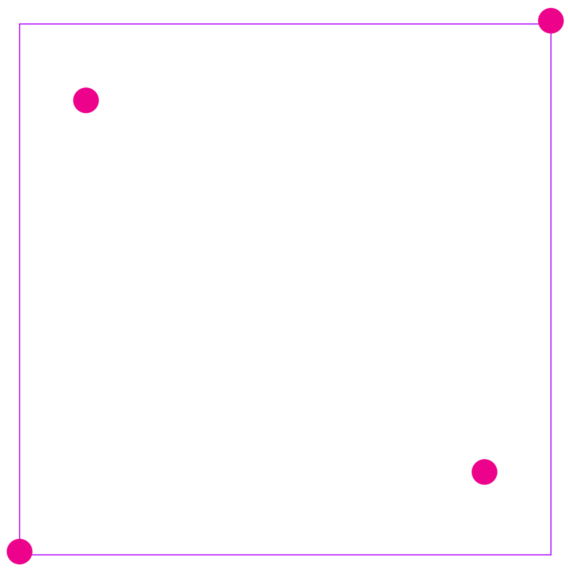

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We thus understand limits of $\lambda > 0$ K-E metrics.

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But what about limits of general Einstein metrics?

Theorem A.

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Theorem B.

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$$g_\infty(J\cdot, J\cdot) = g_\infty$$

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the orbifold Einstein metric g_∞ is conformally Kähler.

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It is false in all higher dimensions!

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A more transparent proof was then given in L '21.

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Conversely, $\lambda > 0$ K-E $\Longrightarrow \det(W^+) > 0$.

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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Then show that the given Einstein metrics g_j below to this universal family for all $j \gg 0$.

And hence that they are actually conformally Kähler!

Many Thanks for the Invitation!

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Mandatory Photo Credit: TransCanada PipeLines Pavilion, Courtesy of the Banff Center.

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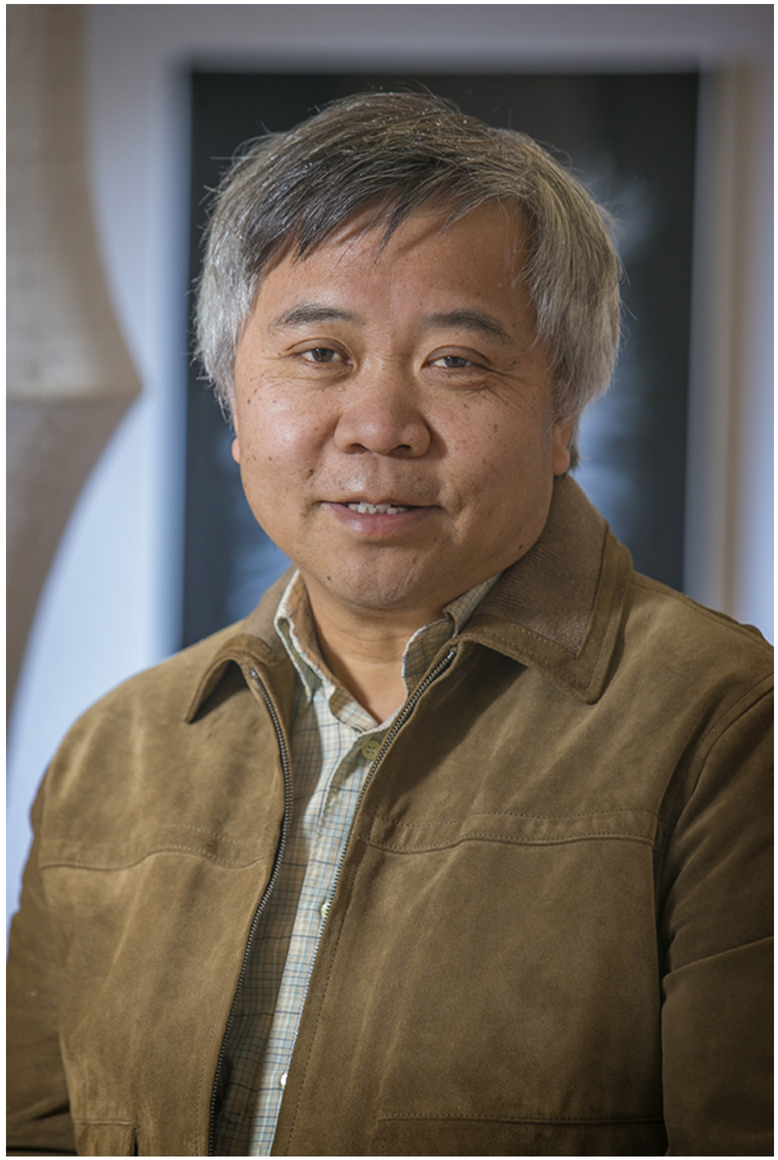

Happy Birthday Xiuxiong!
And Many Happy Returns!